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ARAŞTIRMA MAKALESİ/**RESEARCH ARTICLE**

POLYNOMINAL RINGS SATISFYING THE RADICAL FORMULA

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ABSTRACT

It is not completely known that which non-Noetherian rings satisfy the radical formula In this paper, a necessary and sufficient condition for a polynomial ring to satisfy the radical formula is given.

Key Word: Prime Submodules, Radical Formula.

RADİKAL FORMULAYI SAĞLAYAN POLİNOM HALKALARI

ÖZ

Noether olmayan hangi halkaların radikal formulayı sağladığı tam olarak bilinmemektedir. Bu makalede bir polinom halkasının radikal formulayı sağlaması için gerek ve yeter bir koşul verilecektir.

Anahtar Kelimeler: Asal Altmodül, Radikal Formülü.

1. INTRODUCTION

Let R be a commutative ring and M be an R-module. A proper submodule P of M is called *prime* if whenever $rm \in P$ for some $r \in R$, $m \in M$, then $m \in P$ or $r M \subseteq P$. Let N be a submodule of M with $N \neq M$. The *radical* of N in M, $\text{rad}_M (N)$ is defined to be the intersection of all prime submodules of M containing N. If there is no prime submodule containing N, then we put $\text{rad}_M (N) = M$. The *envelope* of N in M, $E_M (N)$, is defined to be the set

$\{rm : r \in R$ and $m \in M$ such that $r^n m \in N$ for some positive interger $n \geq 1\}$.

We say that *M satisfies the radical formula* (M s.t.r.f.) if for every submodule N of M, the radical of N is the submodule genereted by its envelope, i.e. $\text{rad}_M (N) = < E_M (N) >$. A ring R s.t.r.f. provided that every R-module s.t.r.f..

2. RESULTS

Following work of McCasland and Moore (1986), (1991), (1992) and of Jenkins and Smith (1992), in a series of recent papers Man (1996), (1997a), (1997b) and Man and Leung (1997), have characterised which commutative Noetherian rings s.t.r.f.. In partucular, Man showed that a commutative Noetherian domain s.t.r.f. if and only if R is Dedekind. It is not entirely clear to us which non-Noetherian rings s.t.r.f.. But for a polynominal rings $S [X]$ where S is commutative (not necessarily Noetherian) domain we can say the following:

Theorem 2.1. *Let S be commutative domain. Then the polynomial ring $R = S [X]$ s.t.r.f. if and only if S is a field.*

Proof. ($\Rightarrow$) Suppose R s.t.r.f.. Then the R-module $F = R \oplus R$ s.t.r.f.. Let $0 \neq a \in S$ and let W be the ideal $\sqrt{R_a + RX}$ of R and N be the submodule $W (a,X)$ of F. First we will show that $N = E_F (N)$. Let r, s_1, s_2, belong to R such that $r^k (s_1, s_2) \in N$ for some positive integer k. There exists $w \in W$ such that $r^k (s_1, s_2) = w (a,X)$, i.e. $r^k s_1 = wa$, $r^k s_2 = wX$. It follows that $r^k s_1 X = r^k s_2 a$. If $r = 0$ then $r (s_1, s_2) \in N$. Suppose that $r \neq 0$. Then $s_1 X. = s_2 a$. Since $a \neq 0$ it follows that $s_2 = Xh$ for some $h \in R$. Then $s_1 X = s_2 a = Xha$ gives $s_1 = ha$. Now $r^k (s_1, s_2) = r^k (ha,hX) = r^k h (a,X)$ and hence $r^k h \in W$.

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Clearly $(rh)^k \in W$ and hence $rh \in W$. Thus $r(s_1, s_2) = rh(a, X) \in N$. It follows that $E_F(N) \subseteq N$ and hence $E_F(N) = N$. Since F s.t.r.f. $N = E_F(N) = \langle E_F(N) \rangle = \mathrm{rad}_F(N)$. Now let K be a prime submodule of F such that $N \subseteq K$. Then $W(a, X) \subseteq K$ gives $WF \subseteq K$ or $(a, X) \in K$. In any case $(a, X) \in K$. Thus

$$R(a, X) \subseteq \mathrm{rad}_F(N) = N = W(a, X).$$

There exists $q \in W$ such that $(a, X) = q(a, X)$. In particular, $a = qa$ so that $q = 1$. It follows that $W = R$ and hence $R = Ra + RX$. There exist $f(X), g(X) \in R$ such that $1 = f(x)a + g(X)X$. Then $1 = f(0)a$ and hence a is a unit in S.

($\Leftarrow$) If S is a field then $S[X]$ is a principal ideal domain and hence a Dedekind domain. Thus $R = S[X]$ s.t.r.f by Theorem 9 in Jenkins and Smith (1992).

Corollary 2.2. *Let R be a commutative ring. Then the polynomial ring $R[X_1, ..., X_n]$ does not s.t.r.f. for positive integers $n > 1$.*

Proof. It is easy to check that if the commutative ring R s.t.r.f. then the ring R / I s.t.r.f. where I is a proper ideal of R. Now suppose $R[X_1, ..., X_n]$ s.t.r.f. where $n > 1$. Let $\mathcal{P}$ be any prime ideal of R. Then the ring

$$(R/\mathcal{P})[X_1, ..., X_n] \cong R[X_1, ..., X_n] / \mathcal{P}[X_1, ..., X_n]$$

s.t.r.f.. Let $S = (R/\mathcal{P})[X_1, ..., X_{n-1}]$. Then $S[X_n] \cong (R/\mathcal{P})[X_1, ..., X_n]$, so s.t.r.f. but S is not a field, a contradiction.

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