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THE QUATERNION REPRESENTATION OF STATIC GRAVITATIONAL FIELD: POISSON'S EQUATION

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ABSTRACT

The quaternions are numbers which have division algebra. This property advantages for physicists. So, Quaternions can be used in the each field of physics and, physical quantities can be represented by the quaternions. In this paper, a quaternionic equation which replaces to the two vector equations of static gravitational field is written in 4-dimensions. In addition, Poisson's equation is also defined in quaternionic representation.

Key Word: Quaternion, Poisson's Equation, Static Gravitational Field.

STATİK GRAVİTASYONEL ALANIN KUATERNİON GÖSTERİMİ: POİSSON DENKLEMİ

ÖZ

Kuaternionlar bölüm cebri olan bir sayı sistemidir. Bu özellik fizikçiler için avantaj sağlamaktadır. Kuaternionlar fiziğin her alanında kullanılabildiği gibi fiziksel nicelikler kuaternionlarla gösterilebilir. Bu çalışmada, statik gravitasyonel alana ait iki vektör denklemi yerine geçen 4-boyutta tek bir kuaternion denklemi yazılmıştır. Ayrıca, Poisson denklemi de kuaternionik gösterimde tanımlanmıştır.

Anahtar Kelimeler: Kuaternion, Poisson Denklemi, Statik Gravitasyonel Alan.

1. INTRODUCTION

Complex numbers were a hot subject for research in the early eighteen hundreds. An obvious question was that if a rule for multiplying two numbers together was known, what about multiplying three numbers? For over a decade, this simple question had bothered Hamilton, the big mathematician of his day.

Hamilton had found a long sought-after solution, it was 4-dimension. One of the first things Hamilton did was get rid of the fourth dimension, setting it equal to zero, and calling the result a "proper quaternion". He spent the rest of his life trying to find a use for quaternions. By the end of the nineteenth century, quaternions were viewed as an oversold novelty.

In the early years of this century, Prof. Gibbs of Yale found a use for proper quaternions by reducing the extra fluid surrounding Hamilton's work and adding key ingredients from Rodrigues concerning the application to the rotation of spheres. He ended up with the vector dot product and cross product we know today.

Today, quaternions are of interest to historians of mathematics. Vector analysis performs the daily mathematical routine that could also be done with quaternions.

Quaternions which are very useful numbers in the justification of the postulates in special relativity, quantum and classical mechanics as well as in solving high energy physic's problems can be used to representing of physical quantities. Some of them, for example, are Dimensional - Directional Analysis by a Quaternionic Representation of Physical Quantities (Arenada,1996), General Quaternion Transformation Representation for Robotic Application (Tan and Balchen,1993), and Quaternion Scalar Field is another example (De Leo and Rotelli, 1992).

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2. QUATERNION ALGEBRA

Considering that the physical quantities of Newtonian mechanics are scalars or vectorials. The both types of quantities in the quadri-dimensional vectorial space of the quaternions is possible represented. Then a physical scalar(vectorial) quantity is represented by a scalar(vector) quaternion.

A quaternion is a quantity represented symbolically by Q and defined by the equation (Özdaş and Özdaş,1986):

$$Q = q_0\hat{\lambda}_0 + q_1\hat{\lambda}_1 + q_2\hat{\lambda}_2 + q_3\hat{\lambda}_3 = [q_0,q_1,q_2,q_3]$$
$$Q = \sum_{k=0}^{3} q_k\hat{\lambda}_k, \quad \hat{\lambda}_0 = 1, \quad (k = 0,1,2,3) \tag{1}$$

where the real numbers q_k denote the component of Q relative to the unitary quaternion $\hat{\lambda}_k$ (k=0,1,2,3). The scalar and vectorial parts of Q are designed, respectively, by $(Q)_s$ and $(Q)_v$, and they are defined by,

$$(Q)_s = q_0\hat{\lambda}_0$$
$$(Q)_v = q_1\hat{\lambda}_1 + q_2\hat{\lambda}_2 + q_3\hat{\lambda}_3 \tag{2}$$

A quaternion is a scalar(vector) quaternion if its vectorial(scalar) parts are equal to zero.

The unitary quaternions $\hat{\lambda}_k$ (k=0,1,2,3) satisfy the Hamilton and Taif multiplication table (Arenada,1996):

	$\hat{\lambda}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
$\hat{\lambda}_0$	1	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
$\hat{\lambda}_1$	$\hat{\lambda}_1$	-1	$\hat{\lambda}_3$	$-\hat{\lambda}_2$
$\hat{\lambda}_2$	$\hat{\lambda}_2$	$-\hat{\lambda}_3$	-1	$\hat{\lambda}_1$
$\hat{\lambda}_3$	$\hat{\lambda}_3$	$\hat{\lambda}_2$	$-\hat{\lambda}_1$	-1

The quaternion conjugate Q^* of a given quaternion Q are defined as (Horn, 1987):

$$Q^* = q_0\hat{\lambda}_0 - q_1\hat{\lambda}_1 - q_2\hat{\lambda}_2 - q_3\hat{\lambda}_3 = [q_0, -q_1, -q_2, -q_3] \tag{3}$$

The product of two quaternions Q and P with components q_k and p_k (k=0,1,2,3) is given by (Chou, 1992):

$$QP = \{q_0p_0 - (q_1p_1 + q_2p_2 + q_3p_3)\}\hat{\lambda}_0 + \{q_0p_1 + q_1p_0 + (q_2p_3 - q_3p_2)\}\hat{\lambda}_1$$
$$+ \{q_0p_2 + q_2p_0 + (q_3p_1 - q_1p_3)\}\hat{\lambda}_2 + \{q_0p_3 + q_3p_0 + (q_1p_2 - q_2p_1)\}\hat{\lambda}_3 \tag{4}$$

It must be observed that this product is not commutative (QP+PQ). But the product of quaternion is associative (Harauz,1990):

$$P(QR)=(PQ)R. \tag{5}$$

The inverse Q^{-1} of a quaternion Q whose norm N_Q is different from zero is given by

$$Q^{-1} = \frac{Q^*}{N_Q}. \tag{6}$$

Where NQ = QQ*. The quotient between a quaternion P and a quaternion Q with $N_Q \neq 0$ is defined as (Tanışlı, 1995):

$$\frac{P}{Q} = PQ^{-1} = \frac{PQ^*}{N_Q}. \tag{7}$$

To each vector quaternion P with components $[0,p_1,p_2,p_3]$ a vector P of the Euelidean tridimensional space with components (p_1,p_2,p_3) is associated reciprocally.

If P and Q are the vectors associated, respectively, of the quaternion vectors P and Q, then the scalars and vectorial products of these vectors can be expressed as

$$P.Q=(PQ)_s$$

$$PxQ=(PQ)_v.$$

It must be observed that (Funda and Paul, 1988):

$$PQ = - P.Q + (PxQ) \tag{8}$$

Quaternion notation of ∇ operator in the Hamilton's quaternion can be written as:

$$\nabla = \hat{\lambda}_i \nabla_i \tag{9}$$

Divergence and curl operators are expressed (Dereli,1992),

$$\nabla F(x) = -\nabla \cdot F(x) + \nabla \times F(x) = -divF(x) + curlF(x) \tag{10}$$

where "." and "x" are dot and cross product of two vector quaternions, respectively, and Laplace operator can be defined as follows:

$$N(\nabla) = \nabla_i\nabla_i . \tag{11}$$

3. STATIC GRAVITATIONAL FIELD

It is known from experiments that the gravitational field of a point particle of mass M is given by

$$\vec{g} = -\frac{GM}{r^2}\hat{e}_r \tag{12}$$

where $\hat{e}_r$ is a unit vector drawn outward from the particle. The value of the gravitational constant is

$$G = 6{,}67.10^{-8}cm^3gm^{-1}sec^{-2} = 3{,}42.10^{-8}ft^3slug^{-1}sec^{-2}. \tag{13}$$

It is also known from experiment that the gravitational field has the algebraic properties of a vector. For example, let P be a distance r_1, from mass M_1 and a distance r_2 from M_2 (Figure 1). If $\vec{g}_1$ and $\vec{g}_2$ are the gra-

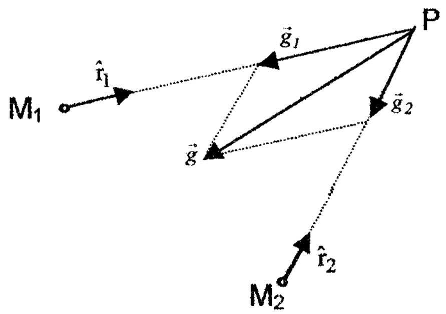

Figure 1. Static Gravitational Field.

vitational fields at P due to M_1 and M_2 separately, then the resultant gravitational field at P is (Bradbury,1968):

$$\vec{g} = \vec{g}_1 + \vec{g}_2 = -\frac{GM_1}{r_1^2}\hat{e}_{r_1} - \frac{GM_2}{r_2^2}\hat{e}_{r_2} \qquad (14)$$

If there are n-point masses present, the net gravitational field is:

$$\vec{g} = -\sum_{\alpha=1}^{n} \frac{GM_\alpha}{r_\alpha^2}\hat{e}_{r_\alpha} \qquad (15)$$

To find the gravitational field of a continuous distribution of matter, the sum must be replaced by an integral. The net gravitational field of the distribution of matter evaluated at the field point is:

$$\vec{g} = -\int \frac{Gdm}{r^2}\hat{e}_r = -\int \frac{G\rho(x_i)\,d\Sigma}{r^2}\hat{e}_r\,; \quad (dm = \rho(x_i)\,d\Sigma) \qquad (16)$$

$\vec{g}$ is also derivable from scalar potential (Φ):

$$\vec{g} = -\vec{\nabla}\Phi, \qquad \Phi = -\int \frac{G\,\rho(x_i)}{r}\,d\Sigma \qquad (17)$$

where ϕ is called the gravitational potential and has dimensions of (force per unit mass)x (distance), or energy per unit mass and Eq.(17) proves that the static gravitational field is a conservation vector field.

It is frequently easier to calculate the gravitational potential by means of Eq.(17) rather then to calculate $\vec{g}$ directly from Eq.(16). If a particle of mass m is placed in gravitational field, it experiences a force $\vec{F} = m\vec{g}$. Its potential energy can be taken as V=mϕ. If there are a number of point masses enclosed by a surtace than

$$\int_\sigma \vec{g}\cdot\hat{n}\,d\sigma = -4\pi G\,(m_1+m_2+m_3+\ldots) \qquad (18)$$

$$= -4\pi G\,(\textit{total mass enclosed by}\,\sigma)$$

If the mass enclosed by σ is in the form of a continuous distribution then to be replaced by

$$\int_\sigma \vec{g}\cdot\hat{n}\,d\sigma = -4\pi G\int_\Sigma \rho\,d\Sigma \qquad \textbf{(19)}$$

where the volume integral is only over the region Σ enclosed by σ. Eq.(19) is known as Gauss' law (Bradbury, 1968).

Gauss divergence theorem, Eq.(19) can be converted to

$$\int_\Sigma \vec{\nabla}\cdot\vec{g}\,d\Sigma = -4\pi G\int_\Sigma \rho\,d\Sigma \qquad (20)$$

This result is an identity applying to any arbitrarily chosen region of integration implying that

$$\vec{\nabla}\cdot\vec{g} = -4\pi G\rho \qquad (21)$$

The mass density ρ is therefore a source function for the gravitational field. More properly it should be called a "sink function" since the lines of $\vec{g}$ always converge toward the matter. Since $\vec{g}$ is conservative, a second fundamental differential equation obeyed by $\vec{g}$ is:

$$\vec{\nabla}\times\vec{g} = 0 \qquad (22)$$

Dot and Cross products of two vectors are defined with a quaternion equation which is the quaternion product of two quaternions. If we use the rule of product of two vector(pure) quaternions which are ∇ and g (Eq. 8), a quaternion equation can be written as follows:

$$\nabla g - [f(x),0,0,0] = 0 \qquad (23)$$

Where $f(x)$ is $4\pi G\rho$. This equation of quaternion replaces both of Eq.(21) and Eq.(22). In addition, the substitution of g = - $\nabla\Phi$ into Eq.(23) leads to

$$\nabla\nabla\Phi + [f(x),0,0,0] = 0 \qquad (24)$$

showing that the basic quaternion equation satisfied by the scalar quaternion of gravitational potential is Poisson's equation. If $f(x)$ is zero in Eq.(24), The equation will be Laplace's equation.

4. CONCLUSIONS

The static gravitational field is conservative and the divergence of which is Gauss' Law for gravitation. We have written a simpler and general way to express the static gravitational field and Poissson's equation as the product of two quaternions. This equation has the same form as the vector equations which was written for conservative field and Divergence theorem. Doing physics with quaternions has very easy, useful and compact representation. It is shown that physical quantities can be represented with quaternions. Also, the test for a conservative field can be done with operator quaternions.

In this study, The defining equations are quaternionic partial differential equations. One of them is Laplace's equation, another is Poisson's equation.

Four Maxwell equations in the electromagnetic theory can be written with two quaternion equations which are electrostatic field and magnetic field.

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