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RESEARCH ARTICLE /**ARAŞTIRMA MAKALESİ**

DISCONTINUITY OF FRACTAL TOPS

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ABSTRACT

In this work we address the question of continuity of the tops function of an iterated function system defined by M. Barnsley and we give some conditions for the tops function to be discontinuous at the points of the attractor with multiple addresses in the code space.

Anahtar Kelimeler : Iterated function system, Code space, Fractal tops.

FRAKTAL TOPS'UN SÜREKSİZLİĞİ

ÖZ

Bu çalışmada itere fonksiyon sistemleri için M. Barnsley tarafından tanımlanan tops fonksiyonunun sürekliliği incelenmiş ve tops fonksiyonunun bazı koşullar altında kod uzayında çoklu adrese sahip noktalarda süreksiz olduğu ispatlanmıştır.

Keywords: İtere fonksiyon sistemi, Kod uzayı, Fraktal tops.

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1. INTRODUCTION

Let (X,d) be a compact metric space and

$$w_n : X \to X, n = 0,1,2,...,N-1$$

be one-to-one contractions on X. The system of functions $\{w_n\}_{n=0}^{N-1}$ is called an iterated function system (IFS) on X. An IFS is also denoted by $\{X, w_0, w_1,..., w_{N-1}\}$. Let $\mathrm{H}(X)$ denote the set of non-empty compact subsets of X and h the Hausdorff metric on $\mathrm{H}(X)$. $(\mathrm{H}(X),h)$ is then a complete metric space and the function

$$W : \mathrm{H}(X) \to \mathrm{H}(X), W(A) = \bigcup_{n=0}^{N-1} w_n(A)$$

is a contraction on $\mathrm{H}(X)$. Accordingly, by the Banach fixed-point theorem an IFS determines a unique, non-empty, compact subset $A \subset X$ satisfying

$$A = W(A) = \bigcup_{n=0}^{N-1} w_n(A).$$

This set A is called the attractor of the IFS. Again by Banach fixed-point theorem, for any $B \in \mathrm{H}(X)$, the sequence of sets

$$B, W(B), W^2(B) = W \circ W(B),..., W^k(B),...$$

converges to the attractor A of the IFS.

On the other hand, there is a so-called code-space representation of the attractor. Consider the set $\Omega = \{\sigma_1\sigma_2...\sigma_n... \mid \sigma_n \in \{0,1,..., N-1\}, n = 1,2,3,...\}$. The function

$$d(\sigma^1, \sigma^2) = \sum_{n=1}^{\infty} \frac{|\sigma_n^1 - \sigma_n^2|}{(N+1)^n}$$

for $\sigma^1 = \sigma_1^1\sigma_2^1...\sigma_n^1... \in \Omega$ and $\sigma^2 = \sigma_1^2\sigma_2^2...\sigma_n^2... \in \Omega$ defines a metric on Ω. This metric space (Ω, d) is called the code-space on symbols $\{0,1,..., N-1\}$.

Let $\sigma = \sigma_1\sigma_2...\sigma_n... \in \Omega$ and A the attractor of the IFS $\{X, w_0, w_1,..., w_{N-1}\}$. It can easily be shown that the set

$$\bigcap_{n=1}^{\infty} w_{\sigma_1} w_{\sigma_2} ... w_{\sigma_n}(A)$$

is a singleton. If we denote this set by $\{x\} \subset A$, then the function

$$\phi : \Omega \to A$$
$$\sigma \mapsto \phi(\sigma) = x$$

can be defined. This function ϕ is continuous and onto, but it need not be one-to-one. If $\phi(\sigma) = x$ holds for a point x of the attractor A, then the sequence $\sigma = \sigma_1\sigma_2...\sigma_n...$ is called an address (or code) of the point x. If ϕ is one-to-one, then every point of the attractor has a unique address, otherwise there are multiple addresses.

We give now an example where all points of the attractor have unique addresses:

Example 1 (Cantor Set).

Let X be the compact interval $[0,1]$ with the Standard metric. Consider the IFS $\{X, w_0, w_1\}$ with the contractions

$$w_0 : [0,1] \to [0,1], w_0(x) = \frac{1}{3}x$$

and

$$w_1 : [0,1] \to [0,1], w_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

The attractor of this IFS is called the Cantor set (denoted by C). (see figure 1.)

Every point of the Cantor set has a single address because of $w_0(C) \cap w_1(C) = \phi$. For example the point $0 \in C$ has the address $000...$ because of

$$\bigcap_{n=1}^{\infty} w_0 w_0 ... w_0(C) = \{0\}$$

and the point $\frac{2}{3} \in C$ has the address $1000...$ by

$$\bigcap_{n=1}^{\infty} w_1 w_0 ... w_0(C) = \left\{\frac{2}{3}\right\}.$$

Figure 1. The Cantor Set

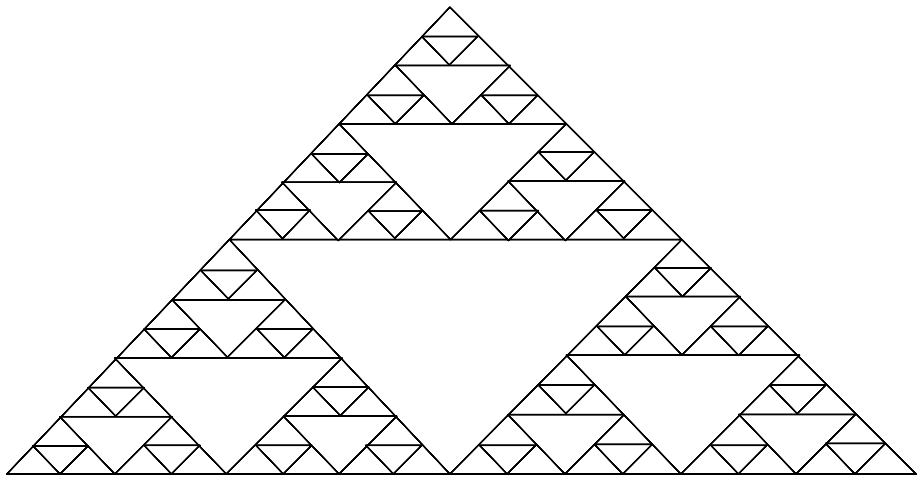

Figure 2. The Sierpinski Gasket

We now give an example where some points of the attractor have multiple addresses:

Example 2 (Sierpinski Gasket):

Let X be the equilateral triangle on the plane (with the standard metric) having the vertices

$P_0 = (0,0), P_1 = (1,0), P_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and w_n $(n = 0,1,2)$ be the contractions on X given by

$$w_n(x) = \frac{x + P_n}{2}.$$

Then the attractor of the IFS $\{X, w_0, w_1, w_2\}$ is called the Sierpinski Gasket. Let us denote it by K (see figure 2).

Because of

$$\bigcap_{n=1}^{\infty} w_k w_k ... w_k(k) = \{P_k\}, k = 0,1,2,$$

the address of the point P_k is $kkk...$, which can be seen to be unique. But the so-called junction points, e. g. $q = \frac{P_0 + P_1}{2}$ have double addresses. For q we have the addresses 0111... and 1000... by the equalities

$$\bigcap_{n=1}^{\infty} w_0 w_1 w_1 ... w_1(K) = \bigcap_{n=1}^{\infty} w_1 w_0 w_0 ... w_0(K) = \{q\}.$$

2. TOPS FUNCTION

The function $\phi : \Omega \to A$ can be inverted if it is one-to-one. But very often it is not one-to-one. In those cases, we define (after Barnsley) the following function $\tau : A \to \Omega$:

Given two different addresses $\sigma = \sigma_1\sigma_2...$ and $\omega = \omega_1\omega_2...$, we define $\sigma < \omega$ if $\sigma_i = \omega_i$ for $i = 1,2,...,k-1$ and $\sigma_k = \omega_k$. We then set $\tau(x) = \max\{\sigma \in \Omega \mid \phi(\sigma) = x\}$. This function τ is called the tops function of the IFS. (Obviously $\tau = \phi^{-1}$ in case ϕ is one-to-one.)

3. DISCONTINUITY OF THE TOPS FUNCTION

It was noted by Barnsley that the tops function is continuous if the attractor is totally disconnected. In that case the points of the attractor have unique addresses , but if this condition is violated, the continuity could be destroyed. Indeed we give now some conditions for multiply-addressed attractors, where the tops function becomes discontinuous.

Theorem 1.

Let $\{X, w_0, w_1, ..., w_{N-1}\}$ be an IFS with the attractor A. Assume $x_0 \in \partial w_j(A) \cap w_i(A)$ for some fixed pair $i < j, i, j = 0,1,...,N-1$ and $x_0 \notin w_k(A)$ for all $k > j.$ Furthermore, assume there exists a sequence $\{x_s\}$ converging to x_0 with $x_s \in w_i(A)$ and $x_s \notin w_t(A)$ for all $t > i.$

Then the tops function τ is not continuous at x_0.

Proof:

Because of $x_0 \in w_j(A)$ and $x_0 \notin w_k(A)$ for all $k > j$, $\tau(x_0) = j\sigma_2\sigma_3...$. On the other hand

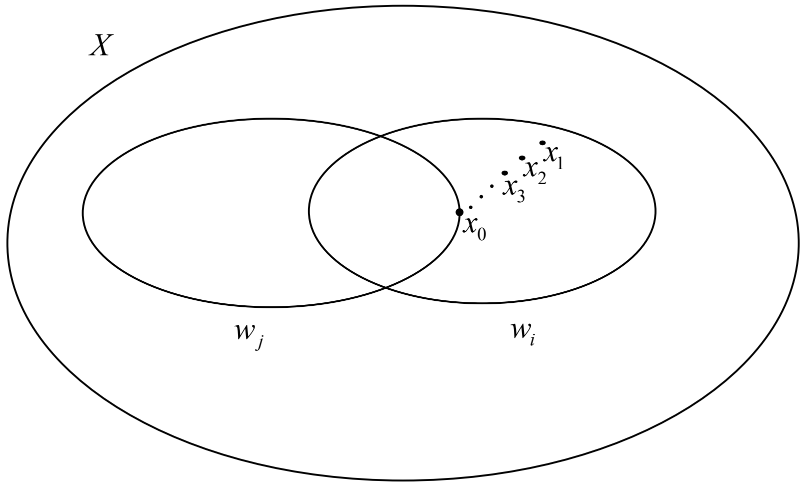

Figure 3. x_0 point and $\{x_s\}$ sequence.

$\tau(x_s) = i\omega_2\omega_3 ... (s = 1,2,3,...)$ because $x_s \in w_i(A)$ and $x_s \notin w_t(A)$ for all $t > i$.

It is obvious that

$$d(\tau(x_0), \tau(x_s)) = \frac{|i-j|}{(N+1)} + \sum_{n=2}^{\infty} \frac{|\sigma_n - \omega_n|}{(N+1)^n} \geq \frac{|i-j|}{(N+1)} > 0.$$

So the sequence $\tau(x_s)$ can not convergence to $\tau(x_0)$, and τ is not continuous at x_0.

We now examplify this property on the Sierpinski Gasket:

Example 3.

Let us consider the point q in example 2.

In the notation of the theorem, we have for the point q, $i=0$, $j=1$. The only $k>1$ is $k=2$ and we have $x_0 \notin w_2(K)$. We use the sequence $\{x_s\}$ with $x_s = (\frac{1}{2} - \frac{1}{2^{s+1}}, 0) \in w_0(K)$ and $x_s \notin w_t(K)$ for $t > 0$ (i.e. $t=1$ and $t=2$). It can be seen that $\tau(q) = 1000...$ and $\tau(x_s) = 0w_2w_3 ....$

The sequence $\tau(x_s)$ can not converge to $\tau(q)$, as

$$d(\tau(x_s), \tau(q)) = \frac{1}{4} + \sum_{n=2}^{\infty} \frac{|w_n|}{4^n} \geq \frac{1}{4}.$$

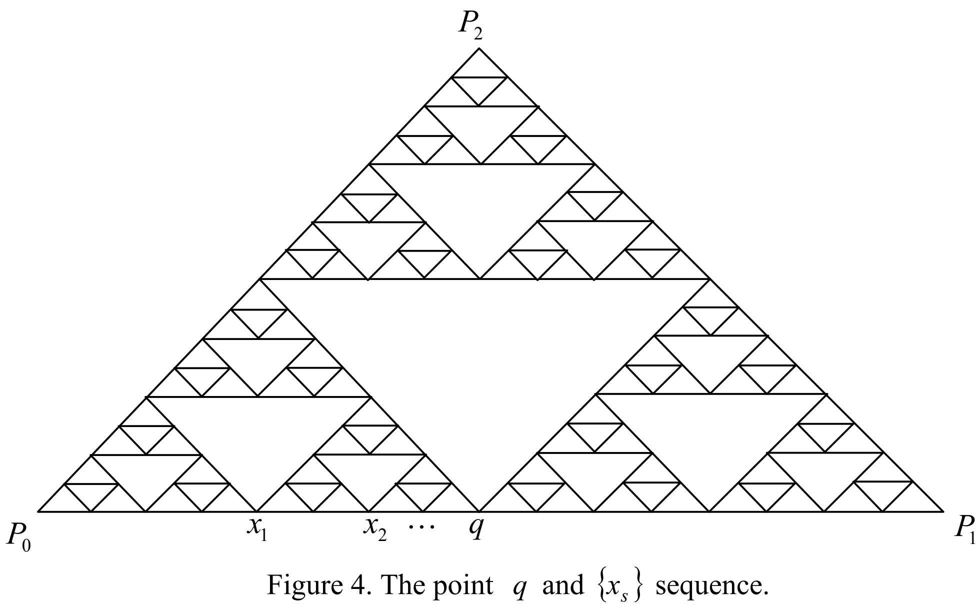

Figure 4. The point q and $\{x_s\}$ sequence.

4. CONCLUSION

We give some conditions for the tops function to be discontinuous at the points of the attractor with multiple addresses in the code space.

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