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THE MAP SHIFTxSHIFT IS CHAOTIC

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ABSTRACT

In this work we consructed a chaotic functions by using the well known chaotic function which name is shift map.

Key Words: Chaos, Topological Transitivity, Sensitive Dependence, Shift Map.

SHIFTxSHIFT DÖNÜŞÜMÜ KAOTİKTİR

ÖZ

Bu çalışmada bilinen bir kaotik fonksiyon örneği olan shift dönüşümünü kullanarak yeni bir kaotik fonksiyon inşa edilmiştir.

Anahtar Kelimeler: Kaos, Topolojik Geçişgenlik, Hassas Bağımlılık, Shift Dönüşümü.

1. INTRODUCTION

The shift map is one of the most famous chaotic functions. In this work firstly we showed that the composition of the shift map with itself is again chaotic. Then we pointed out a relationship between this composite function and product of shift map with itself. Finally using that relationship which is called topological conjugacy we showed that the product function is also chaotic.

2. PRELIMINARIES

Functions which have the three characteristics namely density of periodic points, topological transitivity and sensitive dependence on initial conditions are said to be chaotic. (Devaney 1989 and Holmgren 1994). In the following definition for a function of the form $f: X \to X$, f^k means k times composion of f with itself, that is

$$f^k = f \circ f \circ \ldots \circ f$$

and a point $x \in X$ is called periodic with period n if $f^n(x) = x$ and $f(x) \neq x$, $(1 \leq i \leq n - 1)$. For $x \in X$ the set of the points x, $f(x)$, $f^2(x)$, $f^3(x)$,...

is called the orbit of x.

Definition 1:

Let X be a metric space with metric d. Then the function $f: X \to X$ is chaotic if

1) the set of the periodic points of f is dense in X.

2) f is topologically transitive: for any pair of open non-empty sets U, V, $\subset$ X there exists a positive integer k such that $f^k(U) \cap V \neq \phi$.

3) f is sensitively dependent on initial conditions: there exist a number $\delta > 0$ such that, for any x in X and any neighborhood N of x, there exist a point y in N and a positive integer n such that $d(f^n(x), f^n(y)) > \delta$.

In case of X is an infinite set and $f: X \to X$ is a continuous function, a paper by Banks at. al.(Banks, J., 1992) showed that the hypothesis concerning sensitive dependence is implied by the remaining two conditions.

3. SEQUENCE SPACE AND SHIFT MAP

Definition 2:

The set $\sum_2 = \{(s_0\, s_1\, s_2 \ldots) | s_i = 0 \text{ or } 1\}$ which contains all infinite sequences of 0's and 1's is called

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the sequence space of 0 and 1 or the symbol space of 0 and 1. The metric d on $\sum_2$ is defined by $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$, where $s = s_0 s_1 s_2 \ldots$ and $t = t_0 t_1 t_2 \ldots$ in $\sum_2$.

The shift map $S : \sum_2 \to \sum_2$ is defined by $S(s_0 s_1 s_2 \ldots) = s_1 s_2 \ldots$ and it is chaotic. (Devaney, 1989; Gullick, 1992).

Following proposition shows that the composition of shift map with itself is chaotic also.

Proposition 1:

The composite map

$$S^2 = S \circ S : \sum_2 \to \sum_2$$

$$S^2 (s_0 s_1 s_2 \ldots) = S(S(s_0 s_1 s_2 \ldots)) = s_2 s_3 s_4 \ldots$$

is chaotic.

Proof :

The periodic points for S^2 are repeating sequences of the form

$$s_0 s_1 \ldots s_{2n-1}\ s_0 s_1 \ldots s_{2n-1}\ s_0 s_1 \ldots s_{2n-1} \ldots$$

for $n \geq 1$. Let $t = t_0 t_1 t_2 \ldots$ be any point in $\sum_2$ and $B_\varepsilon(t)$ be any open disc centered at t with radius $\varepsilon > 0$. Now choose n such that $\frac{1}{2^{2n-1}} < \varepsilon$ and define y by

$$y = t_0 t_1 \ldots t_{2n-1}\ t_0 t_1 \ldots t_{2n-1}\ t_0 t_1 \ldots t_{2n-1} \ldots$$

Then y is periodic and $d(t,y) \leq \frac{1}{2^{2n-1}}$. As $\frac{1}{2^{2n-1}} < \varepsilon$, y is belongs to $B_\varepsilon(t)$. So the set of the periodic points of S^2 is dense in $\sum_2$.

Consider the following element of $\sum_2$

$s^* = 00.01.10.11.0000.0001.0010\ldots1111\ldots$ in which $2n$-tuples of 0's and 1's in order for each positive integer n. The orbit of s^* is dense in $\sum_2$. Because some iterate of S^2 aplied to s^* yields a sequence which agrees with any given sequence in an arbitrarily large number of places. It is known that existence of such a dense orbit leads to topological transitivity (Devaney, R., 1989). So S^2 is topologically transitive.

Because of the density of periodic points and topological transitivity of S^2, S^2 is sensitively dependent on initial conditions by the theorem of Banks at al. (Banks, 1992).

4. CHAOTICITY OF SHIFTxSHIFT

We want to show that the product of shift map with itself is chaotic, where product means is the following:

Definition 3:

Let $f : X \to X$ and $g : Y \to Y$ be given functions. The product of f and g is the function $f \times g : X \times Y \to X \times Y$, defined by

$$(f \times g)(x,y) = (f(x), g(y)).$$

Definition 4:

Let $u : X \to X$ and $v : Y \to Y$ be functions. Then u is topologically conjugate to if there is a homeomorphism $h : X \to Y$ such that $h \circ u = v \circ h$. In this case h is called topological conjucacy.

When the functions of X and Y are topologically conjugate, the existence of the homeomorphism from X to Y guarantees that the topologies of the two spaces are identical. The condition $h \circ u = v \circ h$ guarantees that their dynamical properties are the same. One of them is stated in the following theorem (Holmgren, 1994).

Proposition 2:

Let X and Y be metric spaces, $u : X \to X$, $v : Y \to Y$ given functions and $h : X \to Y$ be topological conjugacy of u and v. Then u is chaotic on X if and only if v is chaotic on Y.

We use above theorem to show that $S \times S$ is chaotic on the product space $\sum_2 \times \sum_2$.

Proposition 3:

The product map $S \times S$ on $\sum_2 \times \sum_2$ is topologically conjugate to the composite map $S^2 = S \circ S$ on $\sum_2$ via the function h: $\sum_2 \times \sum_2 \to \sum_2$ defined by

$$h(x_0 x_1 x_2 \ldots, y_0 y_1 y_2 \ldots) = x_0 y_0 x_1 y_1 x_2 y_2 \ldots .$$

Proof:

The continuity and bijectivity of h is clear. The inverse of h is defined by

$$h^{-1}: \sum_2 \to \sum_2 \times \sum_2 ,$$

$$h^{-1}(x_0 x_1 x_2 x_3 \ldots) = (x_0 x_2 \ldots, x_1 x_3)$$

and it is continuos. Now let`s consider the following diagram

$$\begin{tikzcd}
\Sigma_2 \times \Sigma_2 \arrow[r, "S \times S"] \arrow[d, "h"'] & \Sigma_2 \times \Sigma_2 \arrow[d, "h"] \\
\Sigma_2 \arrow[r, "S^2"] & \Sigma_2
\end{tikzcd}$$

Let $(x,y) = (x_0x_1x_2..., y_0y_1y_2...)$ be an arbitrary point in $\sum_2 \times \sum_2$, the image of this point under the composite map $h \circ (S \times S)$ is

$$\begin{aligned} h \circ (S \times S)(x,y) &= h \circ (S \times S)(x_0x_1x_2..., y_0y_1y_2...) \\ &= h(S(x_0x_1x_2..., y_0y_1y_2...), S(y_0y_1y_2...)) \\ &= h(x_1x_2..., y_1y_2...) \\ &= x_1y_1x_2y_2... \end{aligned}$$

the image of the point (x,y) under the other composition is

$$\begin{aligned} S^2 \circ h(x,y) &= S^2 \circ h(x_0x_1x_2..., y_0y_1y_2...) \\ &= S^2(x_0y_0x_1y_1x_2y_2...) \\ &= x_1y_1x_2y_2... \end{aligned}$$

so we get the equality $h \circ (S \times S) = S^2 \circ h$. This completes the proof.

Now we can state our main theorem.

Theorem 1:

The product map $S \times S$ is chaotic on the product space $\sum_2 \times \sum_2$.

The proof of this theorem follows from the successive using of Proposion 1, Proposition 2 and Proposition 3.

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