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Fekete–Szegö problem for a general subclass of analytic functions

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Abstract: In this present investigation, we introduced a certain subclass of starlike and convex functions of complex order b, using a linear multiplier differential operator $D^m_{\lambda,\mu}f(z)$. For this class, the Fekete–Szegö problem is completely solved. Various new special cases are considered.

Key words: Fekete–Szegö problem, analytic functions, starlike and convex functions of complex order, linear multiplier differential operator

1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. And let $\mathcal{S}$ denote the class of functions which are univalent in $\mathcal{U}$. It is well known that for $f \in \mathcal{S}$, $\left|a_3 - a_2^2\right| \leqslant 1$. A classical theorem of Fekete–Szegö (see [7]) states that for $f \in \mathcal{S}$ given by (1.1)

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} 3 - 4\eta & \text{if } \eta \leqslant 0, \\ 1 + 2\exp\left(\frac{-2\eta}{1-\eta}\right) & \text{if } 0 < \eta < 1, \\ 4\eta - 3 & \text{if } \eta \geqslant 1. \end{cases}$$

The latter inequality is sharp in the sense that for each η there exists a function in $\mathcal{S}$ such that the equality holds. Later, Pfluger (see [18]) has considered the complex values of η and provided the inequality

$$\left|a_3 - \eta a_2^2\right| \leqslant 1 + 2\left|\exp\left(\frac{-2\eta}{1-\eta}\right)\right|.$$

To date, several authors have attempted to extend the inequality above to more general classes of analytic functions.

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Given $0 \leqslant \alpha < 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathcal{U}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \;\; z \in \mathcal{U}, \; 0 \leqslant \alpha < 1.$$

On the other hand, a function $f \in \mathcal{A}$ is said to be in the class of convex functions of order α in $\mathcal{U}$, denoted by $\mathcal{C}(\alpha)$, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \;\; z \in \mathcal{U}, \; 0 \leqslant \alpha < 1.$$

A function $f \in \mathcal{A}$ is said to be in the class of starlike functions of complex order $b\,(b \in \mathbb{C} - \{0\})$, denoted by $\mathcal{S}_c^*(b)$, provided that

$$\Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \quad (z \in \mathcal{U}).$$

Furthermore, a function $f \in \mathcal{C}_c(b)$ is convex functions of complex order $b\,(b \in \mathbb{C} - \{0\})$ and type $\alpha\,(0 \leq \alpha < 1)$, that is, $f \in \mathcal{C}_c(b)$, if it satisfies the inequality

$$\Re\left\{1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad (z \in \mathcal{U}).$$

The class $\mathcal{S}_c^*(b)$ of starlike functions of complex order $b\,(b \in \mathbb{C} - \{0\})$ was introduced by Nasr and Aouf [13] while the class $\mathcal{C}_c(b))$ of convex functions of complex order $b\,(b \in \mathbb{C} - \{0\})$ was presented earlier by Wiatrowski [22]. In particular, the classes $\mathcal{S}_c^*(1-\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}_c(1-\alpha) = \mathcal{C}(\alpha)$ are the familiar classes of starlike and convex functions of order α $(0 \leqslant \alpha < 1)$ in $\mathcal{U}$, respectively.

The *linear multiplier differential operator* $D^m_{\lambda,\mu}f$ was defined by Deniz and Orhan in [6] as follows

$$\begin{aligned}
D^0_{\lambda,\mu}f(z) &= f(z)\\
D^1_{\lambda,\mu}f(z) &= D_{\lambda,\mu}f(z) = \lambda\mu z^2(f(z))'' + (\lambda-\mu)z(f(z))' + (1-\lambda+\mu)f(z)\\
D^2_{\lambda,\mu}f(z) &= D_{\lambda,\mu}\left(D^1_{\lambda,\mu}f(z)\right)\\
&\;\;\vdots\\
D^m_{\lambda,\mu}f(z) &= D_{\lambda,\mu}\left(D^{m-1}_{\lambda,\mu}f(z)\right)
\end{aligned}$$

where $\lambda \geqslant \mu \geqslant 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If f is given by (1.1), from the definition of the operator $D^m_{\lambda,\mu}f(z)$ it is easy to see that

$$D^m_{\lambda,\mu}f(z) = z + \sum_{n=2}^{\infty}[1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m a_n z^n. \tag{1.2}$$

It should be remarked that $D^m_{\lambda,\mu}$ is a generalization of many other linear operators considered earlier. In particular, for $f \in \mathcal{A}$ we have the following:

- $D^m_{1,0}f(z) \equiv D^m f(z)$ is investigated by Sãlãgean [21].

- $D^m_{\lambda,0}f(z) \equiv D^m_{\lambda}f(z)$ is studied by Al-Oboudi [2].
- $D^m_{\lambda,\mu}f(z)$ is firstly considered for $0 \leqslant \mu \leqslant \lambda \leqslant 1$, by Rãducanu and Orhan [20].

Now, by making use of the differential operator $D^m_{\lambda,\mu}$, we define a new subclass of analytic functions.

Definition 1.1 *Let b be a nonzero complex number, and let $f \in \mathcal{A}$, such that $D^m_{\lambda,\mu}f(z) \neq 0$ for $z \in \mathcal{U} - \{0\}$. We say that f belongs to $\mathcal{S}_m(b, \lambda, \mu, \nu)$ if*

$$\Re\left(1 + \frac{1}{b}\left(\frac{z(D^m_{\lambda,\mu}f(z))' + \nu z^2(D^m_{\lambda,\mu}f(z))''}{(1-\nu)D^m_{\lambda,\mu}f(z) + \nu z(D^m_{\lambda,\mu}f(z))'} - 1\right)\right) > 0, \quad 0 \leqslant \mu \leqslant \lambda,\ m \in \mathbb{N},\ 0 \leq \nu \leq 1,\ z \in \mathcal{U}.$$

By giving specific values to the parameters m, b, λ, and μ, we obtain the following important subclasses studied by various authors in earlier works, for instance, $\mathcal{S}_m(1-\alpha, 1, 0, 0) = \mathcal{S}_m(\alpha)$ (Sãlãgean [21]), $\mathcal{S}_0(b, 1, 0, 0) = \mathcal{S}^*_c(b)$ (Nasr and Aouf [13]), $\mathcal{S}_1(b, 1, 0, 1) = \mathcal{C}_c(b)$ (Wiatrowski [22], Nasr and Aouf [14]). Indeed, many authors have considered the Fekete–Szegö problem for various subclasses of $\mathcal{A}$, the upper bound for $\left|a_3 - \eta a_2^2\right|$ has been investigated by various authors (see [1, 3–5, 9–12]), see also recent investigations on this subject by [6, 8, 15, 16]. In the present paper we concentrate on the Fekete–Szegö problem for the subclasses $\mathcal{S}_m(b, \lambda, \mu)$ and $\mathcal{C}_m(b, \lambda, \mu)$.

2. Main results

We denote by $\mathcal{P}$ a class of analytic function in $\mathcal{U}$ with $p(0) = 1$ and $\Re p(z) > 0$. In order to derive our main results, we have to recall here the following Lemma (see, [19]).

Lemma 2.1 *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$, then*

$$|c_n| \leqslant 2, \quad for\ n \geqslant 1.$$

If $|c_1| = 2$ then $p(z) \equiv p_1(z) = (1 + \gamma_1 z)/(1 - \gamma_1 z)$ with $\gamma_1 = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore, we have

$$\left|c_2 - \frac{c_1^2}{2}\right| \leqslant 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z + \gamma_1}{1 + \bar{\gamma}_1 \gamma_2 z}}{1 - z\frac{\gamma_2 z + \gamma_1}{1 + \bar{\gamma}_1 \gamma_2 z}},$$

and $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$. Conversely, if $p(z) \equiv p_2(z)$ for some $|\gamma_1| < 1$ and $|\gamma_2| = 1$ then $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ and $\left|c_2 - \frac{c_1^2}{2}\right| \leqslant 2 - \frac{|c_1|^2}{2}$.

Now, consider the functional $\left|a_3 - \eta a_2^2\right|$ for a nonzero complex number b and $\eta \in \mathbb{C}$.

Theorem 2.2 *Let b be a nonzero complex number and $0 \le \nu \le 1$, $\eta \in \mathbb{C}$, $0 \leqslant \mu \leqslant \lambda$. If f, represented in the form (1.1), is in $\mathcal{S}_m(b,\lambda,\mu,\nu)$, then*

$$|a_2| \leqslant \frac{2\,|b|}{(\nu+1)\,A^m}, \tag{2.1}$$

$$|a_3| \leqslant \frac{|b|}{(2\nu+1)\,B^m} \max\{1, |1+2b|\} \tag{2.2}$$

and

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{(2\nu+1)\,B^m} \max\left\{1, \left|1+2b-4\eta b\frac{(2\nu+1)\,B^m}{(\nu+1)^2\,A^{2m}}\right|\right\}, \tag{2.3}$$

where $A = (1+(2\lambda\mu+\lambda-\mu))$ and $B = (1+2(3\lambda\mu+\lambda-\mu))$. Consider the functions

$$\frac{z(\Delta_{\lambda,\mu}^{v,m} f(z)))'}{\Delta_{\lambda,\mu}^{v,m} f(z)} = 1 + b[p_1(z)-1] \tag{2.4}$$

and

$$\frac{z(\Delta_{\lambda,\mu}^{v,m} f(z))'}{\Delta_{\lambda,\mu}^{v,m} f(z)} = 1 + b[p_2(z)-1], \tag{2.5}$$

where p_1, p_2 are given in Lemma 2.1. Equality in (2.1) holds provided that (2.4) is valid. The equality in (2.2) is attained if (2.4) and (2.5) are both satisfied. Similarly, the equality in (2.3) is satisfied for each η given that (2.4) and (2.5) are valid.

Proof Denote $\Delta_{\lambda,\mu}^{v,m} f(z) = (1-\nu)D_{\lambda,\mu}^m f(z) + \nu z(D_{\lambda,\mu}^m f(z))' = z + \beta_2 z^2 + \beta_3 z^3 + \dots$. Then

$$\beta_2 = (\nu+1)\,A^m a_2, \quad \beta_3 = (2\nu+1)\,B^m a_3. \tag{2.6}$$

By definition of the class $\mathcal{S}_m(b,\lambda,\mu,\nu)$, there exists $p \in \mathcal{P}$ such that $\frac{z(\Delta_{\lambda,\mu}^{v,m} f(z))'}{\Delta_{\lambda,\mu}^{v,m} f(z)} = 1 + b(p(z)-1)$, so that

$$\left(\frac{z\left(1+2\beta_2 z + 3\beta_3 z^2 + \dots\right)}{z + \beta_2 z^2 + \beta_3 z^3 + \dots}\right) = 1 - b + b(1 + c_1 z + c_2 z^2 + \dots),$$

which implies the equality

$$z + 2\beta_2 z^2 + 3\beta_3 z^3 + \dots = z + (bc_1 + \beta_2) z^2 + (bc_2 + \beta_2 bc_1 + \beta_3) z^3 + \dots .$$

Equating the coefficients of both sides of the latter we have

$$\beta_2 = bc_1, \quad \beta_3 = \frac{b^2 c_1^2}{2} + \frac{bc_2}{2}, \tag{2.7}$$

so that, on account of (2.6) and (2.7)

$$a_2 = \frac{bc_1}{(\nu+1)\,A^m}, \quad a_3 = \frac{b}{2\,(2\nu+1)\,B^m}(bc_1^2 + c_2). \tag{2.8}$$

Taking into account (2.8) and Lemma 2.1, we obtain

$$|a_2| = \left|\frac{b}{(\nu+1)\,A^m}c_1\right| \leqslant \frac{2\,|b|}{(\nu+1)\,A^m}, \tag{2.9}$$

and

$$\begin{aligned} |a_3| &= \left|\frac{b}{2\,(2\nu+1)\,B^m}\left[c_2 - \frac{c_1^2}{2} + \frac{1+2b}{2}c_1^2\right]\right| \\ &\leqslant \frac{|b|}{2\,(2\nu+1)\,B^m}\left[2 - \frac{|c_1|^2}{2} + |1+2b|\,\frac{|c_1|^2}{2}\right] \\ &= \frac{|b|}{(2\nu+1)\,B^m}\left[1 + |c_1|^2\,\frac{|1+2b|-1}{4}\right] \\ &\leqslant \frac{|b|}{(2\nu+1)\,B^m}\max\{1, [1+|1+2b|-1]\} \end{aligned}$$

resulting in

$$|a_3| \leqslant \frac{|b|}{(2\nu+1)\,B^m}\max\{1, |1+2b|\}.$$

Then, with the aid of Lemma 2.1, we obtain

$$\left|a_3 - \eta a_2^2\right| = \left|\frac{b}{2\,(2\nu+1)\,B^m}(bc_1^2 + c_2) - \eta\frac{b^2c_1^2}{(\nu+1)^2\,A^{2m}}\right|$$

$$\leq \frac{|b|}{2\,(2\nu+1)\,B^m}\left(\left|c_2 - \frac{c_1^2}{2}\right| + \frac{|c_1|^2}{2}\left|1+2b - \frac{4\eta b\,(2\nu+1)\,B^m}{(\nu+1)^2\,A^{2m}}\right|\right) \tag{2.10}$$

$$\begin{aligned} &\leq \frac{|b|}{2\,(2\nu+1)\,B^m}\left(2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left|1+2b - \frac{4\eta b\,(2\nu+1)\,B^m}{(\nu+1)^2\,A^{2m}}\right|\right) \\ &= \frac{|b|}{(2\nu+1)\,B^m}\left[1 + \frac{|c_1|^2}{4}\left(\left|1+2b - \frac{4\eta b\,(2\nu+1)\,B^m}{(\nu+1)^2\,A^{2m}}\right| - 1\right)\right] \\ &\leqslant \frac{|b|}{(2\nu+1)\,B^m}\max\left\{1, \left|1+2b - \frac{4\eta b\,(2\nu+1)\,B^m}{(\nu+1)^2\,A^{2m}}\right|\right\}. \end{aligned}$$

Let us now obtain the accuricies of the estimates in (2.1)–(2.3).

Firstly, in (2.1) the equality holds if $c_1 = 2$. Equivalently, we have $p(z) \equiv p_1(z) = (1+z)/(1-z)$. Therefore, the extremal function in $\mathcal{S}_m(b, \lambda, \mu, \nu)$ is given by

$$\frac{z(\Delta_{\lambda,\mu}^{v,m} f(z))'}{\Delta_{\lambda,\mu}^{v,m} f(z)} = \frac{1+(2b-1)z}{1-z}. \tag{2.11}$$

Next, in (2.2), for the first case, the equality holds if $c_1 = c_2 = 2$. Therefore, the extremal functions in $\mathcal{S}_m(b,\lambda,\mu,\nu)$ is given by (2.11) and for the second case, the equality holds if $c_1 = 0,\ c_2 = 2$. Equivalently, we have $p(z) \equiv p_2(z) = (1+z^2)/(1-z^2)$. Therefore, the extremal function in $\mathcal{S}_m(b,\lambda,\mu,\nu)$ is given by

$$\frac{z(\Delta_{\lambda,\mu}^{v,m} f(z))'}{\Delta_{\lambda,\mu}^{v,m} f(z)} = \frac{1+(2b-1)z^2}{1-z^2}. \tag{2.12}$$

Finally, in (2.3), the equality holds. The extremal function obtained for (2.2) is also valid for (2.3). Thus, the proof of Theorem 2.2 is completed. □

Next we consider the case when η and b are real. In this case, the following theorem holds.

Theorem 2.3 *Let $b > 0$ and let $f \in \mathcal{S}_m(b,\lambda,\mu,\nu)$. For $\eta \in \mathbb{R}$ we have*

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} \frac{b}{(2\nu+1)B^m}\left\{1+2b\left[1-\frac{2\eta(2\nu+1)B^m}{(\nu+1)^2A^{2m}}\right]\right\} & \text{if } \eta \leqslant \frac{(\nu+1)^2A^{2m}}{2(2\nu+1)B^m}, \\ \frac{b}{(2\nu+1)B^m} & \text{if } \frac{(\nu+1)^2A^{2m}}{2(2\nu+1)B^m} \leqslant \eta \leqslant \frac{(1+2b)(\nu+1)^2A^{2m}}{4b(2\nu+1)B^m}, \\ \frac{b}{(2\nu+1)B^m}\left[\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}} - 2b - 1\right] & \text{if } \eta \geqslant \frac{(1+2b)(\nu+1)^2A^{2m}}{4b(2\nu+1)B^m}. \end{cases}$$

where $A = (1+(2\lambda\mu+\lambda-\mu))$ and $B = (1+2(3\lambda\mu+\lambda-\mu))$. For each η, the equality holds for the functions given in equations (2.4) and (2.5).

Proof First, let $\eta \leqslant \frac{(\nu+1)^2A^{2m}}{2(2\nu+1)B^m} \leqslant \frac{(1+2b)(\nu+1)^2A^{2m}}{4b(2\nu+1)B^m}$. In this case it follows from (2.8) and Lemma 2.1 that

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leqslant \frac{b}{2(2\nu+1)B^m}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(1+2b-\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}}\right)\right] \\ &\leqslant \frac{b}{(2\nu+1)B^m}\left[1+2b\left(1-\frac{2\eta(2\nu+1)B^m}{(\nu+1)^2A^{2m}}\right)\right] \end{aligned}$$

Let, now, $\frac{(\nu+1)^2A^{2m}}{2(2\nu+1)B^m} \leqslant \eta \leqslant \frac{(1+2b)(\nu+1)^2A^{2m}}{4b(2\nu+1)B^m}$. Then, using the estimations obtained above, we reach

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{b}{(2\nu+1)B^m}.$$

Finally, for $\eta \geqslant \frac{(1+2b)(\nu+1)^2A^{2m}}{4b(2\nu+1)B^m}$, it follows that

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leqslant \frac{b}{2(2\nu+1)B^m}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}} - 1 - 2b\right)\right] \\ &= \frac{b}{2(2\nu+1)B^m}\left[2 + \frac{|c_1|^2}{2}\left(\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}} - 2 - 2b\right)\right] \\ &\leqslant \frac{b}{(2\nu+1)B^m}\left[\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}} - 2b - 1\right], \end{aligned}$$

which concludes the proof of Theorem 2.3. □

Finally, considering the case of a nonzero complex number b and real η, we obtain:

Theorem 2.4 *Let b be a nonzero complex number and let $f \in \mathcal{S}_m(b,\lambda,\mu,\nu)$. For $\eta \in \mathbb{R}$ we have*

$$\left|a_3-\eta a_2^2\right| \leqslant \begin{cases} \frac{4|b|^2}{(\nu+1)^2A^{2m}}\left[\Re(k_1)-\eta\right]+\frac{|b||\sin\theta|}{(2\nu+1)B^m} & \text{if } \eta \leqslant N_1, \\ \frac{|b|}{(2\nu+1)B^m} & \text{if } N_1 \leqslant \eta \leqslant R_1, \\ \frac{4|b|^2}{(\nu+1)^2A^{2m}}\left[\eta-\Re(k_1)\right]+\frac{|b||\sin\theta|}{(2\nu+1)B^m} & \text{if } \eta \geqslant R_1. \end{cases}$$

where $A=(1+(2\lambda\mu+\lambda-\mu))$ and $B=(1+2(3\lambda\mu+\lambda-\mu))$, $|b|=be^{i\theta}$, $k_1=\frac{(\nu+1)^2A^{2m}}{2(2\nu+1)B^m}+\frac{(\nu+1)^2A^{2m}e^{i\theta}}{4|b|(2\nu+1)B^m}$, $\ell_1=\frac{(\nu+1)^2A^{2m}}{4|b|(2\nu+1)B^m}$, $N_1=\Re(k_1)-\ell_1(1-|\sin\theta|$ and $R_1=\Re(k_1)+\ell_1\left(1-|\sin\theta|\right)$. For each η there is a function in $\mathcal{S}_m(b,\lambda,\mu,\nu)$ such that the equality holds.

Proof From inequality (2.10), we may write

$$\begin{aligned}
\left|a_3-\eta a_2^2\right| &= \frac{|b|}{2(2\nu+1)B^m}\left(\left|c_2-\frac{c_1^2}{2}\right|+\frac{|c_1|^2}{2}\left|1+2b-\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}}\right|\right) \\
&\leqslant \frac{|b|}{2(2\nu+1)B^m}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left|1+2b-\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}}\right|\right] \\
&= \frac{|b|}{2(2\nu+1)B^m}\left[\frac{|c_1|^2}{2}\left(\left|1+2b-\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}}\right|-1\right)+2\right] \\
&= \frac{|b|}{(2\nu+1)B^m}+\frac{|b|}{4(2\nu+1)B^m}\left[\left|\frac{4\eta b(2\nu+1)B^m}{(\nu+1)^2A^{2m}}-2b-1\right|-1\right]|c_1|^2 \\
&= \frac{|b|}{(2\nu+1)B^m}+\frac{|b|^2}{(\nu+1)^2A^{2m}}\left[\left|\eta-\frac{(\nu+1)^2A^{2m}}{2(2\nu+1)B^m}-\frac{(\nu+1)^2A^{2m}}{4b(2\nu+1)B^m}\right|-\frac{(\nu+1)^2A^{2m}}{4|b|(2\nu+1)B^m}\right]|c_1|^2.
\end{aligned}$$

Expressing $|b|=be^{i\theta}$ (or $b=|b|e^{-i\theta}$), $\frac{(\nu+1)^2A^{2m}}{2(2\nu+1)B^m}+\frac{(\nu+1)^2A^{2m}e^{i\theta}}{4|b|(2\nu+1)B^m}=k_1$ and $\frac{(\nu+1)^2A^{2m}}{4|b|(2\nu+1)B^m}=\ell_1$ in the last inequality, we get

$$\begin{aligned}
\left|a_3-\eta a_2^2\right| &\leqslant \frac{|b|}{(2\nu+1)B^m}+\frac{|b|^2}{(\nu+1)^2A^{2m}}\left[|\eta-k_1|-\ell_1\right]|c_1|^2 \\
&\leqslant \frac{|b|}{(2\nu+1)B^m}+\frac{|b|^2}{(\nu+1)^2A^{2m}}\left[|\eta-\Re(k_1)|+\ell_1|\sin\theta|-\ell_1\right]|c_1|^2
\end{aligned}$$

$$=\frac{|b|}{(2\nu+1)B^m}+\frac{|b|^2}{(\nu+1)^2A^{2m}}\left[|\eta-\Re(k_1)|-\ell_1(1-|\sin\theta|)\right]|c_1|^2. \tag{2.13}$$

We consider the following cases for (2.13). Suppose $\eta \leqslant \Re(k_1)$. Then

$$\left|a_3-\eta a_2^2\right| \leqslant \frac{|b|}{(2\nu+1)B^m}+\frac{|b|^2}{(\nu+1)^2A^{2m}}\left[\Re(k_1)-\ell_1(1-|\sin\theta|)-\eta\right]|c_1|^2$$

$$=\frac{|b|}{(2\nu+1)B^m}+\frac{|b|^2}{(\nu+1)^2A^{2m}}\left[N_1-\eta\right]|c_1|^2. \tag{2.14}$$

Let $\eta \leqslant N_1 = \Re(k_1) - \ell_1 (1 - |\sin\theta|)$. On using Lemma 2.1 and $\ell_1 = \frac{(\nu+1)^2 A^{2m}}{4|b|(2\nu+1)B^m}$ in inequality (2.14), we get

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leqslant \frac{|b|}{(2\nu+1) B^m} + \frac{4|b|^2}{(\nu+1)^2 A^{2m}} (\Re(k_1) - \eta) - \frac{4|b|^2}{(\nu+1)^2 A^{2m}} \frac{(\nu+1)^2 A^{2m}}{4|b|(2\nu+1) B^m} (1 - |\sin\theta|) \\ &= \frac{|b|}{(2\nu+1) B^m} + \frac{4|b|^2}{(\nu+1)^2 A^{2m}} (\Re(k_1) - \eta) - \frac{|b|}{(2\nu+1) B^m} (1 - |\sin\theta|) \\ &= \frac{4|b|^2}{(\nu+1)^2 A^{2m}} (\Re(k_1) - \eta) + \frac{|b| |\sin\theta|}{(2\nu+1) B^m}. \end{aligned}$$

If we take $N_1 = \Re(k_1) - \ell_1 (1 - |\sin\theta|) \leqslant \eta \leqslant \Re(k_1)$, then (2.14) gives

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{(2\nu+1) B^m}.$$

Let $\eta \geqslant \Re(k_1)$. It then follows, from (2.13), that

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leqslant \frac{|b|}{(2\nu+1) B^m} + \frac{|b|^2}{(\nu+1)^2 A^{2m}} [\eta - (\Re(k_1) + \ell_1(1 - |\sin\theta|))] |c_1|^2 \\ &= \frac{|b|}{(2\nu+1) B^m} + \frac{|b|^2}{(\nu+1)^2 A^{2m}} [\eta - R_1] |c_1|^2. \end{aligned} \tag{2.15}$$

Let $\eta \leqslant R_1 = \Re(k_1) + \ell_1 (1 - |\sin\theta|)$. On using (2.15) we obtain

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{(2\nu+1) B^m}.$$

Let $\eta \geqslant R_1 = \Re(k_1) + \ell_1 (1 - |\sin\theta|)$. Employing Lemma 2.1 together with $\ell_1 = \frac{(\nu+1)^2 A^{2m}}{4|b|(2\nu+1)B^m}$ in equality (2.15), we obtain

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leqslant \frac{|b|}{(2\nu+1) B^m} + \frac{4|b|^2}{(\nu+1)^2 A^{2m}} (\eta - \Re(k_1)) - \frac{|b|}{(2\nu+1) B^m} (1 - |\sin\theta|) \\ &\leqslant \frac{4|b|^2}{(\nu+1)^2 A^{2m}} (\eta - \Re(k_1)) + \frac{|b| |\sin\theta|}{(2\nu+1) B^m}. \end{aligned}$$

Therefore, the proof is completed. □

Corollary 2.5 *If we take $\lambda = 1$ and $\mu = 0$ in Theorems 2.2–2.4, we have the following results, respectively:*

1. *Let $b \in \mathbb{C}$, $b \neq 0$ and $f \in \mathcal{S}_m(b, \nu)$. Then, for $\eta \in \mathbb{C}$ we have*

$$\begin{aligned} |a_2| &\leqslant \frac{|b|}{(\nu+1) 2^{m-1}}, \\ |a_3| &\leqslant \frac{|b|}{(2\nu+1) 3^m} \max\{1, |1 + 2b|\} \end{aligned}$$

and

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{|b|}{(2\nu+1)\,3^m} \max\left\{1, \left|1 + 2b - 4\eta b \frac{(2\nu+1)}{(\nu+1)^2}\left(\frac{3}{4}\right)^m\right|\right\}.$$

Equality holds for the cases $\lambda = 1$, $\mu = 0$ *of* 2.4 *and* 2.5 *in Theorem* 2.2*.*

2. *Let* $b > 0$ *and* $f \in \mathcal{S}_m(b,\nu)$*. Then, for* $\eta \in \mathbb{R}$ *we have*

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} \frac{b}{(2\nu+1)3^m}\left\{1 + 2b\left[1 - \frac{2\eta(2\nu+1)}{(\nu+1)^2}\left(\frac{3}{4}\right)^m\right]\right\} & \text{if } \eta \leqslant \frac{(\nu+1)^2}{2(2\nu+1)}\left(\frac{4}{3}\right)^m, \\ \frac{b}{(2\nu+1)3^m} & \text{if } \frac{(\nu+1)^2}{2(2\nu+1)}\left(\frac{4}{3}\right)^m \leqslant \eta \leqslant \frac{(1+2b)(\nu+1)^2}{4b(2\nu+1)}\left(\frac{4}{3}\right)^m, \\ \frac{b}{(2\nu+1)3^m}\left[\frac{4\eta b(2\nu+1)}{(\nu+1)^2}\left(\frac{3}{4}\right)^m - 2b - 1\right] & \text{if } \eta \geqslant \frac{(1+2b)(\nu+1)^2}{4b(2\nu+1)}\left(\frac{4}{3}\right)^m. \end{cases}$$

For each η*, the equality holds for the cases* $\lambda = 1$, $\mu = 0$ *of* 2.4 *and* 2.5*.*

3. *Let* $b \in \mathbb{C}$, $b \neq 0$ *and* $f \in \mathcal{S}_m(b,\nu)$*. Then, for* $\eta \in \mathbb{R}$ *we have*

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} \frac{|b|^2}{(\nu+1)^2 4^{m-1}}\left[\Re(k_1) - \eta\right] + \frac{|b||\sin\theta|}{(2\nu+1)3^m} & \text{if } \eta \leqslant N_1, \\ \frac{|b|}{(2\nu+1)3^m} & \text{if } N_1 \leqslant \eta \leqslant R_1, \\ \frac{|b|^2}{(\nu+1)^2 4^{m-1}}\left[\eta - \Re(k_1)\right] + \frac{|b||\sin\theta|}{(2\nu+1)3^m} & \text{if } \eta \geqslant R_1. \end{cases}$$

where $|b| = be^{i\theta}$, $k_1 = \frac{(\nu+1)^2}{2(2\nu+1)}\left(\frac{4}{3}\right)^m - \left(\frac{4}{3}\right)^m \frac{(\nu+1)^2 e^{i\theta}}{4|b|(2\nu+1)}$, $\ell_1 = \left(\frac{4}{3}\right)^m \frac{(\nu+1)^2}{4|b|(2\nu+1)}$, $N_1 = \Re(k_1) - \ell_1\left(1 - |\sin\theta|\right)$ *and* $R_1 = \Re(k_1) + \ell_1\left(1 - |\sin\theta|\right)$*. For each* η*, there is a function in* $\mathcal{S}_m(b,\nu)$ *such that the equality holds.*

For the particular cases of the parameter ν in Theorems 2.2–2.4, the results of the current paper agrees with that of [17].

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