**AN ASYMPTOTIC MODEL FOR THE STONELEY INTERFACIAL WAVES**

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**Master of Science Thesis**

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# ÖZET

**Yüksek Lisans Tezi**

## STONELEY ARAYÜZ DALGALARI İÇİN ASİMPTOTİK MODEL



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Bu master tezi, dinamik ortamlarda Stoneley dalgasının katkısını hesaplamada kullanılan asimptotik modeli elde edebilmek için arayüz dalgaları ile çalışılmıştır. Bu arayüz dalgaları yükler tarafından uyarılmış veya üzerinde yayıldığı arayüz süreksizliğe sahip olabilir.

Bu çalışmada, Stoneley arayüz dalgaları için yavaş zaman pertürbasyonuna dayanan açık bir model elde edilmiştir. Bu model homojen, izotropik ve elastik ortamlarda geçerlidir. Stoneley arayüz dalgası genelleştirilmiş bir Rayleigh dalgası olarak da bilinir. 2006 yılında, Kaplunov et al. tarafından Rayleigh yüzey dalgası için bir açık model geliştirilmiştir. Asimptotik model çeşitli dinamik yüzey problemleri için oldukça kullanışlıdır. Bu sebep ile Stoneley arayüz dalgası için asimptotik bir modelin geliştirilebileceği düşünülmüştür. Bu model sayesinde, sürekli olmayan yerdeğiştirme ve stress bileşenlerinin etkisiyle ortaya çıkan dalganın yayılımı tanımlanabilir ve analiz edilebilir.

Bu çalışmada farklı sınır şartları ile arayüz problemleri incelenmiştir. Önerilen model arayüz üzerinde geçerli olan bir hiperbolik denklem ve ortamın iç kısmında geçerli olan eliptik denklemler yardımıyla ifade edilir. Poisson integral formülü kullanılarak, normal yüklenme durumu için açıklayıcı bir örnek verilmiştir.



**Anahtar Kelimeler:** Stoneley dalgası, asimptotik model, yavaş zaman pertürbasyonu

## ABSTRACT

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This M.Sc. thesis is focused on studying interfacial waves for an asymptotic model evaluating the contribution of the Stoneley wave to the overall dynamic field.These interfacial waves are excited by loads or discontinuities on the interface.

The work deals with the explicit model for Stoneley wave which are based on slow time perturbation. The model is developed in homogeneous, isotropic, elastic media. Stoneley wave is a generalised Rayleigh wave. For the Rayleigh wave, the explicit model was constructed by Kaplunov et al. in 2006. The asymptotic model has been useful to deal with various surface dynamics problems. Therefore we aim to construct an asymptotic model for Stoneley interfacial wave. In essence the model describes propagation of the wave by hyperbolic equation on the interface. The decay over the interior is governed by elliptic equations.

We investigated interfacial problems with different boundary conditions, including the discontinuity in stresses or displacements. The proposed model consist of hyperbolic equation on the interface and elliptic equations on the inside. An illustrative example is presented for the case of normal loading with the solution obtained through the Poisson's integral formula.



**Keywords:** Stoneley wave, asymptotic model, slow time perturbation

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# 1 INTRODUCTION

The mathematical theory of elasticity is one of the active interest areas of applied mathematics. Dating back as far as Galileo's work on [1], the propagation of mechanical disturbances in solids is of interest in many branches of the physical sciences and engineering [2]. It is occupied with an attempt to reduce to calculation the state of strain, or relative displacement, within a solid body which is subject to the action of an equilibrating system of forces, or is in a state of slight internal relative motion [3]. The treatment necessarily involves considerable mathematical analysis [2]. In continuum mechanics, problems of the motion and deformation of substances are rendered amenable to mathematical analysis by introducing the concept of a continuum or continuous medium [2]. Linear elasticity is a simplification of the more general nonlinear theory of elasticity and is a branch continuum mechanics. Among the problems considered in the linearised theory, we can name surface and membrane waves, as well as coated surfaces, moving load problems as areas of interests. In recent years, near-surface characterisations have been the focus of interest as well. With many applications in applied mathematics and engineering problems, this vivid area, either directly or indirectly, contributes to the understanding of natural and technical processes.

The history of the study of wave and vibration phenomena goes back hundreds of years. The first mathematician to consider the nature of rupture was Galileo [1]. After the Galileo's studies, the science of vibrations and waves progressed rapidly. The two great landmarks are the discovery of Hooke's Law in 1660 [4], and the formulation of the general equations by Navier in 1821 [5]. Robert Hooke formulated the law of proportionality between stress and strain for elastic bodies [6]. This law is known as the basis for static and dynamic theory of elasticity. The general equations of equilibrium represented one of the most important developments in mechanics [7]. By the Autumn of 1822 Cauchy had discovered most of the elements of pure theory of elasticity [3]. Poisson investigated the propagation of waves through an elastic solid and he found the longitudinal and transverse wave types [8]. He also developed approximate theories for the vibrations of rods [8]. In 1883, Kirchhoff held that the equations of equilibrium or motion of such a portion could be simplified, for a first approximation, by the omission of kinetic reactions and forces distributed through the volume [9]. In 1888, Rayleigh and Lamb developed the frequency equation for waves in a plate according to exact elasticity theory [10], [11].

Lamb made the first investigation of pulse propagation in a semi infinite solid [12]. In 1914, Hopkinson performed experiments on the propagation of elastic pulses in bars [13]. Many scientists have been interested in wave propagation and made major contributions. Recent activities in the field of wave propagation have dealt with formulating various approximate theories for plates and rods and with the analysis of transient loading situations [7].

Over many decades, surface waves have been investigated by a lot of mathematicians and physicists. For bodies with a surface of material discontinuity there are, however, plane waves which are not uniform in planes of constant phase [2]. These waves, which are called surface waves, propagate parallel to the surface of discontinuity [2]. The first known surface wave is Rayleigh wave [14]. The possibility of a wave traveling along the free surface of an elastic half space such that the disturbance is largely confined to the neighbourhood of the boundary was considered by Rayleigh [15] . The criterion for surface waves is that the displacement decays exponentially with distance from the free surface [2]. Lord Rayleigh showed that their effect decreases rapidly with depth and that their velocity of propagation is smaller than that of body waves [14]. Rayleigh waves are of a particular importance in seismology, since it is these waves that are most destructive in earthquakes [7]. Love waves (1911) were a consequence of a layered construction of the earth, and that they consisted of SH waves trapped in a superficial layer and propagated by multiple reflections within the layer [16]. Discovered by Horace Lamb in 1917, Lamb waves can exist in plate-like thin plate with parallel free boundaries [17], [18]. Interfacial waves propagate between the two different media. If these medium are solid-solid, Stoneley waves appear at the interface [19]. If these mediums are fluid-solid, these waves are named as Schölte waves [20].

The interfacial waves, that propagate along the boundary of two different media, have attracted the attention of scientists. Rayleigh in his famous work considered, for example, the surface waves occuring on the surface of an elastic isotropic half-space (vacuum-solid interaction) and proved the existence of such waves, now named after him [15]. It is well-known that in isotropic solids the particle motion is elliptical and retrograde, for shallow depths, with respect to the direction of propagation [21]. This type of wave finds many applications in industry and technology even today. Propagating disturbances confined to the neighbourhood of a surface occur not only in the vicinity of a free surface but also at the interface of two half-spaces filled with different materials which are called Stoneley interfacial waves [2]. Scholte showed that the range of existence of simple Stoneley waves has been determined by the evaluation of the boundary curves of these waves [20]. Chadwick and Captain [22] treated the propagation of Rayleigh type surface waves on a half spaces of inextensible linear elastic material and the same authors [23] examine the existence of Stoneley type waves at the interface between two media [24]. In linear elasticity, time-harmonic waves guided by a traction-free surface (Rayleigh waves), waves travelling along the interface between two elastic media (Stoneley waves) and waves at a fluid-solid interface (Schölte-Gogoladze waves) are non-dispersive, since each is a solution to a boundary-value problem that contains no natural scale of length or time [25].

Stoneley wave is a generalised Rayleigh wave [19]. The two media will be distinguished by suffixes 1 and 2, and will be supposed in "welded contact" along an infinite plane face and otherwise extending to infinity, so that there is no slipping at the interface, in which an origin and a set of axes of $x$ and $y$ are taken [19].

Over the years, many developments have occurred in this field, with various approaches used [7]. Rayleigh, Love, Timoshenko and other scientists made approximations in the equations of motion and Chree, Morse, Kynch and Green contributed solutions of the exact equations which only approximately satisfy the boundary conditions [7].Poisson's theory is verified as an approximate theory by an application of Kirchhoff's result [3].

In his paper of 1948 [26], Friedlander has given a solution of the surface wave problem in terms of two harmonic functions related through a Hilbert transform. This work has later been advanced by Chadwick (1976) [27], who showed that only a single harmonic function was enough to obtain the solution, where the second harmonic function could be obtained through a relation on the surface to the first one. He also presented a similar formulation for Stoneley wave. Kiselev and Parker showed that the disturbance at all depths may be represented at each instant in terms of a single function harmonic in a half-space [25]. Kiselev and Parker showed, also, how waves that are not time-harmonic have a compact representation provided that they are surface waves [25]. Kaplunov et al. The explicit asymptotic model for the Bleustein-Gulyaev wave are derived [28] . Kaplunov et al. obtained the solution of a surface wave problem (both for Rayleigh and Bluestein-Gulyaev waves) with the use of the mentioned relation between the harmonic functions [29] . The approach used in the mentioned paper exploits a slow time perturbation of the self-similar solutions. In the literature, we took as an example some studies for instance [30]- [33] etc.. Dasgupta [34] examined the Stoneley wave propagation with incompressible medium. Dowaikh and Ogden examined the propagation of interfacial (Stoneley) waves along the boundary between two half-spaces of pre-stressed incompressible isotropic elastic material [35]. Destrade and Fu used the surface-impadence matrix method for the interfacial waves polarized in a plane of symmetry of anisotropic elastic materials [36]. Mendez et al. focused on canonic models for interfacial waves (Rayleigh, Stoneley and Schölte) [21]. It was then thought that a similar approach might be applied to the Stoneley interfacial waves, the main topic of this thesis. Due to two different media with different material properties, it turns out that similar relations between the potentials are more complicated in the case of Stoneley wave. Considering several boundary conditions on the surface of interaction, different problems may be considered. Taking into account the principle of superposition an arbitrary boundary value problem may be solved separately and the solutions may be added to give the full solution. As soon as hyperbolic equations on the boundary are solved, the Poisson's formula may then be applied to obtain the inner solutions in terms of potentials for both of the media. Finally, the stresses and displacements can be obtained using the elastic potentials. Apart from the difficulties in algebraic calculations, the method is straightforward, and gives a better physical understanding of the phenomena considered, even though the obtained solutions are asymptotic ones.

The structure of the thesis is as follows: In the second chapter, we present some background material needed for the foregoing discussions. In the third chapter, the main part of the thesis, we construct an asymptotic model for the Stoneley interfacial wave. The problem will be examined under different types of boundary conditions, namely normal and tangential loading, and horizontal and vertical displacement conditions. Each boundary condition will be investigated separately, and a hyperbolic-elliptic formulation will be obtained. An illustrative example of an impulse point load problem will be presented. The thesis ends with some concluding remarks and ideas of future works.

# 2 BACKGROUND

In following pages, we present this overview as a reminder that it relies on the linearised theory of elasticity, asymptotic methods and some other concepts. An overview of asymptotic approach for obtaining the explicit model from Stoneley waves is given. Finally, we mention basic information which we use including the Cauchy-Riemann identities for plane harmonic function and the Poisson's integral formula for the Dirichlet problem for the Laplace equation.

## 2.1 The Linearized Theory of Elasticity

The theory of elasticity is a branch of continuum mechanics dealing with deformable solid bodies having physical properties analyse the influence and predict the outcomes of the action of external forces on the body. The fundamental "linearising" assumptions of linear elasticity are small displacement from a given deformation and linear relationships between the components of stress and strain.

At the beginning of 19th century the foundations of the elastic wave propagation was developed by many scientists among which are Cauchy, Poisson, Kirchhoff, Stokes, and Rayleigh.

The Stoneley wave, the subject of this thesis, is a generalised surface wave, propagating along the interface of two different media. A bare minimum of background information is therefore given in the following sections.

### 2.1.1 Strain

Consider a continuous medium of volume $V$ and surface $S$ that undergoes deformation. Before deformation, point $P_0$ is located by the vector $X_i$ and $P_1$, a neighbouring point of $P_0$ is located by the vector $dX_i$ from $P_0$. After deformation, $P_0$ goes into $P_0'$ and is located by the vector $x_i$ and $P_1$ goes into $P_1'$ and is located by the vector $dx_i$ relative to $P_0'$. The displacement of $P_0$ to $P_0'$ is measured by the vector $u_i$. The displacement of $P_1$ to $P_1'$ is measured by $\hat{u}_i$. The final volume and surface of the deformed body are $V'$ and $S'$, respectively. These quantities are shown in Fig.1.1.

The relationships between these quantities are given by

$$x_i = X_i + u_i, \qquad u_i + dx_i = \hat{u}_i + dX_i.$$

But, from the first equation we deduce that $dx_i = dX_i + du_i$. Substituting it

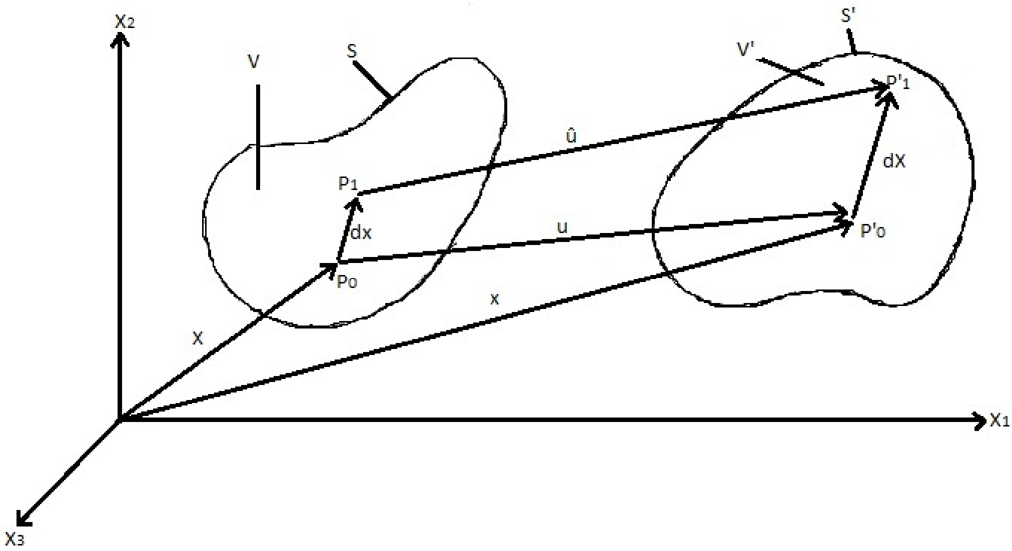


**Figure 1.** Deformation of continuum of volume $V$ into the volume $V'$

in the second equation, we have

$$\hat{u}_i = u_i + du_i$$

To first order, we may express $du_i$ as

$$du_i = u_{i,j} dx_j,$$

which may be presented in the form

$$du_i = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) dx_i + \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) dx_i. \tag{1}$$

We then define the infinitesimal strain and rotation tensors respectively as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \qquad \omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{2}$$

The result (1) emphasises that the kinematics of an arbitrary neighbouring point of $P_0$ is governed by the local strain-gradient field $u_{i,j}$ and that the motion is a combination of local distortion effects $\varepsilon_{ij}$ and also local rigid-body rotation effects $\omega_{ij}$ [7].

#### 2.1.2 Stress

Consider a continuum of volume $V$ and surface $S$ that is acted upon by various forces as shown Fig. 1.2(a). As a result of these forces, tractive forces will act on an arbitrary surface element within the body, as shown in Fig. 1.2(b). The traction vector is given by

$$\mathbf{t} = t_j \mathbf{i}_j, \tag{3}$$

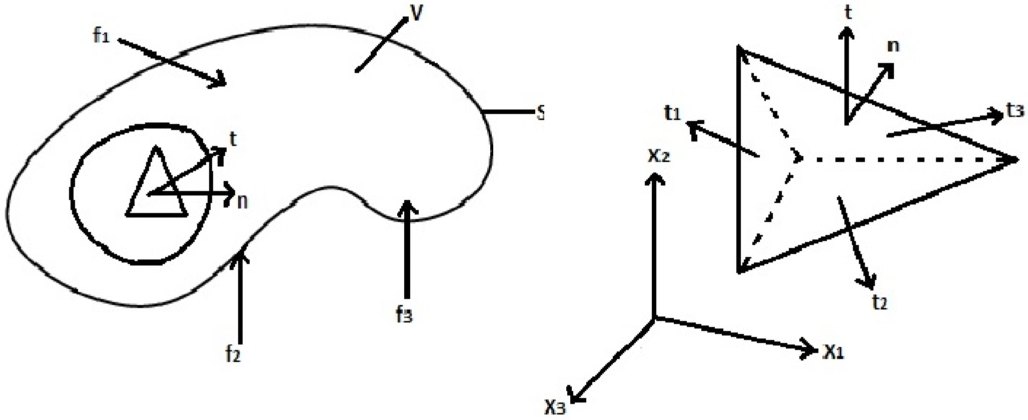


**Figure 2.** (a) A continuum subjected to forces, and (b) a trihedral element of that continuum.

where the traction components $t_j$ serve to define the stress tensor $\tau_{ij}$ by

$$t_i = \tau_{ij} n_j \tag{4}$$

where $\vec{n} = (n_1, n_2, n_3)$ is the normal vector and $l$, $m$, $n$ are unit vectors in directions $x$, $y$, $z$ respectively. In Cartesian frame, these equations take form

$$\begin{aligned} t_x &= \tau_{xx} l + \tau_{xy} m + \tau_{xz} n, \\ t_y &= \tau_{yx} l + \tau_{yy} m + \tau_{yz} n, \\ t_z &= \tau_{zx} l + \tau_{zy} m + \tau_{zz} n. \end{aligned} \tag{5}$$

Now, we will present the basic elasticity equations, scalar and vector potential equations. In addition we will mention dilatational and distortional waves which can propagate in an infinite medium, with each being characterised by a specific velocity [7].

### 2.2 Stress-Strain Relations

In general form, the linear relation between the components of the stress tensor and the components of strain tensor (Hooke's law) is

$$\tau_{ij} = C_{ijkl} \delta ij \varepsilon_{kl}, \tag{6}$$

where

$$C_{ijkl} = C_{jikl} = C_{klij} = C_{ijlk}. \tag{7}$$

It follows that 21 of the 81 components of the tensor $C_{ijkl}$ are indepented. the solid is homogeneous if the coefficients $C_{ijkl}$ do not depend on $\mathbf{x}$. It is isotropic when there are no preferred directions. It can be shown that elastic isotropy implies that the constants $C_{ijkl}$ may be expressed as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{8}$$

where $\delta_{ij}$ is the Kronecker delta, whose components are

$$y = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hooke's law then assumes the well-known form

$$\tau_{ij} = \lambda\varepsilon_{kk}\delta ij + 2\mu\varepsilon_{ij}, \tag{9}$$

where $\lambda$ and $\mu$ are known as the Lamé constants, $\mu$ being the shear modulus [37]. By assuming homogeneity and isotropy, the number of constants reduces from 21 to 2 [7].

### 2.2.1 The Governing Equations

The equations for a homogeneous isotropic elastic solid may be summarised in Cartesian tensor notation as

$$\begin{aligned} \tau_{ij,j} + \rho f_i &= \rho\ddot{u}_i, \\ \tau_{ij} &= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \\ \varepsilon_{ij} &= \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \end{aligned} \tag{10}$$

where $\tau_{ij}$ is the Cauchy stress tensor, $u_i$ is the displacement vector and $\mathbf{f}$ is the body force. The stress tensor is symmetric, so that $\tau_{ij} = \tau_{ji}$. The mass density per unit volume of the material is $\rho$, and $f_i$ is the body force per unit mass of material, $\varepsilon_{ij}$ is the strain tensor that is given by (see eqn.(2)) [7].

Both of elastic constants may be expressed in terms of the other elastic constants that often appear in linear elasticity which are Young's modulus $E$, the Bulk modulus $K$ and the Poisson's ratio $v$. A number of useful relationships among the isotropic constants are summarized in Table 1.1 [37].

The governing equations in terms of displacements are obtained by substituting the expression for strain into the stress-strain relation (9) and that

**Table 1.** Relationship among isotropic elastic constants

| | $E, v$ | $E, \mu$ | $\lambda, \mu$ |
|---|---|---|---|
| $\lambda$ | $\frac{Ev}{(1+v)(1-2v)}$ | $\frac{\mu(E-2\mu)}{3\mu-E}$ | $\lambda$ |
| $\mu$ | $\frac{E}{2(1+v)}$ | $\mu$ | $\mu$ |
| $E$ | $E$ | $E$ | $\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ |
| $K$ | $\frac{E}{3(1-2v)}$ | $\frac{\mu E}{3(3\mu-E)}$ | $\lambda+\frac{2}{3}\mu$ |
| $v$ | $v$ | $\frac{E-2\mu}{2\mu}$ | $\frac{\lambda}{2(\lambda+\mu)}$ |

result into the stress equations of motion, giving **Navier's equations** for the media

$$(\lambda+\mu)u_{j,ji}+\mu u_{i,jj}+\rho f_i=\rho\ddot{u}_i \quad i=1,2,3, \tag{11}$$

where a dot over the displacement components represent a time derivative. The vector equivalent of this expression is

$$(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}+\mu\nabla^2\mathbf{u}+\rho\mathbf{f}=\rho\ddot{\mathbf{u}} \tag{12}$$

where $\mathbf{u}(u,v,w)$ is displacement vector. In terms of rectangular scalar notation, this represents three equations

$$\begin{aligned}(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)+\mu\nabla^2\mathbf{u}+\rho f_x &= \rho\frac{\partial^2 u}{\partial t^2},\\ (\lambda+\mu)\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right)+\mu\nabla^2\mathbf{u}+\rho f_y &= \rho\frac{\partial^2 v}{\partial t^2},\\ (\lambda+\mu)\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}\right)+\mu\nabla^2\mathbf{u}+\rho f_z &= \rho\frac{\partial^2 w}{\partial t^2},\end{aligned} \tag{13}$$

where $u, v, w$ are the particle displacements in the $x, y, z$, directions, respectively. Returning to the vector notation, we note that the dilatation of material is defined by

$$\Delta=\nabla\cdot\mathbf{u}=\varepsilon_x+\varepsilon_y+\varepsilon_z=\varepsilon_{kk}, \tag{14}$$

so that (12) may also be written as

$$(\lambda+\mu)\nabla\Delta+\mu\nabla^2\mathbf{u}+\rho\mathbf{f}=\rho\ddot{\mathbf{u}}. \tag{15}$$

The results (12) and (15) are the most commonly employed forms of the equations. Substituting $\nabla^2\mathbf{u}$ in (12) gives

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times \mathbf{u} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}}. \tag{16}$$

Recalling that the rotation vector $\omega$ is defined by

$$\omega = \frac{1}{2}\nabla \times \mathbf{u},$$

and again using the dilatation $\Delta$, we may express the last result (16) as

$$(\lambda + 2\mu)\nabla\Delta - 2\mu\nabla \times \omega\mathbf{u} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}}.$$

One of the advantages of the last form is that it explicitly displays the dilatation and rotation. A greater advantage is that the result is valid in any curvilinear coordinate system, whereas the results (12) and (15) are valid only in rectangular coordinates.

A decomposition of a vector field into the gradient of a scalar and the curl of a zero-divergence vector is performed due to a theorem by **Helmholtz** [7].

$$\mathbf{u} = \nabla\Phi + \nabla \times \Psi, \qquad \nabla \cdot \Psi = 0 \tag{17}$$

The condition $\nabla \cdot \Psi = 0$ provides the necessary additional condition to uniquely determine the three components of $\mathbf{u}$ from the four components of $\Phi, \Psi$. We also express

$$\mathbf{f} = \nabla f + \nabla \times \mathbf{B}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{18}$$

Thus, the following equations may be written

$$(\lambda + 2\mu)\nabla^2\Phi + \rho f = \rho\ddot{\Phi}, \tag{19}$$

$$\mu\nabla^2\boldsymbol{\Psi} + \rho\mathbf{B} = \rho\ddot{\boldsymbol{\Psi}}, \tag{20}$$

where $\Phi$ and $\boldsymbol{\Psi}$ are the so-called **scalar** and **vector potentials**, respectively [7].

### 2.2.2 Dilatational and Distortional Waves

Consider the governing displacement equations in the absence of body forces, given by

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} = \rho\ddot{\mathbf{u}}. \tag{21}$$

If the vector operation of divergence is performed on the above, we obtain

$$(\lambda + \mu)\nabla \cdot (\nabla\nabla \cdot \mathbf{u}) + \mu\nabla \cdot (\nabla^2\mathbf{u}) = \rho\nabla \cdot \ddot{\mathbf{u}}. \tag{22}$$

Since $\nabla \cdot \nabla \sim \nabla^2$, $\nabla \cdot (\nabla^2 u) = \nabla^2(\nabla \cdot u)$ and $\nabla \cdot \mathbf{u} = \Delta$, the dilatation,(22) reduces to

$$(\lambda + 2\mu)\nabla^2\Delta = \rho\frac{\partial^2\Delta}{\partial t^2}. \tag{23}$$

This is recognised as the wave equation, expressible in the form

$$\nabla^2\Delta = \frac{1}{c_1^2}\frac{\partial^2\Delta}{\partial t^2}, \tag{24}$$

where the propagation velocity $c_1$ is given by

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{25}$$

We thus conclude that a change in volume, or dilatational disturbance, will propagate at the velocity $c_1$.

We now perform the operation of curl on the governing equation (22). Since the curl of the gradient of scalar is zero, this gives

$$\mu\nabla^2\omega = \rho\frac{\partial^2\omega}{\partial t^2}, \tag{26}$$

where $\omega = \nabla \times u/2$ is the previously defined rotation vector. This result is in the form of the vector wave equation and may be expressed as

$$\nabla^2\omega = \frac{1}{c_2^2}\frac{\partial^2\omega}{\partial t^2}, \tag{27}$$

where the propagation velocity $c_2$ is given by

$$c_2 = \sqrt{\frac{\mu}{\rho}}. \tag{28}$$

Thus, rotational waves propagate with a velocity $c_2$ in the medium.

Finally we refer to (19)-(20), the equations that resulted from introducing the scalar and vector potentials $\Phi$ and $\Psi$. If the body forces are zero, we have $\mathbf{f} = \mathbf{B} = 0$, and the two equations again give the scalar and vector wave equations and contain the velocities $c_1$ and $c_2$. The significance of the Helmholtz resolution of $\mathbf{u}$ becomes even more apparent at this stage. The scalar potential is seen to be associated with the dilatation part of the dieturbance, and the vector potential is associated with the rotational part [7].

We have thus found that waves may propagate in the interior of an elastic solid at two different speeds $c_1$ and $c_2$. The ratio of the two wave speeds may be expressed as

$$k = \frac{c_1}{c_2} = \sqrt{\frac{\lambda + 2\mu}{\mu}} = \sqrt{\frac{2 - 2v}{1 - 2v}}. \tag{29}$$

Since $0 \leq v \leq 1/2$ always, we see that $c_1 > c_2$.

A variety of terminology exist for the two wave-types. Dilatational waves are also called irrotational and primary (P) waves. The rotational waves are also called distortional and secondary (S) waves. Other designations frequently used are longitudinal and shear waves [7].

### 2.2.3 Plane Waves

Let us now discuss plane waves which propagate in an infinite elastic solid. A plane displacement wave propagating with phase velocity $c$ in a direction defined by the unit propagating vector $\mathbf{n}$ is represented by

$$u = \mathbf{A}f(\mathbf{n} \cdot \mathbf{x} - ct) \tag{30}$$

or, in index notation,

$$u_i = A_i f(n_i \cdot x_i - ct).$$

In this equation $\mathbf{A}$ and $\mathbf{n}$ are unit vectors defining the directions of motion and propagation, respectively. The vector $x$ denotes the position vector and $\mathbf{n}.\mathbf{x} = \textit{constant}$ describes a plane normal to the unit vector $\mathbf{n}$ equation (2.2.3), thus represents a **plane wave** whose planes of constant phase are normal to $\mathbf{n}$ and propagate with velocity $c$ [2]

The expression for a plane wave, equation (21), is substituted into the homogeneous form of the displacement equation of motion given by equation . By employing the relations (21). By employing the relations, we obtain

$$[\mu\mathbf{n} + (\lambda + \mu)(\mathbf{n}.\mathbf{A})\mathbf{n} - \rho c^2\mathbf{n}]f''(\mathbf{n}\mathbf{x} - ct) = 0$$

or

$$(\mu - \rho c^2)\mathbf{A} + (\lambda + \mu)(\mathbf{n}.\mathbf{A})\mathbf{n} = 0. \tag{31}$$

Since $\mathbf{n}$ and $\mathbf{A}$ are two different unit vectors, equation (31) can be satisfied in two ways only: either $\mathbf{A} = \pm\mathbf{n}$, or $\mathbf{n}.\mathbf{A} = 0$. If $\mathbf{A} = \pm\mathbf{n}$, we have $\mathbf{A}.\mathbf{n} = \pm 1$ and equation (31) yields

$$c = c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

In this case the motion is parallel to the direction of propagation, and the wave is therefore called a longitudinal (dilatational) or L wave.

If $\mathbf{A} \neq \pm\mathbf{n}$, both terms in (31) have to vanish independently, yielding $\mathbf{n}.\mathbf{A} = 0$ and

$$\mathbf{n}.\mathbf{A} = 0 \quad \text{and} \quad c = c_2 = \sqrt{\frac{\mu}{\rho}}.$$

Now the motion is normal to the direction of propagation, and the wave is called a transverse (distortional) or T wave [37]. Thus, plane waves propagate at one or the other velocity in a media [7].

### 2.2.4 Plane Strain

In two-dimensional problems the body forces and components of the stress tensor are independent of one of the coordinates, say $x_3$. The stress equations of motion can be derived from (10) by setting $\partial/\partial x_3 = 0$. We find that the system of equations splits up into two uncoupled systems. These are

$$\tau_{3\beta,\beta} + \rho f_3 = \rho \ddot{u}_3 \tag{32}$$

and

$$\tau_{\alpha\beta,\beta} + \rho f_\alpha = \rho \ddot{u}_\alpha \tag{33}$$

where Greek indices can assume the values 1 and 2 only.

It follows from eqn. (33) that the in-plane displacements $u_\alpha$ depend on $x_1, x_2$ and $t$ only, with regard to the dependence of $u_3$ on the spatial coordinates and time, two separate cases that is plane stress and plane strain are described by eqn. (33). Here we restrict our attention to the plane strain case.

In plane strain case all field variables are indepented of $x_3$ and the displacement in the $x_3-$direction vanishes identically. Hooke's law then yields the following relations:

$$\tau_{\alpha\beta} = \lambda u_{\gamma,\gamma}\delta_{\alpha\beta} + \mu(u_{\alpha-\beta} + u_{\beta,\alpha}), \tag{34}$$

$$\tau_{33} = \lambda u_{\gamma,\gamma}, \tag{35}$$

where Greek indices can assume the values 1 and 2 only. Elimination of $\tau_{\alpha\beta}$ from (33) and (34) leads to

$$\mu u_{\alpha_\beta\beta} + (\alpha + \mu) u_{\beta,\beta\alpha} + \rho f_\alpha = \rho \ddot{u}_\alpha \tag{36}$$

[37].

### 2.2.5 Boundary Conditions

On the surface $S$ of the undeformed body, boundary conditions must be prescribed. The following boundary conditions are most common:

*Displacement boundary conditions* the three components $u_i$ are prescribed on the boundary.

*Traction boundary conditions* the three traction components $t_i$ are prescribed on the boundary at a position with unit normal $\mathbf{n}$. Through Cauchy's formula

$$t_i = \tau_{ji} \cdot n_j$$

this case actually corresponds to conditions on three components of the stress tensor.

*Mixed boundary conditions* are prescribed displacements and traction on different parts of the boundary [2].

### 2.2.6 Surface Waves

A surface (interfacial) wave is a mechanical wave that propagates along the interface of differing media. For example wind waves are generated between the atmosphere and ocean and they propagate at the interface. Rayleigh wave which was investigated by Lord Rayleigh is a surface wave. Rayleigh showed that their effect decreases rapidly with depth and their velocity of propagation is smaller than that of body waves [7]. When there is a boundary, as in the half plane problem, a third type of wave may exist whose effects are confined closely to the surface.

We encounter surface waves in a variety of natural phenomena as well as engineering applications. The most famous of these are the Rayleigh, Lamb,Love, Stoneley and Schölte waves [2]. Rayleigh waves propagate near the surface of solid. While Rayleigh wave propagates at the interface of vacuum-solid, Schölte waves is surface waves created dynamic load at an interface between a solid and a fluid medium. Lamb and Love waves propagate in solids, Love waves are horizontally polarized surface waves, Lamb waves move in the direction perpendicular to the plate it acts on.

A Stoneley wave is a high-amplitude interface wave named after the British seismologist *Robert Stoneley* in 1924 [19]. The wave is of maximum intensity at the interface and decreases exponentially away from it. The two media will be distinguished by suffixes 1 and 2. It will be supposed in "*welded contact*" along an infinite plane face (no slipping) [19].

We may consider displacement components of the form,

$$u_1 = Ae^{-by}\exp[ik(x-ct)]$$
$$u_1 = Be^{-by}\exp[ik(x-ct)]$$
$$u_1 = 0.$$

The real part of $b$ is supposed to be positive, so that the displacements decrease with increasing $y$ and tend to zero as $y$ increases beyond bounds [2].

Given suitable generating conditions, surface waves as well as body waves are generated at a bounding surface. For a two-dimensional geometry the surface waves are essentially one-dimensional, but the body waves are cylindrical and undergo geometrical attenuation. Thus at some distance from the source the disturbance due to the surface wave becomes predominant [2].

### 2.3 Asymptotic Approximation

It is not always possible to find exact analytical solutions for most differential and integral equations. Asymptotic analysis is concerned with both developing techniques and obtaining approximate analytical solutions to such problems.

In 1886, Poincaré gave a precise definition of what is called an asymptotic expansion and laid the foundations of modern asymptotic analysis [38].

Many scientists used several asymptotic expansion methods to evaluate integrals for mathematical and physical problems such as Watson lemma, stationary phase, steepest descent etc. It is allowing an explicit solution in terms of elementary functions [39].

The derivations that will be given in the next chapter are based on perturbing in slow time the self-similar solutions for a homogeneous surface wave [29]. Therefore we give some definitions related to this subject.

**Definition 2.3.1.** *If $f(z)$ and $g(z)$, two functions of a complex number $z$, which may be parameter of the problem or an independent variable defined on some domain $D$, $f(z) = O(g(z))$ as $z \to z_0$ means that there are constants $K$ and $\delta$ such that*

$$|f| \leq K|g| \qquad \textit{for} \quad 0 < |z - z_0| < \delta.$$

*We say that $f$ is "big Oh" of $g$ as $z \to z_0$ [38].*

**Definition 2.3.2.** *If $f(z)$ and $g(z)$, two functions of a complex number $z$, which may be parameter of the problem or an independent variable defined on* *some domain D, $f(z) = o(g(z))$ as $z \to z_0$ means that for every positive $\varepsilon$ there is a $\delta$ (independent of $\varepsilon$) such that*

$$|f| \leq \varepsilon|g| \qquad \textit{for} \quad 0 < |z - z_0| < \delta.$$

*We say that $f$ is "little Oh" of $g$ as $z \to z_0$ [38].*

Thus as long as $g(z)$ is not zero in a neighbourhood of $z_0$, other than possibly $z_0$, $f(z) = O(g(z))$ implies that $f/g \to 0 \quad \text{as} \quad z \to z_0$, while $f(z) = o(g(z))$ implies that $f/g$ is bounded.

We say that $f(z)$ is **asymptotically equivalent** or **equal** to $g(z)$ under the limit $z \to z_0$ if $f$ and $g$ are such that $\lim\limits_{z \to z_0} f/g = 1$. We write

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0 \quad \text{if} \quad \lim_{z \to z_0} \frac{f(z)}{g(z)} = 1.$$

**Definition 2.3.3.** *A finite or infinite sequences of functions $\{\phi_n(z)\}, n = 1, 2, \cdot$ is an **asymptotic sequence** as $z \to z_0$ if, for all $n$,*

$$\phi_{n+1}(z) = o(\phi_n(z)) \quad \textit{as} \quad z \to z_0$$

*that is, $\lim\limits_{z \to z_0} \phi_{n+1}/\phi_n = 0$ [38].*

**Definition 2.3.4.** *If $\{\phi_n(z)\}$ is an asymptotic sequence of functions as $z \to z_0$, we say that $\sum\limits_{n=1} a_n\phi_n(z)$, where the $a_n$ are constant (with the upper limit omitted), is an **asymptotic expansion** or **asymptotic approximation** of the function $f(z)$ if for each $N$*

$$f(z) = \sum_{n=1}^{N} a_n\phi_n(z) + o(\phi_N(z)) \quad \textit{as} \quad z \to z_0$$

*[38].*

Asymptotic methods are frequently used in differential equations, evaluation of certain integrals, obtaining approximate analytical solution to such problems etc. In this thesis, the solution in terms of potentials are sought in terms of asymptotic expansions in the slow time perturbation parameter.

## 2.4 Other Basic Concepts

### 2.4.1 Harmonic Functions, Cauchy-Riemann Equations

If $f(z)$ is defined in finite domain $G \subset \mathbb{C}$, and is differentiable with respect to $z$ at each point of $G$, then $f(z)$ is said to be an **analytic function** in $G$.

A function $f(z) = u(x, y)+iv(x, y)$ is analytic in a domain $G$ if the functions $u(x, y)$ and $v(x, y)$ are differentiable throughout $G$ and the **Cauchy-Riemann differential equations**

$$u_x = v_y, \qquad u_y = -v_x$$

are satisfied [40].

### 2.4.2 Solution of the Dirichlet Problem for a Half Space

Let us consider the case when the domain $D$ is a half-space; for definiteness, let $D$ be the half-space $x_2 > 0$. Here we shall require that the sought-for solution of the Dirichlet problem should be bounded. Let $x(x_1, x_2)$ and $\xi(\xi_1, \xi_2)$ be two points belonging to that half-space and let us take the point $\xi' = (\xi_1, -\xi_2)$ symmetric to the point $\xi$ about the plane $\xi_2 = 0$. We will assume that in the case under consideration the sought-for $u(x)$ of the Dirichlet problem can be represented in form

$$u(x) = \frac{x_2}{\pi} \int_{\xi_2=0} \frac{\phi(\xi_1)}{(\xi_1 - x_1)^2 + x_2^2} d\xi_1 \tag{37}$$

for $\xi_2 > 0$.

Formula (38) expresses the solution of the Dirichlet problem with the boundary condition

$$\lim_{x \to y} u(x) = \phi(y_1); \quad x_2 > 0, \quad y_2 = 0 \tag{38}$$

for the half space $x_2 > 0$; this formula is also called **Poisson's formula** [41], [42].

## 3 ASYMPTOTIC MODEL FOR THE STONELEY WAVE

In this chapter, we construct the asymptotic model for the Stoneley wave. As it is already mentioned in the Introduction, after the discovery of Rayleigh surface wave, Friedlander [26] suggested a solution in terms of a pair of harmonic functions. Over thirty years later, Chadwick [27] demonstrated that the solution may actually be obtained using a single potential, the second of which can be obtained through a relation on the surface.

In our study, we derive an asymptotic model for interfacial Stoneley wave. In 2006, the asymptotic model for Rayleigh surface waves was obtained by Kaplunov et. al, being applied to a member of dynamic problems and allowing significant simplifications of the analysis [29]. In [31], [32] and [43], papers, they derived asymptotic model and used asymptotic model for different problems. Thus we intend to derive an asymptotic model for the interfacial Stoneley wave with the understanding that the model will provide notable simplification, reducing a vector problem of elasticity to a scalar one. We should also mention that the proposed method provides an estimate of the contribution of Stoneley wave to the overall dynamic response and is therefore accurate provided that the interfacial wave is dominant, which would be true for a variety of near-resonant problems.

### 3.1 Statement of the Problem

We consider a plane strain problem for two elastic isotropic half planes assumed to be in contact along an infinite straight line, with the Cartesian axes $Ox$ along the interface (See, Fig. 3.1). The governing equations of motion are written in terms of Lamé elastic potentials as

$$\frac{\partial^2 \varphi_i}{\partial x^2} + \frac{\partial^2 \varphi_i}{\partial y^2} - \frac{1}{c_{1i}^2} \frac{\partial^2 \varphi_i}{\partial t^2} = 0, \tag{39}$$

$$\frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2} - \frac{1}{c_{2i}^2} \frac{\partial^2 \psi_i}{\partial t^2} = 0, \quad i = 1, 2 \tag{40}$$

where $\varphi_i$ and $\psi_i$ denote wave potentials, $c_{1i}$ and $c_{2i}$ are longitudinal and shear wave speeds, respectively, which are represented in terms of elastic constants $\lambda_i, \mu_i$, and density $\rho_i$ in case of medium $i$ as

$$c_{1i}^2 = \frac{\lambda_i + 2\mu_i}{\rho_i}, \qquad c_{2i}^2 = \frac{\mu_i}{\rho_i}, \quad i = 1, 2$$

[7].

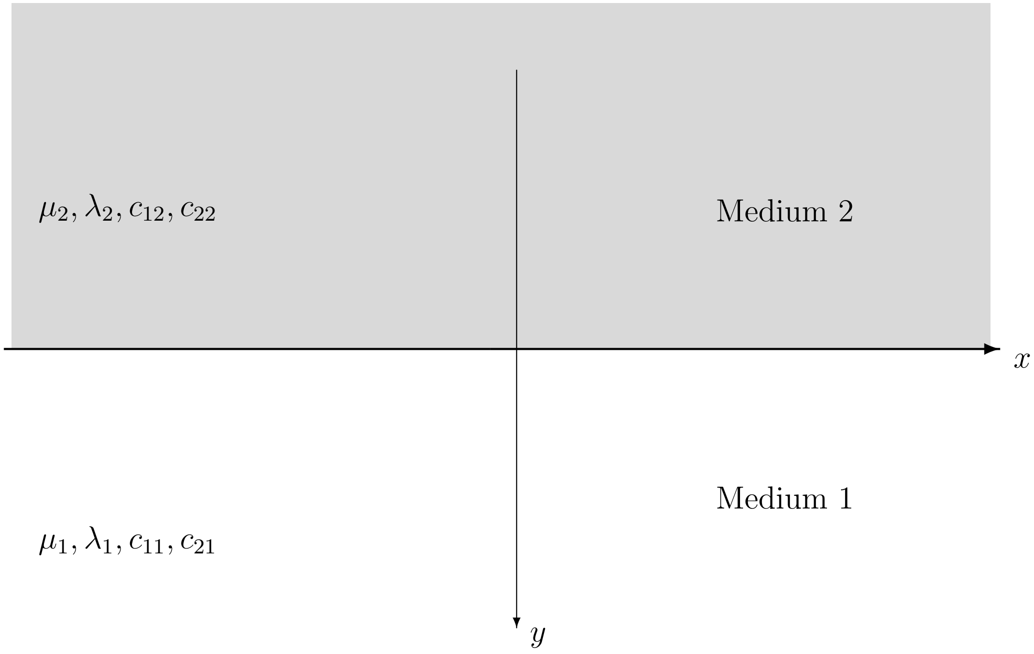


**Figure 3.** Geometry of the problem

The boundary conditions on the interface $y = 0$ are written in the form

$$(u_1 - u_2)\big|_{y=0} = q_1(x,t), \tag{41}$$

$$(w_1 - w_2)\big|_{y=0} = q_2(x,t), \tag{42}$$

$$\left(\sigma_{12}^{(1)} - \sigma_{12}^{(2)}\right)\Big|_{y=0} = p_1(x,t), \tag{43}$$

$$\left(\sigma_{22}^{(1)} - \sigma_{22}^{(2)}\right)\Big|_{y=0} = p_2(x,t), \tag{44}$$

where the functions $q_i(x,t), i = 1,2$, correspond to a jump in the displacements $u_i$ and $w_i$ along the interface, and $p_i(x,t), i = 1,2$ are the normal and tangential loads, respectively. The displacements $u_i$, $w_i$ and the stresses $\sigma_{jk}^{(i)}$ for medium $i$ are expressed in terms of the elastic potentials as

$$u_i = \frac{\partial \varphi_i}{\partial x} + \frac{\partial \psi_i}{\partial y}, \qquad w_i = \frac{\partial \varphi_i}{\partial y} - \frac{\partial \psi_i}{\partial x}, \tag{45}$$

$$\sigma_{12}^{(i)} = \mu_i \left(2\frac{\partial^2 \varphi_i}{\partial x \partial y} - \frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2}\right), \tag{46}$$

$$\sigma_{22}^{(i)} = \lambda_i \frac{\partial^2 \varphi_i}{\partial x^2} + (\lambda_i + 2\mu_i)\frac{\partial^2 \varphi_i}{\partial y^2} - 2\mu_i \frac{\partial^2 \psi_i}{\partial x \partial y}, \qquad i = 1,2. \tag{47}$$

The interfacial boundary conditions are then presented in the form

$$\left[\frac{\partial\varphi_1}{\partial x}-\frac{\partial\varphi_2}{\partial x}+\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial y}\right]\Bigg|_{y=0}=q_1(x,t), \tag{48}$$

$$\left[\frac{\partial\varphi_1}{\partial y}-\frac{\partial\varphi_2}{\partial y}-\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial x}\right]\Bigg|_{y=0}=q_2(x,t), \tag{49}$$

$$\left[2\mu_1\frac{\partial^2\varphi_1}{\partial x\partial y}-2\mu_2\frac{\partial^2\varphi_2}{\partial x\partial y}+\mu_1\left(\frac{\partial^2\psi_1}{\partial y^2}-\frac{\partial^2\psi_1}{\partial x^2}\right)-\right.$$
$$\left.-\mu_2\left(\frac{\partial^2\psi_2}{\partial y^2}-\frac{\partial^2\psi_2}{\partial x^2}\right)\right]\Bigg|_{y=0}=p_1(x,t), \tag{50}$$

$$\left[\lambda_1\frac{\partial^2\varphi_1}{\partial x^2}+(\lambda_1+2\mu_1)\frac{\partial^2\varphi_1}{\partial y^2}-\lambda_2\frac{\partial^2\varphi_2}{\partial x^2}-(\lambda_2+2\mu_2)\frac{\partial^2\varphi_2}{\partial y^2}-\right.$$
$$\left.-2\mu_1\frac{\partial^2\psi_1}{\partial x\partial y}+2\mu_2\frac{\partial^2\psi_2}{\partial x\partial y}\right]\Bigg|_{y=0}=p_2(x,t). \tag{51}$$

It is convenient to separate the boundary conditions ((48)-(51)) into those for a normal load ($q_1=q_2=p_1=0$, $p_2\neq 0$), a tangential load ($q_1=q_2=p_2=0$, $p_1\neq 0$), a vertical jump ($p_1=p_2=q_2=0$, $q_1\neq 0$) and a horizontal jump ($p_1=p_2=q_1=0$, $q_2\neq 0$).

### 3.2 Model for the Stoneley Wave

The asymptotic model provides better physical understanding of the surface wave phenomena and also allows significant simplifications for boundary problems with given surface loading [39]. The self-similar solution in the variables

$$\xi=x-c_st, \qquad y=y, \tag{52}$$

(where $c_s$ denotes the Stoneley wave speed) has been constructed by Chadwick (1976). To incorporate the effect of surface (interfacial) loading, we perturb below this self similar solution in the slow time $\tau=\varepsilon t \quad (\varepsilon\ll 1)$. This perturbation allows us to evaluate Stoneley wave contribution into the overall dynamic response. Here and below we assume that the condition of existence of Stoneley wave [20], are satisfied. The governing equations may be written in terms of the new variables as

$$\frac{\partial^2\varphi_i}{\partial y^2}+\left(1-\frac{c_s^2}{c_{1i}^2}\right)\frac{\partial^2\varphi_i}{\partial\xi^2}+2\varepsilon\frac{c_s}{c_{1i}^2}\frac{\partial^2\varphi_i}{\partial\xi\partial\tau}-\frac{\varepsilon^2}{c_{1i}^2}\frac{\partial^2\varphi_i}{\partial\tau^2}=0, \tag{53}$$

$$\frac{\partial^2\psi_i}{\partial y^2}+\left(1-\frac{c_s^2}{c_{2i}^2}\right)\frac{\partial^2\psi_i}{\partial\xi^2}+2\varepsilon\frac{c_s}{c_{2i}^2}\frac{\partial^2\psi_i}{\partial\xi\partial\tau}-\frac{\varepsilon^2}{c_{2i}^2}\frac{\partial^2\psi_i}{\partial\tau^2}=0 \tag{54}$$

and the boundary conditions with regard to variables $\xi$ and $\tau$ are

$$\left[\frac{\partial\varphi_1}{\partial\xi}-\frac{\partial\varphi_2}{\partial\xi}+\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial y}\right]\Bigg|_{y=0}=q_1(\xi,t), \tag{55}$$

$$\left[\frac{\partial\varphi_1}{\partial y}-\frac{\partial\varphi_2}{\partial y}-\frac{\partial\psi_1}{\partial\xi}+\frac{\partial\psi_2}{\partial\xi}\right]\Bigg|_{y=0}=q_2(\xi,t), \tag{56}$$

$$\Bigg[2\mu_1\frac{\partial^2\varphi_1}{\partial\xi\partial y}-2\mu_2\frac{\partial^2\varphi_2}{\partial\xi\partial y}+\mu_1\left(\frac{\partial^2\psi_1}{\partial y^2}-\frac{\partial^2\psi_1}{\partial\xi^2}\right)-$$
$$-\mu_2\left(\frac{\partial^2\psi_2}{\partial y^2}-\frac{\partial^2\psi_2}{\partial\xi^2}\right)\Bigg]\Bigg|_{y=0}=p_1(\xi,t), \tag{57}$$

$$\Bigg[\lambda_1\frac{\partial^2\varphi_1}{\partial\xi^2}+(\lambda_1+2\mu_1)\frac{\partial^2\varphi_1}{\partial y^2}-\lambda_2\frac{\partial^2\varphi_2}{\partial\xi^2}-(\lambda_2+2\mu_2)\frac{\partial^2\varphi_2}{\partial y^2}-$$
$$-2\mu_1\frac{\partial^2\psi_1}{\partial\xi\partial y}+2\mu_2\frac{\partial^2\psi_2}{\partial\xi\partial y}\Bigg]\Bigg|_{y=0}=p_2(\xi,t). \tag{58}$$

We now search for asymptotic solutions of (53) and (54), along with the boundary conditions (55)-(58) with respect to the wave potentials in the form of asymptotic series

$$\varphi_i=\frac{P_*}{\varepsilon\mu_*}\left(\varphi_0^{(i)}(\xi,y,t)+\varepsilon(\varphi_1^{(i)}(\xi,y,t)+...)\right), \tag{59}$$

$$\psi_i=\frac{P_*}{\varepsilon\mu_*}\left(\psi_0^{(i)}(\xi,y,t)+\varepsilon(\psi_1^{(i)}(\xi,y,t)+...)\right), \qquad i=1,2. \tag{60}$$

where $P_*=\max\{p_1(\xi,t),p_2(\xi,t),q_1(\xi,t),q_2(\xi,t)\}$ and $\mu_*=\max\{\mu_1,\mu_2\}$ .

Substituting (59), (60) into (53) and (54), we arrive at the leading order, at elliptic equations

$$\frac{\partial^2\varphi_0^{(i)}}{\partial y^2}+\left(1-\frac{c_s^2}{c_{1i}^2}\right)\frac{\partial^2\varphi_0^{(i)}}{\partial\xi^2}=0, \tag{61}$$

$$\frac{\partial^2\psi_0^{(i)}}{\partial y^2}+\left(1-\frac{c_s^2}{c_{2i}^2}\right)\frac{\partial^2\psi_0^{(i)}}{\partial\xi^2}=0. \tag{62}$$

Equations (61) and (62) are satisfied by plane harmonic functions of the form

$$\varphi_0^{(i)}=\varphi_0^{(i)}(\xi,\alpha_i y,\tau), \quad \psi_0^{(i)}=\psi_0^{(i)}(\xi,\beta_i y,\tau) \tag{63}$$

where

$$\alpha_i^2=1-\frac{c_s^2}{c_{1i}^2}, \quad \beta_i^2=1-\frac{c_s^2}{c_{2i}^2}.$$

The functions appearing in (63) may also be assumed to contain the scaled variable $\tau$ as a parameter.

At next order, governing equations reduce to the inhomogeneous equations given by

$$\frac{\partial^2\varphi_1^{(i)}}{\partial y^2}+\alpha_i^2\frac{\partial^2\varphi_1^{(i)}}{\partial \xi^2}=-2\left(\frac{1-\alpha_i^2}{c_s}\right)\frac{\partial^2\varphi_0^{(i)}}{\partial\xi\partial\tau}, \tag{64}$$

$$\frac{\partial^2\psi_1^{(i)}}{\partial y^2}+\beta_i^2\frac{\partial^2\psi_1^{(i)}}{\partial \xi^2}=-2\left(\frac{1-\beta_i^2}{c_s}\right)\frac{\partial^2\psi_0^{(i)}}{\partial\xi\partial\tau}. \tag{65}$$

The solution of these inhomogeneous equations may be represented by

$$\varphi_1^{(i)}=\varphi_{10}^{(i)}+y\varphi_{11}^{(i)}, \qquad \psi_1^{(i)}=\psi_{10}^{(i)}+y\psi_{11}^{(i)}$$

where $\varphi_{10}^{(i)},\varphi_{11}^{(i)},\psi_{10}^{(i)}$ and $\psi_{11}^{(i)}$ are harmonic functions. Inserting $\varphi_1^{(i)}$ and $\psi_1^{(i)}$ into equations (64) and (65), we obtain the following equations:

$$\frac{\partial\varphi_{11}^{(1)}}{\partial y}=-\left(\frac{1-\alpha_1^2}{c_s}\right)\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}, \qquad \frac{\partial\psi_{11}^{(1)}}{\partial y}=-\left(\frac{1-\beta_1^2}{c_s}\right)\frac{\partial^2\psi_0^{(1)}}{\partial\xi\partial\tau}, \tag{66}$$

$$\frac{\partial\varphi_{11}^{(2)}}{\partial y}=-\left(\frac{1-\alpha_2^2}{c_s}\right)\frac{\partial^2{\varphi_0}^{(2)}}{\partial\xi\partial\tau}, \qquad \frac{\partial\psi_{11}^{(2)}}{\partial y}=-\left(\frac{1-\beta_2^2}{c_s}\right)\frac{\partial^2\psi_0^{(2)}}{\partial\xi\partial\tau}. \tag{67}$$

Here and below, we employ the Cauchy-Riemann identities

$$\frac{\partial f}{\partial\xi}=\frac{1}{k}\frac{\partial\overline{f}}{\partial y}, \quad \frac{\partial f}{\partial y}=-k\frac{\partial\overline{f}}{\partial\xi}, \quad \overline{\overline{f}}=-f, \qquad y>0, \tag{68}$$

$$\frac{\partial f}{\partial\xi}=-\frac{1}{k}\frac{\partial\overline{f}}{\partial y}, \quad \frac{\partial f}{\partial y}=k\frac{\partial\overline{f}}{\partial\xi}, \quad \overline{\overline{f}}=-f, \qquad y<0 \tag{69}$$

where $f=f(\xi,ky)$ is an arbitrary plane harmonic function and $\overline{f}$ is its harmonic conjugate.

Utilizing the Cauchy-Riemann equations in (66) and (67) the asymptotic expansions may now be written in the form

$$\varphi_1=\frac{P_*}{\varepsilon\mu_*}\left(\varphi_0^{(1)}+\varepsilon\left(\varphi_{10}^{(1)}-y\left(\frac{1-\alpha_1^2}{c_s\alpha_1}\right)\frac{\partial\overline{\varphi}_0^{(1)}}{\partial\tau}\right)\right), \tag{70}$$

$$\psi_1=\frac{P_*}{\varepsilon\mu_*}\left(\psi_0^{(1)}+\varepsilon\left(\psi_{10}^{(1)}-y\left(\frac{1-\beta_1^2}{c_s\beta_1}\right)\frac{\partial\overline{\psi}_0^{(1)}}{\partial\tau}\right)\right), \tag{71}$$

$$\varphi_2=\frac{P_*}{\varepsilon\mu_*}\left(\varphi_0^{(2)}+\varepsilon\left(\varphi_{10}^{(2)}+y\left(\frac{1-\alpha_2^2}{c_s\alpha_2}\right)\frac{\partial\overline{\varphi}_0^{(2)}}{\partial\tau}\right)\right), \tag{72}$$

$$\psi_2=\frac{P_*}{\varepsilon\mu_*}\left(\psi_0^{(2)}+\varepsilon\left(\psi_{10}^{(2)}+y\left(\frac{1-\beta_2^2}{c_s\beta_2}\right)\frac{\partial\overline{\psi}_0^{(2)}}{\partial\tau}\right)\right). \tag{73}$$

The obtained solutions, equations (70)-(73), over the interior allow separate consideration of four different boundary conditions, including the cases of normal, tangential loading, along with vertical and horizontal jumping.

### 3.2.1 Normal Loading

We begin with consideration of the case of normal loading, implying that $q_1, q_2$ and $p_1$ vanish

$$\left[\frac{\partial \varphi_1}{\partial \xi} - \frac{\partial \varphi_2}{\partial \xi} + \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial y}\right]\Bigg|_{y=0} = 0, \tag{74}$$

$$\left[\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial y} - \frac{\partial \psi_1}{\partial \xi} + \frac{\partial \psi_2}{\partial \xi}\right]\Bigg|_{y=0} = 0, \tag{75}$$

$$\left[2\mu_1 \frac{\partial^2 \varphi_1}{\partial \xi \partial y} - 2\mu_2 \frac{\partial^2 \varphi_2}{\partial \xi \partial y} + \mu_1 \left(\frac{\partial^2 \psi_1}{\partial y^2} - \frac{\partial^2 \psi_1}{\partial \xi^2}\right) - \right.$$
$$\left. -\mu_2 \left(\frac{\partial^2 \psi_2}{\partial y^2} - \frac{\partial^2 \psi_2}{\partial \xi^2}\right)\right]\Bigg|_{y=0} = 0, \tag{76}$$

$$\left[\lambda_1 \frac{\partial^2 \varphi_1}{\partial \xi^2} + (\lambda_1 + 2\mu_1) \frac{\partial^2 \varphi_1}{\partial y^2} - \lambda_2 \frac{\partial^2 \varphi_2}{\partial \xi^2} - (\lambda_2 + 2\mu_2) \frac{\partial^2 \varphi_2}{\partial y^2} - \right.$$
$$\left. -2\mu_1 \frac{\partial^2 \psi_1}{\partial \xi \partial y} + 2\mu_2 \frac{\partial^2 \psi_2}{\partial \xi \partial y}\right]\Bigg|_{y=0} = p_2(\xi, t). \tag{77}$$

Substituting the asymptotic solution (70)-(73) into the boundary conditions (55)-(58) we obtain at the leading order on the interface $y = 0$

$$\left[\frac{\partial \varphi_0^{(1)}}{\partial \xi} + \frac{\partial \psi_0^{(1)}}{\partial y} - \frac{\partial \varphi_0^{(2)}}{\partial \xi} - \frac{\partial \psi_0^{(2)}}{\partial y}\right]\Bigg|_{y=0} = 0, \tag{78}$$

$$\left[\frac{\partial \varphi_0^{(1)}}{\partial y} - \frac{\partial \psi_0^{(1)}}{\partial \xi} - \frac{\partial \varphi_0^{(2)}}{\partial y} + \frac{\partial \psi_0^{(2)}}{\partial \xi}\right]\Bigg|_{y=0} = 0, \tag{79}$$

$$\left[2\alpha_1\mu_1 \frac{\partial^2 \varphi_0^{(1)}}{\partial \xi^2} + 2\alpha_2\mu_2 \frac{\partial^2 \varphi_0^{(2)}}{\partial \xi^2} - \mu_1(1+\beta_1^2) \frac{\partial^2 \psi_0^{(1)}}{\partial \xi^2} + \right.$$
$$\left. +\mu_2(1+\beta_2^2) \frac{\partial^2 \psi_0^{(2)}}{\partial \xi^2}\right]\Bigg|_{y=0} = 0, \tag{80}$$

$$\left[-\mu_1(1+\beta_1^2) \frac{\partial^2 \varphi_0^{(1)}}{\partial \xi^2} + \mu_2(1+\beta_2^2) \frac{\partial^2 \varphi_0^{(2)}}{\partial \xi^2} + 2\beta_1\mu_1 \frac{\partial^2 \psi_0^{(1)}}{\partial \xi^2} + \right.$$
$$\left. +2\beta_2\mu_2 \frac{\partial^2 \psi_0^{(2)}}{\partial \xi^2}\right]\Bigg|_{y=0} = 0. \tag{81}$$

Now, we apply Cauchy-Riemann equations for the boundary conditions and take the derivative of the boundary conditions which is related to displacements with $\xi$

$$\left[\frac{\partial^2\varphi_0^{(1)}}{\partial\xi^2} - \beta_1\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi^2} - \frac{\partial^2\varphi_0^{(2)}}{\partial\xi^2} - \beta_2\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi^2}\right]\Bigg|_{y=0} = 0, \tag{82}$$

$$\left[\alpha_1\frac{\partial^2\varphi_0^{(1)}}{\partial\xi^2} - \frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi^2} + \alpha_2\frac{\partial^2\varphi_0^{(2)}}{\partial\xi^2} + \frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi^2}\right]\Bigg|_{y=0} = 0, \tag{83}$$

$$\Bigg[2\alpha_1\mu_1\frac{\partial^2\varphi_0^{(1)}}{\partial\xi^2} + 2\alpha_2\mu_2\frac{\partial^2\varphi_0^{(2)}}{\partial\xi^2} - \mu_1(1+\beta_1^2)\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi^2} + \\ +\mu_2(1+\beta_2^2)\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi^2}\Bigg]\Bigg|_{y=0} = 0, \tag{84}$$

$$\Bigg[-\mu_1(1+\beta_1^2)\frac{\partial^2\varphi_0^{(1)}}{\partial\xi^2} + \mu_2(1+\beta_2^2)\frac{\partial^2\varphi_0^{(2)}}{\partial\xi^2} + 2\beta_1\mu_1\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi^2} + \\ +2\beta_2\mu_2\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi^2}\Bigg]\Bigg|_{y=0} = 0. \tag{85}$$

From (82), (83) and (84), we deduce the following relations for the potentials

$$\frac{\partial\overline{\psi}_0^{(1)}}{\partial\xi}\Bigg|_{y=0} = \frac{\delta_1}{\delta_2}\frac{\partial\varphi_0^{(1)}}{\partial\xi}\Bigg|_{y=0}, \tag{86}$$

$$\frac{\partial\overline{\psi}_0^{(2)}}{\partial\xi}\Bigg|_{y=0} = \frac{\delta_3}{\delta_4}\frac{\partial\varphi_0^{(2)}}{\partial\xi}\Bigg|_{y=0}, \tag{87}$$

$$\frac{\partial\varphi_0^{(2)}}{\partial\xi}\Bigg|_{y=0} = -\frac{\delta_4}{\delta_2}\frac{\partial\varphi_0^{(1)}}{\partial\xi}\Bigg|_{y=0}, \tag{88}$$

where

$$\delta_1 = 2\alpha_1(\mu_1-\mu_2)(1-\alpha_2\beta_2) + \rho_2 c_s^2(\alpha_1+\alpha_2), \tag{89}$$

$$\delta_2 = (2(\mu_1-\mu_2) - \rho_1 c_s^2)(1-\alpha_2\beta_2) + \rho_2 c_s^2(1+\alpha_2\beta_1), \tag{90}$$

$$\delta_3 = 2\alpha_2(\mu_1-\mu_2)(1-\alpha_1\beta_1) - \rho_1 c_s^2(\alpha_1+\alpha_2), \tag{91}$$

$$\delta_4 = -(2(\mu_1-\mu_2) + \rho_2 c_s^2)(1-\alpha_1\beta_1) + \rho_1 c_s^2(1+\alpha_1\beta_2). \tag{92}$$

Equations (82)-(85) constitute a homogeneous system in the wave potential variables and has nontrivial solution only when the determinant of coefficients is zero. The determinant of the system then gives the well-known Stoneley equation (see, Stoneley 1924), i.e,

$$c_s^4\left[(1-\alpha_2\beta_2)\rho_1^2 - (2+\alpha_2\beta_1+\alpha_1\beta_2)\rho_1\rho_2 + (1-\alpha_1\beta_1)\rho_2^2\right] + \\ + 4c_s^2\left[(\mu_1-\mu_2)((1-\alpha_1\beta_1)\rho_2 - (1-\alpha_2\beta_2)\rho_1)\right] + \\ + 4(1-\alpha_1\beta_1)(1-\alpha_2\beta_2)(\mu_1-\mu_2)^2 = 0 \tag{93}$$

When $\rho_2 = 0$, this equation reduces to the ordinary equation for Rayleigh waves

$$(1+\beta_1^2)^2 - 4\alpha_1\beta_1 = 0$$

(see, Kaplunov et al., 2006).

At the next order, the boundary conditions take the form

$$\left[\frac{\partial\varphi_{10}^{(1)}}{\partial\xi} - \frac{\partial\varphi_{10}^{(2)}}{\partial\xi} - \beta_1\frac{\partial\overline{\psi}_{10}^{(1)}}{\partial\xi} - \beta_2\frac{\partial\overline{\psi}_{10}^{(2)}}{\partial\xi} - \frac{1-\beta_1^2}{c_s\beta_1}\frac{\partial\overline{\psi}_0^{(1)}}{\partial\tau} - \frac{1-\beta_2^2}{c_s\beta_2}\frac{\partial\overline{\psi}_0^{(2)}}{\partial\tau}\right]\Bigg|_{y=0} = 0, \tag{94}$$

$$\left[\alpha_1\frac{\partial\varphi_{10}^{(1)}}{\partial\xi} + \alpha_2\frac{\partial\varphi_{10}^{(2)}}{\partial\xi} - \frac{\partial\overline{\psi}_{10}^{(1)}}{\partial\xi} + \frac{\partial\overline{\psi}_{10}^{(2)}}{\partial\xi} + \frac{1-\alpha_1^2}{c_s\alpha_1}\frac{\partial\varphi_0^{(1)}}{\partial\tau} + \frac{1-\alpha_2^2}{c_s\alpha_2}\frac{\partial\varphi_0^{(2)}}{\partial\tau}\right]\Bigg|_{y=0} = 0, \tag{95}$$

$$\left[2\mu_1\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + 2\mu_2\alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - (2\mu_1 - c_s^2\rho_1)\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + (2\mu_2 - c_s^2\rho_2^2)\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} + 2\mu_1\frac{1-\alpha_1^2}{c_s\alpha_1}\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau} - 2\mu_1\frac{1-\beta_1^2}{c_s}\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi\partial\tau} + 2\mu_2\frac{1-\alpha_2^2}{c_s\alpha_2}\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau} + 2\mu_2\frac{1-\beta_2^2}{c_s}\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0, \tag{96}$$

$$\left[-(2\mu_1 - c_s^2\rho_1)\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + (2\mu_2 - c_s^2\rho_2)\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} + 2\mu_1\beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + 2\mu_2\beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} - 2\mu_1\frac{c_s}{c_{21}^2}\frac{1-\alpha_1^2}{c_s}\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau} + 2\mu_1\frac{c_s}{c_{21}^2\beta_1}\frac{\partial^2\psi_0^{(1)}}{\partial\xi\partial\tau} + 2\mu_2\frac{c_s}{c_{22}^2}\frac{1-\alpha_2^2}{c_s}\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau} + 2\mu_2\frac{c_s}{c_{22}^2\beta_2}\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = \frac{p_2\mu_*}{P_*}, \tag{97}$$

which, using the relations (86) and (87) between the potentials, may be rewritten as

$$\left[\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} - \frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} - \beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} - \frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2}\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau} - \frac{c_s\delta_3}{c_{22}^2\beta_2\delta_4}\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0, \tag{98}$$

$$\left[\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + \alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + \frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} + \frac{c_s}{c_{11}^2\alpha_1}\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau} + \right.$$

$$
\left. + \frac{c_s \delta_3}{c_{12}^2 \alpha_2 \delta_4} \frac{\partial^2 \varphi_0^{(2)}}{\partial \xi \partial \tau} \right] \Bigg|_{y=0} = 0, \tag{99}
$$

$$
\left[ 2\mu_1 \alpha_1 \frac{\partial^2 \varphi_{10}^{(1)}}{\partial \xi^2} + 2\mu_2 \alpha_2 \frac{\partial^2 \varphi_{10}^{(2)}}{\partial \xi^2} - (2\mu_1 - c_s^2 \rho_1) \frac{\partial^2 \overline{\psi}_{10}^{(1)}}{\partial \xi^2} + \right.
$$

$$
+ (2\mu_2 + c_s^2 \rho_2) \frac{\partial^2 \overline{\psi}_{10}^{(2)}}{\partial \xi^2} + 2\mu_1 \left( \frac{c_s}{c_{11}^2 \alpha_1} - \frac{c_s \delta_1}{c_{21}^2 \delta_2} \right) \frac{\partial^2 \varphi_0^{(1)}}{\partial \xi \partial \tau} +
$$

$$
\left. + 2\mu_2 \left( \frac{c_s}{c_{12}^2 \alpha_2} + \frac{c_s \delta_3}{c_{22}^2 \delta_4} \right) \frac{\partial^2 \varphi_0^{(2)}}{\partial \xi \partial \tau} \right] \Bigg|_{y=0} = 0, \tag{100}
$$

$$
\left[ -(2\mu_1 - c_s^2 \rho_1) \frac{\partial^2 \varphi_{10}^{(1)}}{\partial \xi^2} + (2\mu_2 + c_s^2 \rho_2) \frac{\partial^2 \varphi_{10}^{(2)}}{\partial \xi^2} + 2\mu_1 \beta_1 \frac{\partial^2 \overline{\psi}_{10}^{(1)}}{\partial \xi^2} + \right.
$$

$$
+ 2\mu_2 \beta_2 \frac{\partial^2 \overline{\psi}_{10}^{(2)}}{\partial \xi^2} - 2\mu_1 \left( \frac{c_s}{c_{21}^2} - \frac{c_s \delta_1}{c_{21}^2 \beta_1 \delta_2} \right) \frac{\partial^2 \varphi_0^{(1)}}{\partial \xi \partial \tau} -
$$

$$
\left. - 2\mu_2 \left( \frac{c_s \delta_4}{c_{22}^2 \delta_2} + \frac{c_s \delta_3}{c_{22}^2 \beta_2 \delta_2} \right) \frac{\partial^2 \varphi_0^{(2)}}{\partial \xi \partial \tau} \right] \Bigg|_{y=0} = \frac{p_2 \mu_*}{P_*}. \tag{101}
$$

Hence, employing the relation between $\phi_{10}^{(1)}$ and $\phi_{10}^{(2)}$ (see, eqn. (88)), (98)-(101) further reduce to

$$
\left[ \frac{\partial^2 \varphi_{10}^{(1)}}{\partial \xi^2} - \frac{\partial^2 \varphi_{10}^{(2)}}{\partial \xi^2} - \beta_1 \frac{\partial^2 \overline{\psi}_{10}^{(1)}}{\partial \xi^2} - \beta_2 \frac{\partial^2 \overline{\psi}_{10}^{(2)}}{\partial \xi^2} - \right.
$$

$$
\left. - \left( \frac{c_s \delta_1}{c_{21}^2 \beta_1 \delta_2} - \frac{c_s \delta_3}{c_{22}^2 \beta_2 \delta_2} \right) \frac{\partial^2 \varphi_0^{(1)}}{\partial \xi \partial \tau} \right] \Bigg|_{y=0} = 0, \tag{102}
$$

$$
\left[ \alpha_1 \frac{\partial^2 \varphi_{10}^{(1)}}{\partial \xi^2} + \alpha_2 \frac{\partial^2 \varphi_{10}^{(2)}}{\partial \xi^2} - \frac{\partial^2 \overline{\psi}_{10}^{(1)}}{\partial \xi^2} + \frac{\partial^2 \overline{\psi}_{10}^{(2)}}{\partial \xi^2} + \right.
$$

$$
\left. + \left( \frac{c_s}{c_{11}^2 \alpha_1} - \frac{c_s \delta_4}{c_{12}^2 \alpha_2 \delta_2} \right) \frac{\partial^2 \varphi_0^{(1)}}{\partial \xi \partial \tau} \right] \Bigg|_{y=0} = 0, \tag{103}
$$

$$
\left[ 2\mu_1 \alpha_1 \frac{\partial^2 \varphi_{10}^{(1)}}{\partial \xi^2} + 2\mu_2 \alpha_2 \frac{\partial^2 \varphi_{10}^{(2)}}{\partial \xi^2} - (2\mu_1 - c_s^2 \rho_1) \frac{\partial^2 \overline{\psi}_{10}^{(1)}}{\partial \xi^2} + \right.
$$

$$
+ (2\mu_2 - c_s^2 \rho_2) \frac{\partial^2 \overline{\psi}_{10}^{(2)}}{\partial \xi^2} +
$$

$$
\left. + \left[ 2\mu_1 \left( \frac{c_s}{c_{11}^2 \alpha_1} - \frac{c_s \delta_1}{c_{21}^2 \delta_2} \right) - 2\mu_2 \left( \frac{c_s \delta_4}{{c_{12}}^2 \alpha_2 \delta_2} + \frac{c_s \delta_3}{c_{22}^2 \delta_2} \right) \right] \frac{\partial^2 {\varphi_0}^{(1)}}{\partial \xi \partial \tau} \right] \Bigg|_{y=0} = 0, \tag{104}
$$

$$
\left[ -(2\mu_1 - c_s^2 \rho_1) \frac{\partial^2 \varphi_{10}^{(1)}}{\partial \xi^2} + (2\mu_2 - c_s^2 \rho_2) \frac{\partial^2 \varphi_{10}^{(2)}}{\partial \xi^2} + 2\mu_1 \beta_1 \frac{\partial^2 \overline{\psi}_{10}^{(1)}}{\partial \xi^2} + \right.
$$

$$+2\mu_2\beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}+$$

$$+\left[-2\mu_1\left(\frac{c_s}{c_{21}^2}-\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2}\right)-2\mu_2\left(\frac{c_s\delta_4}{c_{22}^2\delta_2}+\frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right)\right]\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0}=\frac{p_2\mu_*}{P_*}. \tag{105}$$

Equation (102) may now be written in terms of the derivatives of $\phi_{10}^{(1)}, \phi_{10}^{(2)}, \overline{\psi}_{10}^{(2)}$ and $\phi_0^{(1)}$ upon solving the linear system (102)-(105):

$$\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2}\Bigg|_{y=0}=\Bigg[\frac{1}{\beta_1}\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}-\frac{1}{\beta_1}\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2}-\frac{\beta_2}{\beta_1}\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}-$$
$$-\left(\frac{c_s\delta_1}{c_{21}^2\beta_1^2\delta_2}-\frac{c_s\delta_3}{c_{22}^2\beta_1\beta_2\delta_2}\right)\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0}, \tag{106}$$

The potential $\overline{\psi}_{10}^2$ is

$$\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}\Bigg|_{y=0}=\Bigg[\frac{1-\alpha_1\beta_1}{\beta_1+\beta_2}\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}-\frac{1+\alpha_2\beta_1}{\beta_1+\beta_2}\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2}-$$
$$-\frac{\beta_1}{\beta_1+\beta_2}\left[\frac{c_s\delta_1}{c_{21}^2\beta_1^2\delta_2}-\frac{c_s\delta_3}{c_{22}^2\beta_1\beta_2\delta_2}+\frac{c_s}{c_{11}^2\alpha_1}-\frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right]\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} \tag{107}$$

Inserting relations (106) and (107) into (104) we get, after straightforward but lengthy calculations,

$$\left[\frac{2\mu_1\alpha_1\beta_1-2\mu_1+c_s^2\rho_1}{\beta_1}+\frac{((2\mu_2-c_s^2\rho_2)\beta_1+(2\mu_1-c_s^2\rho_1)\beta_2)(1-\alpha_1\beta_1)}{\beta_1(\beta_1+\beta_2)}\right]\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}+$$
$$+\Bigg[\frac{2\mu_2\alpha_2\beta_1+2\mu_1-{c_s}^2\rho_1}{\beta_1}-$$
$$-\frac{((2\mu_2-c_s^2\rho_2)\beta_1+(2\mu_1-c_s^2\rho_1)\beta_2)(1+\alpha_2\beta_1)}{\beta_1(\beta_1+\beta_2)}\Bigg]\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2}-$$
$$-\left[\frac{(2\mu_2-c_s^2\rho_2)\beta_1+(2\mu_1-c_s^2\rho_1)\beta_2}{(\beta_1+\beta_2)}\right]\times$$
$$\times\left[\frac{c_s\delta_1}{c_{21}^2{\beta_1}^2\delta_2}-\frac{c_s\delta_3}{c_{22}^2\beta_1\beta_2\delta_2}+\frac{c_s}{c_{11}^2\alpha_1}-\frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right]\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}+$$
$$+\Bigg[(2\mu_1-c_s^2\rho_1)\left(\frac{c_s\delta_1}{c_{21}^2{\beta_1}^2\delta_2}-\frac{c_s\delta_3}{c_{22}^2\beta_1\beta_2\delta_2}\right)+2\mu_1\left(\frac{c_s}{c_{11}^2\alpha_1}-\frac{c_s\delta_1}{c_{21}^2\delta_2}\right)-$$
$$-2\mu_2\left(\frac{c_s\delta_4}{{c_{12}}^2\alpha_2\delta_2}+\frac{c_s\delta_3}{c_{22}^2\delta_2}\right)\Bigg]\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}=0. \tag{108}$$

Hence, the quantity $\partial^2\varphi_0^{(2)}/\partial\xi^2$ can now be given at the surface $y = 0$ as

$$\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} =$$
$$-\frac{(2\mu_1\alpha_1\beta_1 - 2\mu_1 + c_s^2\rho_1)(\beta_1+\beta_2) + ((2\mu_2 - c_s^2\rho_2)\beta_1 + (2\mu_1 - c_s^2\rho_1)\beta_2)(1-\alpha_1\beta_1)}{(2\mu_2\alpha_2\beta_1 + 2\mu_1 - c_s^2\rho_1)(\beta_1+\beta_2) - ((2\mu_2 - c_s^2\rho_2)\beta_1 + (2\mu_1 - c_s^2\rho_1)\beta_2)(1+\alpha_2\beta_1)}\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}+$$
$$+\frac{\beta_1((2\mu_2 - c_s^2\rho_2)\beta_1 + (2\mu_1 - c_s^2\rho_1)\beta_2)}{(2\mu_2\alpha_2\beta_1 + 2\mu_1 - c_s^2\rho_1)(\beta_1+\beta_2) - ((2\mu_2 - c_s^2\rho_2)\beta_1 + (2\mu_1 - c_s^2\rho_1)\beta_2)(1+\alpha_2\beta_1)}\times$$
$$\times\left(\frac{c_s\delta_1}{c_{21}^2\beta_1^2\delta_2} - \frac{c_s\delta_3}{c_{22}^2\beta_1\beta_2\delta_2} + \frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right)\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}+$$
$$+\frac{\beta_1+\beta_2}{(2\mu_2\alpha_2\beta_1 + 2\mu_1 - c_s^2\rho_1)(\beta_1+\beta_2) - ((2\mu_2 - c_s^2\rho_2)\beta_1 + (2\mu_1 - c_s^2\rho_1)\beta_2)(1+\alpha_2\beta_1)}\times$$
$$\times\left[(2\mu_1 - c_s^2\rho_1)\left(\frac{c_s\delta_1}{c_{21}^2\beta_1^2\delta_2} - \frac{c_s\delta_3}{c_{22}^2\beta_1\beta_2\delta_2}\right) + 2\mu_1\left(\frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_1}{c_{21}^2\delta_2}\right) -\right.$$
$$\left.-2\mu_2\left(\frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2} + \frac{c_s\delta_3}{c_{22}^2\delta_2}\right)\right]\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}. \qquad (109)$$

Repeating the same procedure for equation (105), we find

$$\left[\frac{c_s^2\rho_1(\beta_1+\beta_2) + 2\beta_2(\mu_2-\mu_1)(1-\alpha_1\beta_1)}{\beta_1+\beta_2}\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}+\right.$$
$$+\frac{(2\mu_2 - 2\mu_1 - c_s^2\rho_2)(\beta_1+\beta_2) - 2\beta_2(\mu_2-\mu_1)(1+\alpha_2\beta_1)}{\beta_1+\beta_2}\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2}-$$
$$-\left[\frac{2\beta_2(\mu_2-\mu_1)}{\beta_1+\beta_2}\left(\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2} - \frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2} + \frac{c_s\beta_1}{c_{11}^2\alpha_1} - \frac{c_s\beta_1\delta_4}{c_{12}^2\alpha_2\delta_2}\right)+\right.$$
$$+2\mu_1\left(\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2} - \frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right) + 2\mu_1\left(\frac{c_s}{c_{21}^2} - \frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2}\right)+$$
$$\left.\left.+2\mu_2\left(\frac{c_s}{c_{22}^2} + \frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right)\right]\frac{\partial^2{\varphi_0}^{(1)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = \frac{p_2\mu_*}{P_*}. \quad (110)$$

If we now substitute $\partial^2\varphi_{10}^{(2)}/\partial\xi^2$ from equation (109) into equation (110) and rearrange the terms, we get

$$\frac{c_s^2\rho_1(\beta_1+\beta_2) + (2\mu_2\beta_2 - 2\mu_1\beta_2)(1-\alpha_1\beta_1)}{\beta_1+\beta_2}\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}-$$
$$-\left[\frac{(2\mu_2 - 2\mu_1 - c_s^2\rho_2)(\beta_1+\beta_2) - (2\mu_2\beta_2 - 2\mu_1\beta_2)(1+\alpha_2\beta_1)}{\beta_1+\beta_2}\right]\times$$
$$\times\left[\frac{(2\mu_1\alpha_1\beta_1 - 2\mu_1 + c_s^2\rho_1)(\beta_1+\beta_2) + ((2\mu_2 - c_s^2\rho_2)\beta_1 + (2\mu_1 - c_s^2\rho_1)\beta_2)(1-\alpha_1\beta_1)}{(2\mu_2\alpha_2\beta_1 + 2\mu_1 - c_s^2\rho_1)(\beta_1+\beta_2) - ((2\mu_2 - c_s^2\rho_2)\beta_1 + (2\mu_1 - c_s^2\rho_1)\beta_2)(1+\alpha_2\beta_1)}\right]\times$$
$$\times\frac{\partial^2{\varphi_{10}}^{(1)}}{\partial\xi^2}+$$
$$+\frac{(2\mu_2 - 2\mu_1 - c_s^2\rho_2)(\beta_1+\beta_2) - (2\mu_2\beta_2 - 2\mu_1\beta_2)(1+\alpha_2\beta_1))}{(2\mu_2\alpha_2\beta_1 + 2\mu_1 - c_s^2\rho_1)(\beta_1+\beta_2) - ((2\mu_2 - c_s^2\rho_2)\beta_1 + (2\mu_1 - c_s^2\rho_1)\beta_2)(1+\alpha_2\beta_1)}\times$$

$$\times\left[\frac{\beta_1}{\beta_1+\beta_2}\right]\times$$

$$\times\Bigg\{((2\mu_2-c_s^2\rho_2)\beta_1+(2\mu_1-c_s^2\rho_1)\beta_2)\left[\frac{c_s\delta_1}{c_{21}^2{\beta_1}^2\delta_2}-\frac{c_s\delta_3}{c_{22}^2\beta_1\beta_2\delta_2}+\frac{c_s}{c_{11}^2\alpha_1}-\frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right]+$$
$$-(2\mu_1-c_s^2\rho_1)(\beta_1+\beta_2)\left(\frac{c_s\delta_1}{c_{21}^2{\beta_1}^2\delta_2}-\frac{c_s\delta_3}{c_{22}^2\beta_1\beta_2\delta_2}\right)-2\mu_1(\beta_1+\beta_2)\left(\frac{c_s}{c_{11}^2\alpha_1}-\frac{c_s\delta_1}{c_{21}^2\delta_2}\right)-$$
$$+2\mu_2(\beta_1+\beta_2)\left(\frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}+\frac{c_s\delta_3}{c_{22}^2\delta_2}\right)\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}\Bigg\}-$$
$$-\left[\frac{2\beta_2(\mu_2-\mu_1)}{\beta_1+\beta_2}\left(\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2}-\frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2}+\frac{c_s\beta_1}{c_{11}^2\alpha_1}-\frac{c_s\beta_1\delta_4}{c_{12}^2\alpha_2\delta_2}\right)+\right.$$
$$+2\mu_1\left(\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2}-\frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right)+2\mu_1\left(\frac{c_s}{c_{21}^2}-\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2}\right)+$$
$$\left.+2\mu_2\left(\frac{c_s\delta_4}{c_{22}^2\delta_2}+\frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right)\right]\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}\bigg|_{y=0}=\frac{p_2\mu_*}{P_*}\tag{111}$$

It is easy to see that the coefficient of $\partial^2\varphi_{10}^{(1)}/\partial\xi^2$ is the Stoneley equation (93) with a nonzero denominator, and, therefore, is zero. The remaning terms can be rewritten to give

$$\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}\bigg|_{y=0}=\frac{\delta_2}{2c_s\mathcal{B}}\frac{p_2\mu_*}{P_*}\tag{112}$$

where $\delta_2$ is given by (90) and the material coefficient

$$\begin{aligned}\mathcal{B}=2&\left[\frac{\beta_2^2c_{22}^2+{\alpha_2}^2c_{12}^2}{c_{22}^2c_{12}^2\alpha_2\beta_2}(1-\alpha_1\beta_1)+\frac{\beta_1^2c_{21}^2+\alpha_1^2c_{11}^2}{c_{11}^2c_{21}^2\alpha_1\beta_1}(1-\alpha_2\beta_2)\right](\mu_1-\mu_2)^2+\\&+\frac{1}{2}c_s^4\left[\frac{\beta_2^2c_{22}^2+\alpha_2^2c_{12}^2}{c_{22}^2c_{12}^2\alpha_2\beta_2}\rho_1^2+\left(\frac{\beta_1^2c_{21}^2+\alpha_2^2c_{12}^2}{c_{21}^2c_{12}^2\alpha_2\beta_1}+\frac{\beta_2^2c_{22}^2+\alpha_1^2c_{11}^2}{c_{11}^2c_{22}^2\alpha_1\beta_2}\right)\rho_1\rho_2+\right.\\&\left.+\frac{\beta_1^2c_{21}^2+\alpha_1^2c_{11}^2}{c_{11}^2c_{21}^2\alpha_1\beta_1}\rho_2^2\right]+\\&+2c_s^2((1-\alpha_2\beta_2)\rho_1^2-(2+\alpha_2\beta_1+\alpha_1\beta_2)\rho_1\rho_2+(1-\alpha_1\beta_1)\rho_2^2)+\\&+2c_s^2(\mu_1-\mu_2)\left(-\frac{\beta_2^2c_{22}^2+\alpha_2^2c_{12}^2}{c_{22}^2c_{12}^2\alpha_2\beta_2}\rho_1+\frac{\beta_1^2c_{21}^2+\alpha_1^2c_{11}^2}{c_{21}^2c_{11}^2\alpha_1\beta_1}\rho_2\right)+\\&+4(\mu_1-\mu_2)((1-\alpha_1\beta_1)\rho_2-(1-\alpha_2\beta_2)\rho_1).\end{aligned}\tag{113}$$

From (112) we finally have

$$\frac{2\varepsilon}{c_s}\frac{\partial^2\varphi_a^{(1)}}{\partial\xi\partial\tau}\bigg|_{y=0}=\frac{\delta_2}{c_s^2\mathcal{B}}p_2\tag{114}$$

where $\varphi_a^{(1)}$ is an approximate solution

$$\varphi_a^{(i)}=\frac{P_*}{\varepsilon\mu}\varphi_0^{(i)},\qquad\psi_a^{(i)}=\frac{P_*}{\varepsilon\mu}\psi_0^{(i)},\tag{115}$$

Changing back to the original variables, we can easily write down the leading order operator identity as

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} = \frac{2\varepsilon}{c_s}\frac{\partial^2}{\partial \xi \partial \tau}, \tag{116}$$

Then, transforming equation (114) to original variables, we obtain

$$\frac{\partial^2 \varphi_s^{(1)}}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2 \varphi_s^{(1)}}{\partial t^2} = \frac{\delta_2}{c_s^2 \mathcal{B}} p_2. \tag{117}$$

where $\varphi_s^{(i)} = \varphi_a^{(i)}\big|_{y=0}$. This hyperbolic equation corresponds to propagation of Stoneley wave along the interface. The boundary condition for the second pair of potentials $\psi^{(i)}$ can be found from equation (114)

$$\begin{aligned} \left.\frac{\partial \psi_a^{(1)}}{\partial x}\right|_{y=0} &= \frac{\delta_1}{\delta_2 \alpha_1}\left.\frac{\partial \varphi_a^{(1)}}{\partial y}\right|_{y=0}, \\ \left.\frac{\partial \varphi_a^{(2)}}{\partial x}\right|_{y=0} &= -\frac{\delta_4}{\delta_2}\left.\frac{\partial \varphi_a^{(1)}}{\partial x}\right|_{y=0}, \\ \left.\frac{\partial \psi_a^{(2)}}{\partial x}\right|_{y=0} &= -\frac{\delta_3}{\delta_4 \alpha 2}\left.\frac{\partial \varphi_a^{(1)}}{\partial y}\right|_{y=0}. \end{aligned} \tag{118}$$

These equations show that the potentials are related to each other. This equation also shows that the wave potentials are related to each other by means of a Hilbert transforms as has been shown in Chadwick (1976). To obtain the interior field we use equations (61) and (62)

$$\begin{aligned} \frac{\partial^2 \varphi_a^{(1)}}{\partial y^2} + \alpha_1^2 \frac{\partial^2 \varphi_a^{(1)}}{\partial x^2} &= 0 \\ \frac{\partial^2 \varphi_a^{(2)}}{\partial y^2} + \alpha_2^2 \frac{\partial^2 \varphi_a^{(2)}}{\partial x^2} &= 0 \\ \frac{\partial^2 \psi_a^{(1)}}{\partial y^2} + \beta_1^2 \frac{\partial^2 \psi_a^{(1)}}{\partial x^2} &= 0 \\ \frac{\partial^2 \psi_a^{(2)}}{\partial y^2} + \beta_2^2 \frac{\partial^2 \psi_a^{(2)}}{\partial x^2} &= 0 \end{aligned} \tag{119}$$

The obtained equations will simplify the further investigations in that as the governing equation is solved in terms of one of the potentials the remaining potentials may easily be found using the obtained relations (118). Using equations (86)-(87) displacement components can be given explicitly once the potentials are found. Therefore in order to obtain the full solution of the considered problem it suffices to solve a single equation for one of the potentials along the interface of the two media.

Thus, the asymptotic model for normal loading is obtained with hyperbolic equation on the interface and elliptic equations for the interior. If we solve a Drichlet problem, we may find the interior field.

### 3.2.2 Tangential Loading

In this section we will consider the case of tangential loading only, i.e. we will take the normal stress (cf. eqns. (55)-(58)) to be zero with continuous displacements which correspond to $q_1 = q_2 = 0$, that

$$\left[\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\xi} + \frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial y}\right]\Bigg|_{y=0} = 0, \tag{120}$$

$$\left[\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial y} - \frac{\partial\psi_1}{\partial\xi} + \frac{\partial\psi_2}{\partial\xi}\right]\Bigg|_{y=0} = 0, \tag{121}$$

$$\begin{aligned}\Bigg[2\mu_1\frac{\partial^2\varphi_1}{\partial\xi\partial y} - 2\mu_2\frac{\partial^2\varphi_2}{\partial\xi\partial y} + \mu_1\left(\frac{\partial^2\psi_1}{\partial y^2} - \frac{\partial^2\psi_1}{\partial\xi^2}\right) - \\ -\mu_2\left(\frac{\partial^2\psi_2}{\partial y^2} - \frac{\partial^2\psi_2}{\partial\xi^2}\right)\Bigg]\Bigg|_{y=0} = p_1(\xi,t),\end{aligned} \tag{122}$$

$$\begin{aligned}\Bigg[\lambda_1\frac{\partial^2\varphi_1}{\partial\xi^2} + (\lambda_1+2\mu_1)\frac{\partial^2\varphi_1}{\partial y^2} - \lambda_2\frac{\partial^2\varphi_2}{\partial\xi^2} - (\lambda_2+2\mu_2)\frac{\partial^2\varphi_2}{\partial y^2} - \\ -2\mu_1\frac{\partial^2\psi_1}{\partial\xi\partial y} + 2\mu_2\frac{\partial^2\psi_2}{\partial\xi\partial y}\Bigg]\Bigg|_{y=0} = 0.\end{aligned} \tag{123}$$

In what follows, the calculations are very similar to the case of normal loading and therefore only the necessary will be presented here. Employing the asymptotic approach, the boundary conditions at the leading order along the interface $y = 0$ reduce to the same equations as for the normal loading case (see, eqns. (82)-(85)). The relations between the potentials can therefore be rewritten as

$$\frac{\partial\varphi_0^{(1)}}{\partial\xi}\Bigg|_{y=0} = \frac{\delta_2}{\delta_1}\frac{\partial\overline{\psi}_0^{(1)}}{\partial\xi}\Bigg|_{y=0}, \tag{124}$$

$$\frac{\partial\varphi_0^{(2)}}{\partial\xi}\Bigg|_{y=0} = \frac{\delta_4}{\delta_3}\frac{\partial\overline{\psi}_0^{(2)}}{\partial\xi}\Bigg|_{y=0}, \tag{125}$$

$$\frac{\partial\psi_0^{(2)}}{\partial\xi}\Bigg|_{y=0} = -\frac{\delta_3}{\delta_1}\frac{\partial\psi_0^{(1)}}{\partial\xi}\Bigg|_{y=0}. \tag{126}$$

where on the left hand sides the potentials $\psi^{(i)}, i = 1,2$ are preferred and $\delta_i, i = 1,...,4$ are given (see, eqns. (89)-(92)) which are equivalent to (86)-(88).

At next order, boundary equations differ from the normal loading case in the normal and tangential components. We arrive at the following equations:

$$\left[\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} - \frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} - \beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} - \frac{1-\beta_1^2}{c_s\beta_1}\frac{\partial^2\overline{\psi}_{0}^{(1)}}{\partial\xi\partial\tau} - \right.$$
$$\left. - \frac{1-\beta_2^2}{c_s\beta_2}\frac{\partial^2\overline{\psi}_{0}^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0, \tag{127}$$

$$\left[\alpha_1\frac{\partial^2{\varphi_{10}}^{(1)}}{\partial\xi^2} + \alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + \frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} + \frac{1-\alpha_1^2}{c_s\alpha_1}\frac{\partial^2\varphi_{0}^{(1)}}{\partial\xi\partial\tau} + \right.$$
$$\left. + \frac{1-\alpha_2^2}{c_s\alpha_2}\frac{\partial^2\varphi_{0}^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0, \tag{128}$$

$$\left[2\mu_1\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + 2\mu_2\alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \mu_1(1+\beta_1^2)\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + \mu_2(1+\beta_2^2)\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} + \right.$$
$$+2\mu_1\frac{1-\alpha_1^2}{c_s\alpha_1}\frac{\partial^2\varphi_{0}^{(1)}}{\partial\xi\partial\tau} - 2\mu_1\frac{1-\beta_1^2}{c_s}\frac{\partial^2\overline{\psi}_{0}^{(1)}}{\partial\xi\partial\tau} +$$
$$\left. +2\mu_2\frac{1-\alpha_2^2}{c_s\alpha_2}\frac{\partial^2\varphi_{0}^{(2)}}{\partial\xi\partial\tau} + 2\mu_2\frac{1-\beta_2^2}{c_s}\frac{\partial^2\overline{\psi}_{0}^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = \frac{p_1\mu_*}{P_*}, \tag{129}$$

$$\left[-\mu_1(1+\beta_1^2)\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + \mu_2(1+\beta_2^2)\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} + 2\mu_1\beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + 2\mu_2\beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}\right.$$
$$-2(\lambda_1+2\mu_1)\frac{1-\alpha_1^2}{c_s}\frac{\partial^2\varphi_{0}^{(1)}}{\partial\xi\partial\tau} + 2\mu_1\frac{1-\beta_1^2}{c_s\beta_1}\frac{\partial^2\varphi_{0}^{(1)}}{\partial\xi\partial\tau} +$$
$$\left. -2\mu_1\frac{c_s}{c_{21}^2}\frac{\partial^2\varphi_{0}^{(2)}}{\partial\xi\partial\tau} + 2\mu_2\frac{c_s}{c_{22}^2}\frac{\partial^2\overline{\psi}_{0}^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0. \tag{130}$$

On using the relations (89)-(92) between the potentials $\varphi_0^{(1)}$, $\varphi_0^{(2)}$ and $\overline{\psi}_0^{(2)}$ the latter equations may be reduced to

$$\left[\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} - \frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} - \beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}\right.$$
$$\left. + \left(-\frac{c_s}{c_{21}^2\beta_1} + \frac{c_s\delta_3}{c_{22}^2\beta_2\delta_1}\right)\frac{\partial^2\overline{\psi}_{0}^{(1)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0, \tag{131}$$

$$\left[\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + \alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + \frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}\right.$$
$$\left. + \left(\frac{c_s\delta_2}{c_{11}^2\alpha_1\delta_1} - \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_1}\right)\frac{\partial^2\overline{\psi}_{0}^{(1)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0, \tag{132}$$

$$\left[2\mu_1\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + 2\mu_2\alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - (2\mu_1 - c_s^2\rho_1)\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + \right.$$

$$+(2\mu_2 - c_s^2\rho_2)\frac{\partial^2 \overline{\psi}_{10}^{(2)}}{\partial \xi^2} -$$

$$-\left[2\mu_1\left(\frac{c_s}{c_{21}^2} - \frac{c_s\delta_2}{c_{11}^2\alpha_1\delta_1}\right) + 2\mu_2\left(\frac{c_s\delta_3}{c_{22}^2\delta_1} + \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_1}\right)\right]\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} = \frac{p_1\mu_*}{P_*}, \tag{133}$$

$$\Bigg[(c_s^2\rho_1 - 2\mu_1)\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + (2\mu_2 - c_s^2\rho_2)\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} + 2\mu_1\beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + 2\mu_2\beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} +$$

$$+\left[2\mu_1\left(\frac{c_s}{c_{21}^2\beta_1} - \frac{c_s\delta_2}{c_{21}^2\delta_1}\right) - 2\mu_2\left(\frac{c_s\delta_3}{c_{22}^2\beta_2\delta_1} + \frac{c_s\delta_4}{c_{22}^2\delta_1}\right)\right]\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} = 0. \tag{134}$$

Following the same steps for the case of normal loading, equations (131)-(134) may be solved for the variables $\phi_{10}^{(1)}$ and $\overline{\psi}_0^{(1)}$. The coefficient of $\phi_{10}^{(1)}$ turns out to give the Stoneley equation and therefore is zero. The solution in the transformed variables is thus obtained as

$$\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi\partial\tau}\Bigg|_{y=0} = \frac{\delta_2}{2c_s\mathcal{B}}\frac{p_1\mu_*}{P_*}. \tag{135}$$

Equation (135) is transformed to the original variables, and we obtain after some manipulation

$$\frac{\partial^2\psi_s^{(1)}}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2\psi_s^{(1)}}{\partial t^2} = \frac{\delta_2}{c_s^2\mathcal{B}}p_1. \tag{136}$$

which leads to the following hyperbolic equation for the potential. In the tangential case, similarly we get elliptic equations (119).

### 3.2.3 Vertical Jumping

In this section we assume that displacement boundary conditions ($p_1$ and $p_2$ are both zero, as well as $q_2 = 0$). In what follows, the asymptotic model will be obtained under these boundary conditions, we have

$$\left[\frac{\partial\varphi_1}{\partial\xi} + \frac{\partial\psi_1}{\partial y} - \frac{\partial\varphi_2}{\partial\xi} - \frac{\partial\psi_2}{\partial y}\right]\Bigg|_{y=0} = q_1(\xi, t), \tag{137}$$

$$\left[\frac{\partial\varphi_1}{\partial y} - \frac{\partial\psi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial y} + \frac{\partial\psi_2}{\partial\xi}\right]\Bigg|_{y=0} = 0, \tag{138}$$

$$\Bigg[2\mu_1\frac{\partial^2\varphi_1}{\partial\xi\partial y} - 2\mu_2\frac{\partial^2\varphi_2}{\partial\xi\partial y} - \mu_1\left(\frac{\partial^2\psi_1}{\partial y^2} - \frac{\partial^2\psi_1}{\partial\xi^2}\right) -$$

$$+\mu_2\left(\frac{\partial^2\psi_2}{\partial y^2} - \frac{\partial^2\psi_2}{\partial\xi^2}\right)\Bigg]\Bigg|_{y=0} = 0, \tag{139}$$

$$\Bigg[\lambda_1\frac{\partial^2\varphi_1}{\partial\xi^2} + (\lambda_1 + 2\mu_1)\frac{\partial^2\varphi_1}{\partial y^2} - \lambda_2\frac{\partial^2\varphi_2}{\partial\xi^2} - (\lambda_2 + 2\mu_2)\frac{\partial^2\varphi_2}{\partial y^2} -$$

$$-2\mu_1\frac{\partial^2\psi_1}{\partial\xi\partial y}+2\mu_2\frac{\partial^2\psi_2}{\partial\xi\partial y}\Bigg]\Bigg|_{y=0}=0. \qquad (140)$$

Following equation (141)-(144) the boundary conditions at leading order give

$$\left[\frac{\partial\varphi_0^{(1)}}{\partial\xi}-\beta_1\frac{\partial\overline{\psi}_0^{(1)}}{\partial\xi}-\frac{\partial\varphi_0^{(2)}}{\partial\xi}-\beta_2\frac{\partial\overline{\psi}_0^{(2)}}{\partial\xi}\right]\Bigg|_{y=0}=0, \qquad (141)$$

$$\left[\alpha_1\frac{\partial\varphi_0^{(1)}}{\partial\xi}-\frac{\partial\overline{\psi}_0^{(1)}}{\partial\xi}+\alpha_2\frac{\partial\varphi_0^{(2)}}{\partial\xi}+\frac{\partial\overline{\psi}_0^{(2)}}{\partial\xi}\right]\Bigg|_{y=0}=0, \qquad (142)$$

$$\Bigg[2\alpha_1\mu_1\frac{\partial^2\varphi_0^{(1)}}{\partial\xi^2}+2\alpha_2\mu_2\frac{\partial^2\varphi_0^{(2)}}{\partial\xi^2}-\mu_1(1+\beta_1^2)\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi^2}+$$

$$+\mu_2(1+\beta_2^2)\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi^2}\Bigg]\Bigg|_{y=0}=0, \qquad (143)$$

$$\Bigg[-\mu_1(1+\beta_1^2)\frac{\partial^2\varphi_0^{(1)}}{\partial\xi^2}+\mu_2(1+\beta_2^2)\frac{\partial^2\varphi_0^{(2)}}{\partial\xi^2}+2\beta_1\mu_1\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi^2}+$$

$$+2\beta_2\mu_2\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi^2}\Bigg]\Bigg|_{y=0}=0 \qquad (144)$$

where we made use of the Cauchy-Riemann equations. Hence, we arrive at the same relations (86)-(88).

At next order of the boundary equations are

$$\Bigg[\frac{\partial\varphi_{10}^{(1)}}{\partial\xi}-\frac{\partial\varphi_{10}^{(2)}}{\partial\xi}-\beta_1\frac{\partial\overline{\psi}_{10}^{(1)}}{\partial\xi}-\beta_2\frac{\partial\overline{\psi}_{10}^{(2)}}{\partial\xi}-\frac{1-\beta_1^2}{c_s\beta_1}\frac{\partial\overline{\psi}_0^{(1)}}{\partial\tau}-$$

$$-\frac{1-\beta_2^2}{c_s\beta_2}\frac{\partial\overline{\psi}_0^{(2)}}{\partial\tau}\Bigg]\Bigg|_{y=0}=\frac{q_1\mu_*}{P_*}, \qquad (145)$$

$$\Bigg[\alpha_1\frac{\partial\varphi_{10}^{(1)}}{\partial\xi}+\alpha_2\frac{\partial\varphi_{10}^{(2)}}{\partial\xi}-\frac{\partial\overline{\psi}_{10}^{(1)}}{\partial\xi}+\frac{\partial\overline{\psi}_{10}^{(2)}}{\partial\xi}+\frac{1-\alpha_1^2}{c_s\alpha_1}\frac{\partial\varphi_0^{(1)}}{\partial\tau}+$$

$$+\frac{1-\alpha_2^2}{c_s\alpha_2}\frac{\partial\varphi_0^{(2)}}{\partial\tau}\Bigg]\Bigg|_{y=0}=0, \qquad (146)$$

$$\Bigg[2\mu_1\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}+2\mu_2\alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2}-\mu_1(1+\beta_1^2)\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2}+$$

$$+\mu_2(1+\beta_2^2)\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}+2\mu_1\frac{c_s}{c_{11}^2\alpha_1}\frac{\partial\varphi_0^{(1)}}{\partial\tau}-$$

$$-2\mu_1\frac{c_s}{c_{21}^2}\frac{\partial\overline{\psi}_0^{(1)}}{\partial\xi\partial\tau}+2\mu_2\frac{c_s}{c_{12}^2\alpha_2}\frac{\partial\varphi_0^{(2)}}{\partial\xi\partial\tau}+2\mu_2\frac{c_s}{c_{22}^2}\frac{\partial\overline{\psi}_0^{(2)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0}=0, \qquad (147)$$

$$\Bigg[(-2\mu_1+c_s^2\rho_1)\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}+(2\mu_2-c_s^2\rho_2)\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2}+2\mu_1\beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2}+$$

$$+2\mu_2\beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} - 2\mu_1\frac{c_s}{c_{21}^2}\frac{\partial^2\varphi_0^{(1)}}{\partial^2\tau} -$$

$$+2\mu_1\frac{c_s}{c_{21}^2\beta_1}\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi\partial\tau} + 2\mu_2\frac{c_s}{c_{22}^2}\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau} + 2\mu_2\frac{c_s}{c_{22}^2\beta_2}\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} = 0. \tag{148}$$

In order to simplify these equations we make use of (86)-(88) and the dipslacement boundary conditions are derivatived with respect to the $\xi$ . On doing so we obtain as

$$\Bigg[\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} - \frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} - \beta_2\frac{\partial\overline{\psi}_{10}^{(2)}}{\partial^2\xi^2} +$$

$$+\left(-\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2} + \frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right)\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} = \frac{\mu_*}{P_*}\frac{\partial q_1}{\partial\xi}, \tag{149}$$

$$\Bigg[\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + \alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + \frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} +$$

$$+\left(\frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right)\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} = 0, \tag{150}$$

$$\Bigg[2\mu_1\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + 2\mu_2\alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - \mu_1(1+\beta_1^2)\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} + \mu_2(1+\beta_2^2)\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} +$$

$$+\left(\frac{2\mu_1 c_s}{c_{11}^2\alpha_1} - \frac{2\mu_1 c_s\delta_1}{c_{21}^2\delta_2} - \frac{2\mu_2 c_s\delta_4}{c_{12}^2\alpha_2\delta_2} - \frac{2\mu_2 c_s\delta_3}{c_{22}^2\delta_2}\right)\frac{\partial\varphi_0^{(2)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} = 0, \tag{151}$$

$$\Bigg[(-2\mu_1 + c_s^2\rho_1)\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + (2\mu_2 - c_s^2\rho_2)\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} + 2\mu_1\beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2} +$$

$$+2\mu_2\beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} +$$

$$+\left(-\frac{2\mu_1 c_s}{c_{21}^2} + \frac{2\mu_1 c_s\delta_1}{c_{21}^2\beta_1\delta_2} - \frac{2\mu_2 c_s\delta_4}{c_{22}^2\delta_2} - \frac{2\mu_2 c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right)\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} = 0. \tag{152}$$

Inserting $\partial^2\overline{\psi}_{10}^{(1)}/\partial\xi^2$ from equation (150) into the equations (149), (151) and (152) gives

$$\Bigg[(1-\alpha_1\beta_1)\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} - (1+\alpha_2\beta_1)\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} - (\beta_1+\beta_2)\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} -$$

$$-\left[\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2} - \frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2} + \beta_1\left(\frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right)\right]\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau}\Bigg]\Bigg|_{y=0} = \frac{\mu_*}{P_*}\frac{\partial q_1}{\partial\xi}, \tag{153}$$

$$\Bigg[c_s^2\rho_1\alpha_1\frac{\partial\varphi_{10}^{(1)}}{\partial\xi} + (2\mu_2\alpha_2 - 2\mu_1\alpha_2 + c_s^2\rho_1\alpha_2)\frac{\partial\varphi_{10}^{(2)}}{\partial\xi} -$$

$$-(2\mu_1 - c_s^2\rho_1 - 2\mu_2 + c_s^2\rho_2)\frac{\partial\overline{\psi}_{10}^{(2)}}{\partial\xi} + \left[\frac{2\mu_1 c_s}{c_{11}^2\alpha_1} - \frac{2\mu_1 c_s\delta_1}{c_{21}^2\delta_2} - \frac{2\mu_2 c_s\delta_4}{c_{12}^2\alpha_2\delta_2} + \right.$$
$$\left.\left. +(-2\mu_1 + c_s^2\rho_1)\left(\frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right) - \frac{2\mu_2 c_s\delta_3}{c_{22}^2\delta_2}\right]\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0, \qquad (154)$$

$$\left[(-2\mu_1 + c_s^2\rho_1 + 2\mu_1\beta_1\alpha_1)\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} + (2\mu_1\beta_1\alpha_2 + 2\mu_2 - c_s^2\rho_2)\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} + \right.$$
$$+(2\mu_2\beta_2 + 2\mu_1\beta_1)\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2} + \left[2\mu_1\beta_1\left(\frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_4}{\alpha_2 c_{12}^2\delta_2}\right) - \right.$$
$$\left.\left. - \frac{2\mu_1 c_s}{c_{21}^2} + \frac{2\mu_1 c_s\delta_1}{c_{21}^2\beta_1\delta_2} - \frac{2\mu_2 c_s\delta_4}{c_{22}^2\delta_2} - \frac{2\mu_2 c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right]\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0} = 0. \qquad (155)$$

Using $\partial^2\overline{\psi}_{10}^{(2)}/\partial\xi^2$ from (154) and inserting into eqns. (153) and (155), we obtain

$$\frac{(1-\alpha_1\beta_1)(2(\mu_1-\mu_2)+c_s^2(\rho_2-\rho_1)) - (\beta_1+\beta_2)c_s^2\rho_1\alpha_1}{2(\mu_1-\mu_2)+c_s^2(\rho_2-\rho_1)}\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} -$$
$$\left[(1+\alpha_2\beta_2) + \frac{(\beta_1+\beta_2)(2(\mu_2-\mu_1)\alpha_2 + c_s^2\rho_1\alpha_2)}{2(\mu_1-\mu_2)+c_s^2(\rho_2-\rho_1)}\right]\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} -$$
$$-\left\{\frac{\beta_1+\beta_2}{2(\mu_1-\mu_2)+c_s^2(\rho_2-\rho_1)}\left[\frac{2\mu_1 c_s}{c_{11}^2\alpha_1} - \frac{2\mu_1 c_s\delta_1}{c_{21}^2\delta_2} - \frac{2\mu_2 c_s\delta_4}{c_{12}^2\alpha_2\delta_2} + \right.\right.$$
$$\left. +(-2\mu_1 + c_s^2\rho_1)\left(\frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right) - \frac{2\mu_2 c_s\delta_3}{c_{22}^2\delta_2}\right] +$$
$$\left.\left[\frac{c_s\delta_1}{c_{21}^2\beta_1\delta_2} - \frac{c_s\delta_3}{c_{22}^2\beta_2\delta_2} + \beta_1\left(\frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right)\right]\right\}\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}\Bigg|_{y=0} = \frac{\mu_*}{P_*}\frac{\partial q_1}{\partial\xi}, \qquad (156)$$

$$\left[(-2\mu_1 + c_s^2\rho_1 + 2\mu_1\beta_1\alpha_1) + \frac{2(\mu_2\beta_2 + \mu_1\beta_1)c_s^2\rho_1\alpha_1}{2(\mu_1-\mu_2)+c_s^2(\rho_2-\rho_1)}\right]\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2} +$$
$$+\left[(2\mu_1\beta_1\alpha_2 + 2\mu_2 - c_s^2\rho_2) + \frac{2(\mu_2\beta_2 + \mu_1\beta_1)(2(\mu_2-\mu_1) + c_s^2\rho_1\alpha_2)}{2(\mu_1-\mu_2)+c_s^2(\rho_2-\rho_1)}\right]\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2} +$$
$$\left\{\frac{2(\mu_2\beta_2 + \mu_1\beta_1)}{2(\mu_1-\mu_2)+c_s^2(\rho_2-\rho_1)}\left[\frac{2\mu_1 c_s}{c_{11}^2\alpha_1} - \frac{2\mu_1 c_s\delta_1}{c_{21}^2\delta_2} - \frac{2\mu_2 c_s\delta_4}{c_{12}^2\alpha_2\delta_2} + \right.\right.$$
$$\left. -(2\mu_1 - c_s^2\rho_1)\left(\frac{c_s}{c_{11}^2\alpha_1} - \frac{c_s\delta_4}{c_{12}^2\alpha_2\delta_2}\right) - \frac{2\mu_2 c_s\delta_3}{c_{22}^2\delta_2}\right] + \left[\frac{2\mu_1\beta_1 c_s}{c_{11}^2\alpha_1} - \right.$$
$$\left.\left. -\frac{c_s\delta_4}{2\mu_1\beta_1\alpha_2 c_{12}^2\delta_2} - \frac{2\mu_1 c_s}{c_{21}^2} + \frac{2\mu_1 c_s\delta_1}{c_{21}^2\beta_1\delta_2} - \frac{2\mu_2 c_s\delta_4}{c_{22}^2\delta_2} - \frac{2\mu_2 c_s\delta_3}{c_{22}^2\beta_2\delta_2}\right]\right\}\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}\Bigg|_{y=0} = 0. \qquad (157)$$

Eliminating $\partial^2\phi_{10}^{(2)}/\partial\xi^2$ from the last two equations gives a differential equation containing the potentials $\partial^2\phi_{10}^{(1)}/\partial\xi^2$ and $\partial^2\phi_0^{(1)}/\partial\xi\partial\tau$. As before, it is not difficult to verify that the coefficient of $\partial^2\phi_{10}^{(1)}/\partial\xi^2$ is a multiple is the Stoneley equation and therefore vanishes. The remaining term is therefore

$$\left.\frac{\partial^2\phi_0^{(1)}}{\partial\xi\partial\tau}\right|_{y=0} = \frac{\delta_2 D_1}{2c_s\mathcal{B}D_2}\frac{\mu_*}{P_*}\frac{\partial q_1}{\partial\xi} \tag{158}$$

where

$$\begin{aligned} D_1 = {}&(2\mu_1\beta_1\alpha_2 + 2\mu_2 - c_s^2\rho_2)(2(\mu_1-\mu_2) + c_s^2(\rho_2-\rho_1))+ \\ &+ (2\mu_2\beta_2 + 2\mu_1\beta_1)(2(\mu_2-\mu_1)\alpha_2 + c_s^2\rho_1\alpha_2), \qquad (159)\\ D_2 = {}&(1+\alpha_2\beta_2)(2(\mu_1-\mu_2) + c_s^2(\rho_2-\rho_1))+ \\ &+ (\beta_1+\beta_2)(2(\mu_2-\mu_1)\alpha_2 + c_s^2\rho_1\alpha_2). \qquad (160)\end{aligned}$$

If we transform original variables and use equation (158), we may express

$$\frac{\partial^2\varphi_s^{(1)}}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2\varphi_s^{(1)}}{\partial t^2} = \frac{\delta_2 D_1}{c_s^2\mathcal{B}D_2}\frac{\partial q_1}{\partial\xi}. \tag{161}$$

This concludes our investigation of the vertical jumping case. In the interior field, the same elliptic equations (119) are obtained . Thus, we have derived asymptotic model consisting of hyperbolic and elliptic equations for the vertical jumping.

### 3.2.4 Horizontal Jumping

The case of horizontal jumping, corresponds to taking all the functions but $q_2$ to be zero in the boundary conditions

$$\left.\left[\frac{\partial\varphi_1}{\partial\xi} + \frac{\partial\psi_1}{\partial y} - \frac{\partial\varphi_2}{\partial\xi} - \frac{\partial\psi_2}{\partial y}\right]\right|_{y=0} = 0, \tag{162}$$

$$\left.\left[\frac{\partial\varphi_1}{\partial y} - \frac{\partial\psi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial y} + \frac{\partial\psi_2}{\partial\xi}\right]\right|_{y=0} = q_2(\xi,t), \tag{163}$$

$$\begin{aligned}\Bigg[2\mu_1\frac{\partial^2\varphi_1}{\partial\xi\partial y} - 2\mu_2\frac{\partial^2\varphi_2}{\partial\xi\partial y} - \mu_1\left(\frac{\partial^2\psi_1}{\partial y^2} - \frac{\partial^2\psi_1}{\partial\xi^2}\right) - \\ \left.+\mu_2\left(\frac{\partial^2\psi_2}{\partial y^2} - \frac{\partial^2\psi_2}{\partial\xi^2}\right)\Bigg]\right|_{y=0} = 0, \qquad (164)\end{aligned}$$

$$\begin{aligned}\Bigg[\lambda_1\frac{\partial^2\varphi_1}{\partial\xi^2} + (\lambda_1+2\mu_1)\frac{\partial^2\varphi_1}{\partial y^2} - \lambda_2\frac{\partial^2\varphi_2}{\partial\xi^2} - (\lambda_2+2\mu_2)\frac{\partial^2\varphi_2}{\partial y^2} - \\ \left.-2\mu_1\frac{\partial^2\psi_1}{\partial\xi\partial y} + 2\mu_2\frac{\partial^2\psi_2}{\partial\xi\partial y}\Bigg]\right|_{y=0} = 0. \qquad (165)\end{aligned}$$

The leading order equations for the vertical jumping may be expressed as follows:

$$\left.\left[\frac{\partial\varphi_0^{(1)}}{\partial\xi} - \beta_1\frac{\partial\overline{\psi}_0^{(1)}}{\partial\xi} - \frac{\partial\varphi_0^{(2)}}{\partial\xi} - \beta_2\frac{\partial\overline{\psi}_0^{(2)}}{\partial\xi}\right]\right|_{y=0} = 0, \tag{166}$$

$$\left[\alpha_1\frac{\partial\varphi_0^{(1)}}{\partial\xi}-\frac{\partial\overline{\psi}_0^{(1)}}{\partial\xi}+\alpha_2\frac{\partial\varphi_0^{(2)}}{\partial\xi}+\frac{\partial\overline{\psi}_0^{(2)}}{\partial\xi}\right]\Bigg|_{y=0}=0, \tag{167}$$

$$\left[2\alpha_1\mu_1\frac{\partial^2\varphi_0^{(1)}}{\partial\xi^2}+2\alpha_2\mu_2\frac{\partial^2\varphi_0^{(2)}}{\partial\xi^2}-\mu_1(1+\beta_1^2)\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi^2}+\right.$$
$$\left.+\mu_2(1+\beta_2^2)\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi^2}\right]\Bigg|_{y=0}=0, \tag{168}$$

$$\left[-\mu_1(1+\beta_1^2)\frac{\partial^2\varphi_0^{(1)}}{\partial\xi^2}+\mu_2(1+\beta_2^2)\frac{\partial^2\varphi_0^{(2)}}{\partial\xi^2}+2\beta_1\mu_1\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi^2}+\right.$$
$$\left.+2\beta_2\mu_2\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi^2}\right]\Bigg|_{y=0}=0 \tag{169}$$

which reduce to (86)-(88). In the case of vertical jumping, the next order equations are written as

$$\left[\frac{\partial\varphi_{10}^{(1)}}{\partial\xi}-\frac{\partial\varphi_{10}^{(2)}}{\partial\xi}-\beta_1\frac{\partial\overline{\psi}_{10}^{(1)}}{\partial\xi}-\beta_2\frac{\partial\overline{\psi}_{10}^{(2)}}{\partial\xi}-\frac{1-\beta_1^2}{c_s\beta_1}\frac{\partial\overline{\psi}_0^{(1)}}{\partial\tau}-\right.$$
$$\left.-\frac{1-\beta_2^2}{c_s\beta_2}\frac{\partial\overline{\psi}_0^{(2)}}{\partial\tau}\right]\Bigg|_{y=0}=0, \tag{170}$$

$$\left[\alpha_1\frac{\partial\varphi_{10}^{(1)}}{\partial\xi}+\alpha_2\frac{\partial\varphi_{10}^{(2)}}{\partial\xi}-\frac{\partial\overline{\psi}_{10}^{(1)}}{\partial\xi}+\frac{\partial\overline{\psi}_{10}^{(2)}}{\partial\xi}+\frac{1-\alpha_1^2}{c_s\alpha_1}\frac{\partial\varphi_0^{(1)}}{\partial\tau}\right.$$
$$\left.+\frac{1-\alpha_2^2}{c_s\alpha_2}\frac{\partial\varphi_0^{(2)}}{\partial\tau}\right]\Bigg|_{y=0}=0, \tag{171}$$

$$\left[2\mu_1\alpha_1\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}+2\mu_2\alpha_2\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2}-\mu_1(1+\beta_1^2)\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2}+\right.$$
$$+\mu_2(1+\beta_2^2)\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}+2\mu_1\frac{c_s}{c_{11}^2\alpha_1}\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}-$$
$$\left.-2\mu_1\frac{c_s}{c_{21}^2}\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi\partial\tau}+2\mu_2\frac{c_s}{c_{12}^2\alpha_2}\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau}+2\mu_2\frac{c_s}{c_{22}^2}\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0}=0, \tag{172}$$

$$\left[(-2\mu_1+c_s^2\rho_1)\frac{\partial^2\varphi_{10}^{(1)}}{\partial\xi^2}+(2\mu_2-c_s^2\rho_2)\frac{\partial^2\varphi_{10}^{(2)}}{\partial\xi^2}+2\mu_1\beta_1\frac{\partial^2\overline{\psi}_{10}^{(1)}}{\partial\xi^2}+\right.$$
$$+2\mu_2\beta_2\frac{\partial^2\overline{\psi}_{10}^{(2)}}{\partial\xi^2}-2\mu_1\frac{c_s}{c_{21}^2}\frac{\partial^2\varphi_0^{(1)}}{\partial\xi\partial\tau}+$$
$$\left.+2\mu_1\frac{c_s}{c_{21}^2\beta_1}\frac{\partial^2\overline{\psi}_0^{(1)}}{\partial\xi\partial\tau}+2\mu_2\frac{c_s}{c_{22}^2}\frac{\partial^2\varphi_0^{(2)}}{\partial\xi\partial\tau}+2\mu_2\frac{c_s}{c_{22}^2\beta_2}\frac{\partial^2\overline{\psi}_0^{(2)}}{\partial\xi\partial\tau}\right]\Bigg|_{y=0}=0, \tag{173}$$

Performing similar procedure to boundary conditions are in the previously considered case of horizontal jump, we result in

$$\left.\frac{\partial^2 \phi_0^{(1)}}{\partial \xi \partial \tau}\right|_{y=0} = \frac{\delta_2 D_4}{2 c_s \mathcal{B} D_3} \frac{\mu_*}{q_*} \frac{\partial q_2}{\partial \xi}, \tag{174}$$

where

$$\begin{aligned} D_3 &= (1-\alpha_2\beta_2)(2\mu_1 - c_s^2\rho_1 + 2\mu_2\alpha_2\beta_1)- \\ &\qquad - (1+\alpha_2\beta_1)(2\mu_2(1-\alpha_2\beta_2) - c_s^2\rho_2), \qquad (175)\\ D_4 &= c_s^2\rho_2\beta_2(2\mu_1 - c_s^2\rho_1 + 2\mu_2\alpha_2\beta_1)+ \\ &\qquad + (2\mu_1 - 2\mu_2 + c_s^2\rho_2)\beta_2(2\mu_2(1-\alpha_2\beta_2) - c_s^2\rho_2). \qquad (176) \end{aligned}$$

We have for horizontal jump

$$\frac{\partial^2 \varphi_s^{(1)}}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2 \varphi_s^{(1)}}{\partial t^2} = \frac{\delta_2 D_4}{c_s^2 \mathcal{B} D_3}\frac{\partial q_2}{\partial \xi}. \tag{177}$$

with

$$\begin{aligned} &\frac{\partial^2 \varphi_a^{(1)}}{\partial y^2} + \alpha_1^2 \frac{\partial^2 \varphi_a^{(1)}}{\partial x^2} = 0 \\ &\frac{\partial^2 \varphi_a^{(2)}}{\partial y^2} + \alpha_2^2 \frac{\partial^2 \varphi_a^{(2)}}{\partial x^2} = 0 \\ &\frac{\partial^2 \psi_a^{(1)}}{\partial y^2} + \beta_1^2 \frac{\partial^2 \psi_a^{(1)}}{\partial x^2} = 0 \\ &\frac{\partial^2 \psi_a^{(2)}}{\partial y^2} + \beta_2^2 \frac{\partial^2 \psi_a^{(2)}}{\partial x^2} = 0. \end{aligned} \tag{178}$$

The latter equations is similar to that obtained before for the normal load.

Thus we have constructed asymptotic models for four different cases of boundary conditions. In case of an arbitrary boundary value problem, the latter may be cast into one or more of these cases and therefore the problem may be reduced to separate problems the solutions of which may be obtained from the models acquired above. Through the application of the superposition principle the full asymptotic solution may then easily be obtained.

### 3.3 Model Example

As an example, let us consider normal load in the form of a point instantaneous impulse

$$p_2(x,t) = p_0\delta(x)\delta(t). \tag{179}$$

In this case, the 1D wave equation on the interface is given by

$$\frac{\partial^2\varphi_s^{(1)}}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2\varphi_s^{(1)}}{\partial t^2} = \frac{\delta_2}{c_s^2\mathcal{B}}p_0\delta(x)\delta(t) \tag{180}$$

which has a well-known solution [?]

$$\varphi_s^{(1)} = \begin{cases} \dfrac{\delta_2 p_0}{\mathcal{B}}\dfrac{1}{2c_s} & \text{if } |x| < c_s t, \\ 0 & \text{if } |x| > c_s t, \end{cases}$$

or

$$\varphi_s^{(1)} = \frac{\delta_2 p_0}{2\mathcal{B}c_s}[H(x+c_st) - H(x-c_st)] \tag{181}$$

where $H(x)$ is unit step function. The elliptic problem for equation (119) is solved for using the poisson formula $y > 0$ (see [45] and [30] paper for calculations)

$$\varphi_a^{(1)} = \frac{\delta_2 p_0}{2\mathcal{B}c_s\pi}\left[\tan^{-1}\left(\frac{x+c_st}{\alpha_1 y}\right) - \tan^{-1}\left(\frac{x-c_st}{\alpha_1 y}\right)\right].s \tag{182}$$

Next, we calculate using equations (118) and relations (86)-(88)

$$\psi_a^{(1)} = -\frac{\delta_1 p_0}{4\mathcal{B}c_s\pi}\left[\log\left((x+c_st)^2+\beta_1^2y^2\right) - \log\left((x-c_st)^2+\beta_1^2y^2\right)\right] \tag{183}$$

$$\phi_a^{(1)} = -\frac{\delta_4 p_0}{2\mathcal{B}c_s\pi}\left[\frac{\alpha_1 y}{\alpha_1^2y^2+(x+c_st)^2} - \frac{\alpha_1 y}{\alpha_1^2y^2+(x-c_st)^2}\right] \tag{184}$$

$$\psi_a^{(2)} = \frac{\delta_2\delta_3\alpha_1 p_0}{4\delta_4\alpha_2\mathcal{B}c_s\pi}\left[\log\left((x+c_st)^2+\beta_1^2y^2\right) - \log\left((x-c_st)^2+\beta_1^2y^2\right)\right]. \tag{185}$$

Similar results were obtained by Kaplunov et al. [29] paper.

Thus we have derived an explicit asymptotic model for the Stoneley wave. The decay away from the interface is described by elliptic equations (119). The propagation of the wave along the interface in case of each type of nonhomogeneous boundary conditions (55)-(58) is governed by hyperbolic equation (117), with the potential being related by (86)-(88).

# 4 CONCLUSIONS AND FUTURE WORK

This M.Sc. thesis is very important with regard to analysis of elastic surface waves. The asymptotic approach provides significant simplification in comparison to obtaining exact solutions, through integral transforms, because it reduces the vector problem of the elasticity to a scalar problem for an elliptic equation. One of the results is in the derivation of model, demonstrating the dual hyperbolic-elliptic nature of a interfacial wave.The formulated model consist of a hyperbolic equation describing wave propagating along the interface with $c_s$ ( Stoneley wave speed), and four elliptic equations for the interior. The interior field may be found by solving a Dirichlet problem. The mathematical model of an interface is more advanced than other methods, since singularities are associated with Stoneley wave only. The model can be especially useful for the solution of the problems with a major contributing involving interfacial wave phenomena.

The approach allows various generalizations including those for 3D problems, anisotropic and prestressed bodies, curved surfaces and other interfacial waves (Schölte interfacial wave etc.). It is also possible to examine the near resonant effect of various moving loads problems.

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