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CYCLES IN 2-FACTORIZATIONS OF K_n

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ABSTRACT

This work studies cycles in 2-factorizations of K_n (undirected complete graph with n vertices) and gives a complete solution (with three possible exceptions) of the problem of constructing 2-factorizations of K_n containing a specified number of 8-cycles, for both n even and odd.

Key Words: Complete graph, 2-factorization, Cycle

TAM GRAFLARIN ÖZEL PARÇALANIŞLARINDAKİ DÖNGÜLER

ÖZ

Bu çalışmada n köşeli tam graflardaki döngüler problemi işlenmekte, tek ve çift köşeli tam graflardaki 8-döngü sayısı problemine (üç olası istisna ile) çözüm verilmektedir.

Anahtar Kelimeler: Tam graf, 2-faktör örtülüşü, Döngü

1. INTRODUCTION

A 2-factor of the complete undirected graph K_n is a collection of vertex disjoint cycles which span the vertex set of K_n. A 2-factorization of order n is a pair (S, F), where F is a collection of edge disjoint 2-factors of K_n (with vertex set S) which partitions the edge set of K_n.

Of course, a 2-factorization of K_n exists if and only if n is odd and in this case the number of 2-factors is $(n-1)/2$.

A smallest cycle in K_n is a 3-cycle and a largest cycle is a Hamiltonian cycle (a cycle of length n). The most extensively studied 2-factorizations are Kirkman Triple systems (in which all cycles have length 3) and Hamiltonian decompositions (in which all cycles have length n). It is well known that Kirkman triple systems exist precisely when $n \equiv 3 \pmod 6$ (Ray-Chaudri and Wilson, 1971) and Hamiltonian decompositions exist for all odd n (Lucas, 1983).

In (Dejter et al., 1997) I. J. Dejter, F. Franek, E. Mendelsohn, and A. Rosa looked at the problem of constructing 2-factorizations of K_n containing a specified number of 3-cycles. Modulo a few exceptions they gave a complete solution for $n \equiv 1$ or $3 \pmod 6$. The problem remains open for $n \equiv 5 \pmod 6$.

In (Dejter et al., 1998) I.J. Dejter, C.C. Lindner, and A. Rosa gave a complete solution of the problem of constructing 2-factorizations of K_n containing a specified number of 4-cycles. In (Adams and Billington) P. Adams and E. J. Billington gave a complete solution of the problem of constructing 2-factorizations of K_n containing a specified number of 6-cycles.

Of course K_{2n} can not be 2-factored, for the simple reason that each vertex has odd degree. However, if we remove a 1-factor from the edge set of K_{2n}, things are different. Hence we have the following definition. A 2-factorization of K_{2n} is a triple

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(S, F, I), where I is a 1-factor of the edge set of K_{2n} and F is a collection of edge disjoint 2-factors of K_{2n} which partitions $E(K_{2n}) \setminus I$, with vertex set S.

In (Adams et al.) P. Adams, E. J. Billington, I. J. Dejter, and C. C. Lindner gave a complete solution of the problem of constructing 2-factorizations of K_{2n} containing a specified number of 4-cycles.

In (Adams and Billington) P. Adams and E. J. Billington gave a complete solution of the problem of constructing 2-factorizations of K_{2n} containing a specified number of 6-cycles.

The next unsettled case of constructing 2-factorizations of K_n containing a specified number of cycles of even length is for 8-cycles. In this work we give a complete solution (with 3 possible exceptions) of the problem of constructing 2-factorizations of K_n containing a specified number of 8-cycles. To be specific let $Q(n)$ denote the set of all x such that there exists a 2-factorization of K_n containing x 8-cycles and let

$$FC(n) = \begin{cases} \{0,1,...,8k(2k-1)\} & \text{if } n = 16k+1, \\ \{0,1,...,2k(8k+1)\} & \text{if } n = 16k+3, \\ \{0,1,...,2k(8k+2)\} & \text{if } n = 16k+5, \\ \{0,1,...,2k(8k+3)\} & \text{if } n = 16k+7, \\ \{0,1,...,8k(2k+1)\} & \text{if } n = 16k+9, \\ \{0,1,...,(2k+1)(8k+5)\} & \text{if } n = 16k+11, \\ \{0,1,...,(2k+1)(8k+6)\} & \text{if } n = 16k+13, \\ \{0,1,...,(2k+1)(8k+7)\} & \text{if } n = 16k+15. \end{cases}$$

We will show that $Q(n) = FC(n)$ for all odd n, with the possible exceptions $47 \in FC(33)$. Now, let

$$FC(n) = \begin{cases} \{0,1,...,2k(8k-1)\} & \text{if } n = 16k, \\ \{0,1,...,8k(2k-1)\} & \text{if } n = 16k+2, \\ \{0,1,...,2k(8k+1)\} & \text{if } n = 16k+4, \\ \{0,1,...,2k(8k+2)\} & \text{if } n = 16k+6, \\ \{0,1,...,(2k+1)(8k+3)\} & \text{if } n = 16k+8, \\ \{0,1,...,8k(2k+1)\} & \text{if } n = 16k+10, \\ \{0,1,...,(2k+1)(8k+5)\} & \text{if } n = 16k+12, \\ \{0,1,...,(2k+1)(8k+6)\} & \text{if } n = 16k+14. \end{cases}$$

Then we will show that $Q(n) = FC(n)$ for all even n, with the possible exceptions $45 \in FC(34)$ and $47 \in FC(34)$.

We will organize our results into 6 sections: a general recursive construction for $n \equiv 9, 11, 13$, and 15 (mod 16), a general recursive construction for $n \equiv 1, 3, 5$, and 7 (mod 16), a general recursive construction for $n \equiv 0$ or 8 (mod 16), a general recursive construction for $n \equiv 10$ (mod 16), a general recursive construction for $n \equiv 2, 4, 6, 12$ or 14 (mod 16), and a conclusion.

2. n≡ 9,11,13 or 15 (mod 16)

The following construction is the principal tool used in this section.

Construction A:

Write $n = tv + r$, where t is odd and v is even and $r \in \{1,3,5,7\}$. Let $X = \{1,2,...,t\}$, $V = \{1,2,...,v\}$, and Z be a set of size r. Further, let $(X, \circ)$ be an idempotent commutative quasigroup of order t (Lindner and Rodger, 1997) and set $S = Z \cup (X \times V)$.

Define a collection F of 2-factors of K_{tv+r} as follows:

(1) Let $(Z \cup (\{1\} \times \{1,2,...,v\}), F_1)$ be a 2-factorization of K_{v+r}, where

$$F_1 = \{f_{1_1}, f_{1_2}, ..., f_{(v+r-1)/2}\}$$

(2) For each $x \in X \setminus \{1\}$, let $(Z \cup (\{x\} \times \{1,2,...,v\}), F_x)$ be a 2-factorization of K_{v+r} containing either 0 or $maxFC(v+r)$ 8-cycles and containing a sub-2-factorization of order r, where $maxFC(v+r)$ is the largest value in the set $FC(v+r)$. Let $F_x = \{f_{x_1}, f_{x_2}, ..., f_{x_{(v+r-1)/2}}\}$, where the last $(r-1)/2$ 2-factors contain the sub-2-factorization of order r.

(3) For each pair $a \neq b \in X$ such that $a \circ b = b \circ a = x$, let $(K_{a,b}, f_x(a,b))$ be any 2-factorization of $K_{v,v}$ with parts $\{a\} \times \{1,2,...,v\}$ and $\{b\} \times \{1,2,...,v\}$, where $f_x(a,b) = \{f_{x_1}(a,b), f_{x_2}(a,b), ..., f_{x_{v/2}}(a,b)\}$.

(4) Each of $\{f_{x_i}\} \cup \{f_{x_i}(a,b) | a \circ b = b \circ a = x\}$, where $i = 1, 2, ..., v/2$ is a 2-factor of K_{tv+r}.

(5) Piece together the remaining $(r-1)/2$ 2-factors of F_1, along with the remaining $(r-1)/2$ 2-factors of each F_x, for $x = 2, 3, ..., t$, making sure to delete the cycles belonging to the sub-2-factorization from each of the remaining 2-factors in each F_x.

(6) For each $x \in X$, place the $v/2$ 2-factors in (4) in F as well as the 2-factors in (5).

The union of the 2-factors in (6) gives a total of $\sum_{x \in X}(v/2) + (r-1)/2 = (tv + r - 1)/2$ 2-factors which form a 2-factorization of K_{tv+r} with vertex set S.

Corollary 1. *Construction A gives a 2-factorization of K_{tv+r} containing exactly $\sum_{i=1}^{t(t-1)/2} n_i + \sum_{i=1}^{t} m_i$ 8-cycles, where $n_i \in Q(K_{v,v})$, $m_1 \in Q(v+r)$, and $m_i \in \{0, maxFC(v+r)\}$ for $i = 2, 3, ..., t$.*

It is easy to see that $Q(n) \subseteq FC(n)$ for odd n. Now, with Construction A and Corollary 1 we will show that $FC(n) \subseteq Q(n)$ for the cases $n \equiv 9, 11, 13$, and 15 (mod 16). In each of the following cases we will take $t = 2k + 1$ and $v = 8$.

n≡9 (mod16)

Lemma 2. $Q(9) = FC(9)$.

Proof. S. Küçükçifçi, 2000.

Lemma 3. *$K_{8,8}$ can be 2-factorized into*

$$\{0,1,2,3,4,5,6,7,8\}$$

8-cycles.

Proof. S. Küçükçifçi, 2000.

Lemma 4. $FC(16k+9) \subseteq Q(16k+9)$.

Proof. Take $r = 1$ in Construction A. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$ Corollary 1 gives $FC(16k+9) \subseteq Q(16k+9)$.

n≡11 (mod16)

Lemma 5. $Q(11) = FC(11)$, *where the 2-factorizations of* K_{11} *having* 0 *8-cycles and* 5 *8-cycles contain a cycle of length* 3.

Proof. S. Küçükçifçi, 2000.

Lemma 6. $FC(16k+11) \subseteq Q(16k+11)$.

Proof. Take $r = 3$ in Construction A. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$, $Q(11) = FC(11)$ and $m_i \in \{0,5\}$ for $i = 2,3,...,t$, Corollary 1 gives $FC(16k+11) \subseteq Q(16k+11)$.

n≡13 (mod16)

Lemma 7. $Q(13) = FC(13)$, *where the 2-factorizations of* K_{13} *having* 0 *and* 6 *8-cycles contain sub-2-factorizations of order* 5.

Proof. S. Küçükçifçi, 2000.

Lemma 8. $FC(16k+13) \subseteq Q(16k+13)$.

Proof. Take $r = 5$ in Construction A. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$, $Q(13) = FC(13)$ and $m_i \in \{0,6\}$ for $i = 2,3,...,t$, Corollary 1 gives $FC(16k+13) \subseteq Q(16k+13)$.

n≡15 (mod16)

Lemma 9. $Q(15) = FC(15)$, *where the 2-factorizations of* K_{15} *having* 0 *or* 7 *8-cycles contain a sub-2-factorization of order* 7.

Proof. S. Küçükçifçi, 2000.

Lemma 10. $FC(16k+15) \subseteq Q(16k+15)$.

Proof. Take $r = 7$ in Construction A. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$, $Q(15) = FC(15)$ and $m_i \in \{0,7\}$ for $i = 2,3,...,t$, Corollary 1 gives $FC(16k+15) \subseteq Q(16k+15)$.

3. n≡ 1,3,5 or 7 (mod 16)

We will begin with the following construction.

Construction B:

Write $n = tv + r$, where v and t are even and $r \in \{1,3,5,7\}$. Let $X = \{1,2,...,t\}$, $V = \{1,2,...,v\}$, and Z be a set of size r. Further, let $(X, \circ)$ be a commutative quasigroup of order $t \geq 6$ with holes $H = \{h_1, h_2, ..., h_{t/2}\}$ of size 2 (Lindner and Rodger, 1997) and set $S = Z \cup (X \times V)$.

Define a collection F of 2-factors of K_{tv+r} as follows:

(1) For the hole $h_1 \in H$, let $(Z \cup (h_1 \times \{1,2,...,v\}), F_1)$ be any 2-factorization of K_{2v+r}, where $F_1 = \{f_{1_1}, f_{1_2}, ..., f_{1_{v+(r-1)/2}}\}$.

(2)For each hole $h_i \in H \setminus \{h_1\}$, let $(Z \cup (h_i \times \{1,2,...,v\}), F_i)$ be any 2-factorization of K_{2v+r} having either 0 or $maxFC(2v+r)$ 8-cycles and containing a sub-2-factorization of order r, where $maxFC(2v+r)$ is the largest value in the set $FC(2v+r)$. Let $F_i = \{f_{i_1}, f_{i_2}, ..., f_{i_{v+(r-1)/2}}\}$, where the last $(r-1)/2$ 2-factors contain the sub-2-factorization of order r.

(3) For each $x \in X$, set $F(x) = \{\{a,b\} | a \neq b, a \circ b = b \circ a = x$, and a and b do not belong to the hole containing $x\}$. Denote by $(K_{a,b}, f_x(a,b))$, $\{a,b\} \in F(x)$, any 2-factorization of $K_{v,v}$ with parts $\{a\} \times \{1,2,...,v\}$ and $\{b\} \times \{1,2,...,v\}$, where $f_x(a,b) = \{f_{x_1}(a,b), f_{x_2}(a,b), ..., f_{x_{v/2}}(a,b)\}$.

(4) For each hole $h_i = \{x,y\} \in H$, each of the following is a 2-factor of K_{tv+r}:

$$\begin{cases} \{f_{i_j}\} \cup \{f_{x_j}(a,b) | \{a,b\} \in F(x)\}, & j = 1,2,...,v/2, \\ \{f_{i_k}\} \cup \{f_{y_j}(c,d) | \{c,d\} \in F(y)\}, & j = 1,2,...,v/2 \text{ and} \\ \quad k = v/2, (v/2)+1, ..., v. & \end{cases}$$

(5) Piece together the remaining $(r-1)/2$ 2-factors of F_1, along with the remaining $(r-1)/2$ 2-factors of each F_x, for $x = 2,3,...,t$, making sure to delete the cycles belonging to the sub-2-factorization from each of the remaining 2-factors in each F_x.

(6) For each hole in H, place the v 2-factors in (4) in F as well as the 2-factors in (5).

The union of the 2-factors in (6) gives a total of $\sum_{h \in H}(v) + (r-1)/2 = (tv+r-1)/2$ 2-factors which form a 2-factorization of K_{tv+r} with vertex set S.

Corollary 11. *Construction B gives a 2-factorization of* K_{tv+r} *containing exactly* $\sum_{i=1}^{t(t-2)/2} n_i + \sum_{i=1}^{t/2} m_i$ *8-cycles, where* $n_i \in Q(K_{v,v})$, $m_1 \in Q(2v+r)$, *and* $m_i \in \{0, maxFC(2v+r)\}$ *for* $i = 2,3,...,t/2$.

We will now use Construction B and Corollary 11 to show that $FC(n) \subseteq Q(n)$ for the cases $n \equiv 1,3,5$ and 7 (mod 16).

n≡1 (mod16)

Lemma 12. $Q(17) = FC(17)$.

Proof. S. Küçükçifçi, 2000.

Lemma 13. *$K_{10,10}$ can be 2-factorized into* 0 *or* 10 *8-cycles.*

Proof. S. Küçükçifçi, 2000.

Lemma 14. *K_{33} can be 2-factorized into $FC(33) \setminus \{47\}$ 8-cycles.*

Proof. S. Küçükçifçi, 2000.

Lemma 15. *$FC(16k+1) \subseteq Q(16k+1)$, with the possible exception of $47 \in FC(33)$.*

Proof. Take $r = 1$, $t = 2k$ and $v = 8$ in Construction B. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$ and $Q(17) = FC(17)$, Corollary 11 gives $FC(16k+1) \subseteq Q(16k+1)$ for $k \geq 3$. Lemmas 12 and 14 complete the proof.

n≡3 (mod16)

Lemma 16. *$K_{6,6}$ can be 2-factorized into* 0, 1, *or* 3 *8-cycles.*

Proof. S. Küçükçifçi, 2000.

Lemma 17. $Q(19) = FC(19)$.

Proof. S. Küçükçifçi, 2000.

Lemma 18. $FC(16k+3) \subseteq Q(16k+3)$.

Proof. Take $r = 3$, $t = 4k$ and $v = 4$ in Construction B. Since $n_i \in \{0,2\}$, $m_1 \in Q(11)$ and $m_i \in \{0,5\}$ for $i = 2,3,...,2k$, Corollary 11 gives $FC(16k+3) \subseteq Q(16k+3)$ for $k \geq 2$. Lemma 17 completes the proof.

n≡5 (mod16)

Lemma 19. $Q(21) = FC(21)$.

Proof. S. Küçükçifçi, 2000.

Lemma 20. $FC(16k+5) \subseteq Q(16k+5)$.

Proof. Take $r = 5$, $t = 4k$ and $v = 4$ in Construction B. Since $n_i \in \{0,2\}$, $m_1 \in Q(13)$ and $m_i \in \{0,6\}$ for $i = 2,3,...,2k$, Corollary 11 gives $FC(16k+5) \subseteq Q(16k+5)$ for $k \geq 2$. Lemma 19 completes the proof.

n≡7 (mod16)

Lemma 21. *$Q(23) = FC(23)$, where the 2-factorizations of K_{23} having* 0 *and* 22 *8-cycles contain sub-2-factorizations of order 7.*

Proof. S. Küçükçifçi, 2000.

Lemma 22. *$K_{12,12}$ can be 2-factorized into* 0 *or* 18 *8-cycles.*

Proof. S. Küçükçifçi, 2000.

Lemma 23. $Q(39) = FC(39)$.

Proof. S. Küçükçifçi, 2000.

Lemma 24. $FC(16k+7) \subseteq Q(16k+7)$.

Proof. Take $r = 7$, $t = 2k$ and $v = 8$ in Construction B. Since $n_i \in \{0,1,2,3,4,5,6,7,8\}$, $m_1 \in Q(23)$ and $m_i \in \{0,22\}$ for $i = 2,3,...,k$, Corollary 11 gives $FC(16k+7) \subseteq Q(16k+7)$ for $k \geq 3$. Lemmas 21 and 23 complete the proof.

Now in the next three sections we will solve the problem when n is even.

4. n≡ 0 or 8 (mod 16)

We will begin with the following construction.

Construction C:

Write $n = 4t$, where t is even. Let $X = \{1,2,...,t\}$ and set $S = X \times \{1,2,3,4\}$. Let F be a 1-factorization of K_t (Lindner and Rodger, 1997), where $F = \{f_1, f_2, ..., f_{t-1}\}$.

Define a collection F^* of $2t-1$ 2-factors of K_{4t} as follows:

(1) For each $\{x,y\} \in f_1$, let $(\{x,y\} \times \{1,2,3,4\}, f_1(x,y), I(x,y))$ be any 2-factorization of K_8 (Example 2.2), where $f_1(x,y) = \{f_{1_1}(x,y), f_{1_2}(x,y), f_{1_3}(x,y)\}$ and $I(x,y) = \{\{(x,1),(y,1)\},\{(x,2),(y,2)\},\{(x,3),(y,3)\},\{(x,4),(y,4)\}\}$.

(2) For each $(a,b) \in f_i$, $i = 2,3,...,t-1$, let $(K_{a,b}, f_i(a,b)) = \{f_{i_1}(a,b), f_{i_2}(a,b)\}$ be any 2-factorization of $K_{4,4}$ with parts $\{a\} \times \{1,2,3,4\}$ and $\{b\} \times \{1,2,3,4\}$.

(3) Each of $\{f_{1_i}(x,y) | \{x,y\} \in f_1, \ i = 1,2,3\}$ is a 2-factor of K_{4t}.

(4) Each of $\{f_{i_j}(a,b) | \{a,b\} \in f_i, \ i \in \{2,3,...,t-1\}, \ j \in \{1,2\}\}$ is a 2-factor of K_{4t}.

(5) Place the 3 2-factors in (3) and the $2(t-2)$ 2-factors in (4) in F^*.

(F^* contains $2(t-2)+3 = 2t-1$ 2-factors.)

(6) Let $I = \{I(x,y) | \{x,y\} \in f_1\}$.

Then (S, F^*, I) is a 2-factorization of K_{4t}.

Corollary 25. *Construction C gives a 2-factorization of K_{4t} containing exactly $\sum_{i=1}^{t(t-2)/2} n_i + \sum_{i=1}^{t/2} m_i$ 8-cycles, where $n_i \in Q(K_{4,4})$, $m_i \in Q(8)$.*

It is easy to see that $Q(n) \subseteq FC(n)$ for even n. Now, with Construction C and Corollary 25 we will show that $FC(n) \subseteq Q(n)$ for the cases $n \equiv 0$ and 8 (mod 16). In order to do this we will need the following example.

Lemma 26. $Q(8) = FC(8)$.

Proof. S. Küçükçifçi.

n≡0 (mod16)

Lemma 27. $FC(16k) \subseteq Q(16k)$.

Proof. Take $t = 4k$ in Construction C. Since $Q(8) = \{0,1,2,3\}$ and $Q(K_{4,4}) = \{0,2\}$, Corollary 25 gives $FC(16k) \subseteq Q(16k)$.

n≡8 (mod16)

Lemma 28. $FC(16k+8) \subseteq Q(16k+8)$.

Proof. Take $t = 4k+2$ in Construction C. Corollary 25 gives $FC(16k+8) \subseteq Q(16k+8)$.

5. n≡ 10 (mod 16)

The following construction will take care of the case $n \equiv 10$ (mod 16).

Construction D:

Write $n = tv+r$, where t is odd and v is even and $r \in \{2,4,6\}$. Let $X = \{1,2,...,t\}$, $V = \{1,2,...,v\}$, and Z be a set of size r. Further, let $(X, \circ)$ be an idempotent commutative quasigroup of order t (Lindner and Rodger, 1997) and set $S = Z \cup (X \times V)$.

Define a collection F of 2-factors of K_{tv+r} as follows:

(1) Let $(Z \cup (\{1\} \times \{1,2,...,v\}), F_1)$ be a 2-factorization of K_{v+r}, where $F_1 = \{f_{1_1}, f_{1_2}, ..., f_{(v+r)/2-1}\}$ and the edges of the 1-factor of Z belong to I_1.

(2) For each $x \in X \setminus \{1\}$, let $(Z \cup (\{x\} \times \{1,2,...,v\}), F_x, I_x)$ be a 2-factorization of K_{v+r} having either 0 or $maxFC(v+r)$ 8-cycles and containing a sub-2-factorization of order r, where $maxFC(v+r)$ is the largest value in the set $FC(v+r)$. Let $F_x = \{f_{x_1}, f_{x_2}, ..., f_{x_{(v+r)/2-1}}\}$, where the last $r/2-1$ 2-factors contain the sub-2-factorization of order r and the edges of the 1-factor of Z belong to I_x.

(3) For each pair $a \neq b \in X$ such that $a \circ b = b \circ a = x$, let $(K_{a,b}, f_x(a,b))$ be any 2-factorization of $K_{v,v}$ with parts $\{a\} \times \{1,2,...,v\}$ and $\{b\} \times \{1,2,...,v\}$, where $f_x(a,b) = \{f_{x_1}(a,b), f_{x_2}(a,b), ..., f_{x_{v/2}}(a,b)\}$.

(4) Each of $\{f_{x_i}\} \cup \{f_{x_i}(a,b) | a \circ b = b \circ a = x\}$, where $i = 1,2,...,v/2$ is a 2-factor of K_{tv+r}.

(5) Piece together the remaining $r/2-1$ 2-factors of F_1, along with the remaining $r/2-1$ 2-factors of each F_x, for $x = 2,3,...,t$, making sure to delete the cycles belonging to the sub-2-factorization from each of the remaining 2-factors in each F_x.

(6) For each $x \in X$, place the $v/2$ 2-factors in (4) in F as well as the 2-factors in (5).

(7) Let $I = \{I_x | x \in X\}$.

The union of the 2-factors in (6) gives a total of $\sum_{x \in X}(v/2) + r/2 - 1 = (tv+r-2)/2$ 2-factors which form a 2-factorization of K_{tv+r} with vertex set S.

Corollary 29. *Construction D gives a 2-factorization of K_{tv+r} containing exactly $\sum_{i=1}^{t(t-1)/2} n_i + \sum_{i=1}^{t} m_i$ 8-cycles, where $n_i \in Q(K_{v,v})$, $m_1 \in Q(v+r)$, and $m_i \in \{0, maxFC(v+r)\}$ for $i = 2,3,...,t$.*

We will now use Costruction D and Corollary 29 to show that $FC(n) \subseteq Q(n)$ for the case $n \equiv 10$ (mod 16).

Lemma 30. $FC(16k+10) \subseteq Q(16k+10)$.

Proof. Take $r = 2$, $t = 2k+1$ and $v = 8$ in Construction B. Since any 2-factorization of K_{10} contains 0 8-cycles and $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$ (Küçükçifçi, 2000), Corollary 29 gives $FC(16k+10) \subseteq Q(16k+10)$.

6. n≡ 2,4,6,12 or 14 (mod 16)

The following construction will take care of the remaining cases.

Construction E:

Write $n = tv + r$, where v and t are even and $r \in \{2,4,6\}$. Let $X = \{1,2,...,t\}$, $V = \{1,2,...,v\}$, and Z be a set of size r. Further, let $(X, \circ)$ be a commutative quasigroup of order $t \geq 6$ with holes $H = \{h_1, h_2, ..., h_{t/2}\}$ of size 2 (Lindner and Rodger, 1997) and set $S = Z \cup (X \times V)$.

Define a collection F of 2-factors of K_{tv+r} as follows:

(1) For the hole $h_1 \in H$, let $(Z \cup (h_1 \times \{1,2,...,v\}), F_1, I_1)$ be any 2-factorization of K_{2v+r}, where $F_1 = \{f_{1_1}, f_{1_2}, ..., f_{1_{v+(r-2)/2}}\}$ and the edges of the 1-factor of Z belong to I_1.

(2)For each hole $h_i \in H \setminus \{h_1\}$, let $(Z \cup (h_i \times \{1,2,...,v\}), F_i, I_i)$ be any 2-factorization of K_{2v+r} having either 0 or $maxFC(2v+r)$ 8-cycles and containing a sub-2-factorization of order r. Let $F_i = \{f_{i_1}, f_{i_2}, ..., f_{i_{v+(r-2)/2}}\}$, where the last $(r-2)/2$ 2-factors contain the sub-2-factorization of order r and the edges of the 1-factor of Z belong to I_i.

(3) For each $x \in X$, set $F(x) = \{\{a,b\} | a \neq b, a \circ b = b \circ a = x$, and a and b do not belong to the hole containing $x\}$. Denote by $(K_{a,b}, f_x(a,b))$, $\{a,b\} \in F(x)$, any 2-factorization of $K_{v,v}$ with

parts $\{a\} \times \{1,2,...,v\}$ and $\{b\} \times \{1,2,...,v\}$, where $f_x(a,b) = \{f_{x_1}(a,b), f_{x_2}(a,b), ..., f_{x_{v/2}}(a,b)\}$.

(4) For each hole $h_i = \{x,y\} \in H$, each of the following is a 2-factor of K_{tv+r}:

$$\begin{cases} \{f_{i_j}\} \cup \{f_{x_j}(a,b) | \{a,b\} \in F(x)\}, & j = 1,2,...,v/2, \\ \{f_{i_k}\} \cup \{f_{y_j}(c,d) | \{c,d\} \in F(y)\}, & j = 1,2,...,v/2 \text{ and} \\ k = v/2, (v/2)+1, ..., v. & \end{cases}$$

(5) Piece together the remaining $(r-2)/2$ 2-factors of F_1, along with the remaining $(r-2)/2$ 2-factors of each F_x, for $x = 2,3,...,t$, making sure to delete the cycles belonging to the sub-2-factorization from each of the remaining 2-factors in each F_x.

(6) For each hole in H, place the v 2-factors in (4) in F as well as the 2-factors in (5).

(7) Let $I = \{I_x | x \in X\}$.

The union of the 2-factors in (6) gives a total of $\sum_{h \in H}(v) + (r-2)/2 = (tv+r-2)/2$ 2-factors which form a 2-factorization of K_{tv+r} with vertex set S.

Corollary 31. *Construction E gives a 2-factorization of K_{tv+r} containing exactly $\sum_{i=1}^{t(t-2)/2} n_i + \sum_{i=1}^{t/2} m_i$ 8-cycles, where $n_i \in Q(K_{v,v})$, $m_1 \in Q(2v+r)$, and $m_i \in \{0, maxFC(2v+r)\}$ for $i = 2,3,...,t/2$.*

Now with Construction E and Corollary 31 we will show that $FC(n) \subseteq Q(n)$ for the cases $n \equiv 2,4,6,12$ and 14 (mod 16).

n≡2 (mod16)

Lemma 32. $Q(18) = FC(18)$.

Proof. S. Küçükçifçi.

Lemma 33. *K_{34} can be 2-factorized into $FC(34) \setminus \{45,47\}$ 8-cycles.*

Proof. S. Küçükçifçi.

Lemma 34. *$FC(16k+2) \subseteq Q(16k+2)$, with the possible exceptions of $45 \in FC(34)$ and $47 \in FC(34)$.*

Proof. Take $r = 2$, $t = 2k$ and $v = 8$ in Construction E. Since $Q(K_{8,8}) = \{0,1,2,3,4,5,6,7,8\}$ and $Q(18) = FC(18)$, Corollary 31 gives $FC(16k+2) \subseteq Q(16k+2)$ for $k \geq 3$. Lemmas 32 and complete the proof.

n≡4 (mod16)

Lemma 35. *$Q(12) = FC(12)$, where the 2-factorizations of K_{12} having 0 and 5 8-cycles contain a 4-cycle.*

Proof. S. Küçükçifçi.

Lemma 36. $Q(20) = FC(20)$.

Proof. S. Küçükçifçi.

Lemma 37. $FC(16k+4) \subseteq Q(16k+4)$.

Proof. Take $r = 4$, $t = 4k$ and $v = 4$ in Construction E. Since $K_{4,4}$ can be 2-factorized into 0 or 2 8-cycles and $Q(12) = FC(12)$, Corollary 31 gives $FC(16k+4) \subseteq Q(16k+4)$ for $k \geq 2$. Lemmas 35 and 36 complete the proof.

n≡6 (mod16)

Lemma 38. *$Q(14) = FC(14)$, where each of the 2-factorizations of K_{14} having 0 and 6 8-cycles contains sub-2-factorizations of order 6 and the 2-factorization of K_{14} having 4 8-cycles contains a sub-2-factorization of order 4.*

Proof. S. Küçükçifçi.

Lemma 39. $Q(22) = FC(22)$.

Proof. S. Küçükçifçi.

Lemma 40. $FC(16k+6) \subseteq Q(16k+6)$.

Proof. Take $r = 6$, $t = 4k$ and $v = 4$ in Construction E. Since $K_{4,4}$ can be 2-factorized into 0 or 2 8-cycles and $Q(14) = FC(14)$, Corollary 31 gives $FC(16k+4) \subseteq Q(16k+4)$ for $k \geq 2$. Lemmas 38 and 39 complete the proof.

n≡12 (mod16)

Lemma 41. $FC(16k+12) \subseteq Q(16k+12)$.

Proof. Take $r = 4$, $t = 4k+2$ and $v = 4$ in Construction E. Corollary 31 and Lemma 35 give $FC(16k+12) \subseteq Q(16k+12)$.

n≡14 (mod16)

Lemma 42. $FC(16k+14) \subseteq Q(16k+14)$.

Proof. Take $r = 6$, $t = 4k+2$ and $v = 4$ in Construction E. Corollary 31 and Lemma 38 give $FC(16k+14) \subseteq Q(16k+14)$.

7. CONCLUSION

We summarize our results with the following theorem.

Theorem 43. *$Q(n) = FC(n)$ for all odd n with the possible exceptions of $47 \in FC(33)$ and even n with the possible exceptions of $45 \in FC(34)$ and $47 \in FC(34)$.*

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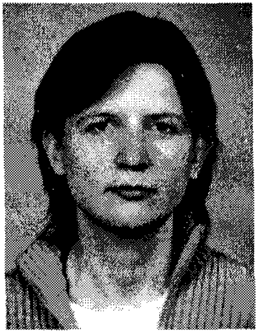

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