ARAŞTIRMA MAKALESİ/**RESEARCH ARTICLE**

ON THE TOTAL TORSION OF REGULARLY HOMOTOPIC CURVES
İlhan KARAKILIÇ[1]

ABSTRACT

It is well known that the torsion of a closed spherical curve is zero. Using this fact the equality of regularly homotopic spherical curves are proved. This is done by the polygonal secant approximation method.

Key Words: Torsion, Homotopy, Spherical

REGÜLER HOMOTOPİK KÜRESEL EĞRİLERİN TOPLAM BURULMASI ÜZERİNE

ÖZ

Küresel eğrilerin toplam burulmalarının sıfıra eşit olduğu sonucu ve eğrilere poligonal yaklaşım metodu kullanılarak, regüler homotopik eğrilerin toplam burulmalarının birbirine eşit olduğu kanıtlanmıştır.

Anahtar Kelimeler: Burulma, Homotopi, Küresel

1. PRELIMINARIES

The Torsion of a Curve : Let P and Q denotes the neighbor points on a curve α of class C^3, such that $P = \alpha(s)$ and $Q = \alpha(s+h)$, where $h > 0$. And also note that P and Q are not rectification points (that is the points with non zero curvatures).

It is clear that the binormals at P and Q, $b(s)$ and $b(s+h)$, are normals to the osculating planes at P and Q respectively.

If a rotation superposes the osculating plane at P onto the osculating plane at Q then $b(s)$ will be send into $b(s+h)$. Thus the angle between the osculating planes is also the angle between the binormals, let us denote it by θ.

And the limit of ratio $\frac{\theta}{h}$ as $h \to 0$ is called the torsion of the curve at P; $\tau(P) = \underset{h \to 0}{Lim}\frac{\theta}{h}$. Here $\tau(P)$ measures the extent to which α fails to lie in its osculating plane at s.

The Torsion of a Polygonal Curve : With the above description of smooth torsion we now define the polygonal torsion of α. Smooth torsion is a function of points, but the polygonal torsion is a function of line segments, $\sigma_i = \{(1-t)v_i + tv_{i+1} : t \in [0,1]\}$, of a polygonal curve in R^3 with vertices v_i.

In other words, the polygonal torsion τ of α associates to each segment σ_i of α a real number $\tau(\sigma_i) = \tau_i$.

Definition 1. *If $\sigma_{i-1}, \sigma_i, \sigma_{i+1}$ are coplanar then $\tau(\sigma_i = 0.$If $\sigma_{i-1}, \sigma_i, \sigma_{i+1}$ are not coplanar, then $\tau(\sigma_i) = \frac{\theta_i}{|v_{i+1} - v_i|}$. Where θ_i is the (undirected) angle between the binormals $b_i = \frac{(v_i - v_{i-1}) \times (v_{i+1} - v_i)}{|(v_i - v_{i-1}) \times (v_{i+1} - v_i)|}$ and $b_{i+1} = \frac{(v_{i+1} - v_i) \times (v_{i+2} - v_{i+1})}{|(v_{i+1} - v_i) \times (v_{i+2} - v_{i+1})|}$. For the sign of θ_i see Goetz(1970).*

2. THE TOTAL TORSION OF A REGULAR CLOSED CURVE ON S^2

Following theorem is several times proved by different methods. But the polygonal secant approximation (see Penna (1980)) is the focus of the study. Hence the results of the proof will be used later.

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Theorem 2. *The total torsion $\int \tilde{\tau} ds$ of a closed unit speed regular curve $\tilde{\alpha} : R \to S^2$ on the unit 2-sphere S^2 is zero. (tilda over an alphabet is used for the regular case)*

By definition of polygonal torsion, $\tau(\sigma_i) = \frac{\theta_i}{|v_{i+1} - v_i|}$, the following result is studied.

Proposition 3. *Let $\{\alpha_i\}$ be a sequence of polygonal secant approximations to the closed regular curve $\tilde{\alpha}$ such that the vertices of α_i are the vertices of α_j for $i < j$, and such that α_i approaches $\tilde{\alpha}$ uniformly as i tends to ∞. Then for each s_0 the torsion $\tilde{\tau}(s_0) = \underset{i \to \infty}{Lim} \tau(\sigma_i)$ where, for each i, σ_i is the segment of α_i for which $\tilde{\alpha}(s_0)$ lies between v_i and v_{i+1}. Now, if α is sufficiently close polygonal secant approximation to $\tilde{\alpha}$, then we may approximate the total torsion of $\tilde{\alpha}$. $\int \tilde{\tau} ds \cong \sum_i \tau_i |v_{i+1} - v_i| = \sum_i \theta_i$ The summation is taken over all segments α_i of α_0.*

The following proposition is the key to the proof of the theorem 1.

Proposition 4. *Let α be a closed polygonal curve in R^3 whose vertices all lie on S^2 and for which the lengths of all segments σ_i are equal (this condition is the polygonal analog of unit speed), then $\sum_i \theta_i$, the summation is taken over all segments σ_i of α, is an integral multiple of 2π.*

The proof is done simply by taking the projections of the binormals and directed angles θ_i, between these binormals on the unit circle.

Now we can put a corollary for the proof of theorem 1.

Corollary 5. *If α_u, $u \in [0,1]$ is a continuous deformation from α_0 to α_1 and at each stage of the deformation, satisfies the assumptions of the previous proposition (the common length of the segments is allowed to vary from one stage of the deformation to another), then $\sum_i \theta_i$ is the same both α_0 and α_1.*

Proof. Let $f : [0,1] \to Z$ be the functions for the curves α_u, $u \in [0,1]$, which gives the integer that is obtained from the total torsion (integer from the multiple of 2π) for that curve. For example; let the total torsion of α_i, $i \in [0,1]$ is $\int \tau ds = 2\pi n$, then $f(i) = n$.

Since f is continuous, $f([0,1])$ is a connected set in Z. Then $f([0,1])$ must be an integer.

As the result, $f(0) = f(1) = m$ (m is any integer). $f(0) = f(1)$ implies $\sum_i \theta_i$ is same for both α_0 and α_1.

It is obvious that we can get a planer curve by the use of this continuous deformation. Since the total torsion of a planer curve is zero, then the integral multiple of 2π arising in proposition is zero.

The main ideas are sketched here, for rigorous proofs see also Penna (1980).

3. THE TOTAL TORSION OF REGULARLY HOMOTOPIC SPHERICAL CURVES

Definition 6. *The regular curves on a manifold M are said to be regularly homotopic and the homotopy $g_v : I \to M$ can be chosen such that for each $v \in I$, g_v is a regular curve, $g_v'(0) = g_0'(0)$, $g_v'(1) = g_0'(1)$.*

Also a new curve, by gluing the end points of two curves, may be defined as follows.

Definition 7. *Let α and β be curves on S^2 with $\alpha(1) = \beta(0)$. The product of α and β is the curve $\alpha.\beta$, defined by;*

$$(\alpha.\beta)(t) = \alpha\beta = \begin{cases} \alpha(2t) & , \quad 0 \le t \le 0.5 \\ \beta(2t-1) & , \quad 0.5 \le t \le 1 \end{cases}$$

Definition 8. *If α is a curve, the same curve with opposite direction α^{-1} can be defined as; $\alpha^{-1}(t) = \alpha(1-t)$.*

By definition, it is clear that if two curves are homotopic, that is; $\alpha_0 \approx \alpha_1$, then, by defining $\alpha_1^{-1}(t) = \alpha_1(1-t)$, we can get $\alpha_0(1) = \alpha_1(1)$. And also $\alpha_1^{-1}(0) = \alpha_0(1)$ implies the product $\alpha_0.\alpha_1^{-1}$ is a closed curve with $\alpha_1^{-1}(1) = \alpha_0(0)$.

Following theorem is the main result of this study.

Theorem 9. *If two unit speed regular curves α_0 and α_1 are regularly homotopic on S^2, then the corresponding total torsions of these curves are equal.*
(We deal with regular homotopy to hinder the problem of having different derivatives at end points of two homotopic curves)

Proof. By the definition, using the curves α_0 and α_1, one can obtain a regular unit speed closed curve on S^2 by the product $\alpha_0.\alpha_1^{-1}$.

Let β be the curve of the product $\alpha_0.\alpha_1^{-1}$, that is $\beta = \alpha_0.\alpha_1^{-1}$. Then the total torsion of β is $\int_\beta \tau ds$.

Since $\int_\beta \tau ds \cong \sum_i \theta_i$, then $\sum_i \theta_i$ can be written as $\sum_i \theta_i = \sum_j \theta_j + \sum_k \theta_k$ where $\sum_j \theta_j$ and $\sum_k \theta_k$ are the summations of the directed angles between the binormals of the curves α_0 and α_1^{-1}, respectively.

It is clear that the directed angles between the binormals of the curve α_1 have opposite directions of angles between the binormals of the curve α_1^{-1}. Then the directed angles between the binormals of the curve α_1 can be represented as $-\sum_k \theta_k$.

Since a closed unit speed regular curve β on the unit 2-sphere S^2 is formed by the product $\alpha_0.\alpha_1^{-1}$ then $\int_\beta \tau ds = 0$.

Hence, $\int_\beta \tau ds \cong \sum_i \theta_i$ (which is given by linear secant approximation). And, by theorem 1, $\sum_i \theta_i = 0$.

This result simply implies; $\sum_i \theta_i = \sum_j \theta_j + \sum_k \theta_k = 0$ or $\sum_j \theta_j = -\sum_k \theta_k$, where $-\sum_k \theta_k = \int_{\alpha_1} \tau ds$ and $\sum_j \theta_j = \int_{\alpha_0} \tau ds$.

Then $\int_{\alpha_1} \tau ds = \int_{\alpha_0} \tau ds$.

In other words; the total torsion of regularly homotopic unit speed curves on S^2 are equal.

REFERENCES

Goetz, A. (1970). *Introduction to Differential Geometry.* Addison-Wesley.

Rotman, J.J. (1988). *An Introduction to Algebraic Topology.* Springer-Verlag.

Penna, M. A. (1980). *Total Torsion.* Ph.D. thesis at the University of Illinois Urbana.

Smale, S. (1958). Regular Curves on Riemannian Manifolds. *Trans. Amer. Math. Soc.* 81, 492-512.
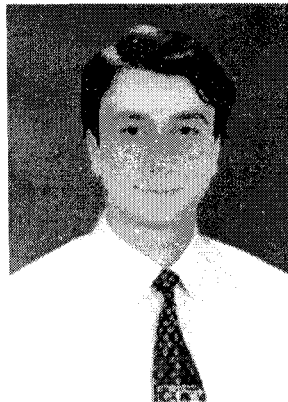
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