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Monopole Equations on R^8: The Energy Integral*

Ayşe Hümeyra BİLGE
Department of Mathematics, Istanbul Technical University,
Istanbul-TURKEY
e-mail: bilge@itu.edu.tr
Tekin DERELİ
Department of Physics, Middle East Technical University,
Ankara-TURKEY
e-mail: tekin@dereli.physics.metu.edu.tr
Şahin KOÇAK
Department of Mathematics, Anadolu University,
Eskişehir-TURKEY
e-mail: skocak@vm.baum.anadolu.edu.tr

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Abstract

We discuss the Seiberg-Witten theory in four dimensions and possible generalizations to eight dimensions. We propose an action which is minimized by a set of monopole equations, previously obtained as a generalization of Seiberg-Witten theory to eight manifolds with Spin(7) holonomy.

1. Introduction

One of the main problems in low dimensional topology is the determination of diffeomorphism invariants of 4-manifolds. The Donaldson theory provides diffeomorphism invariants constructed using solutions of Yang-Mills equations with SU(2) gauge group, through tedious computations. The Seiberg-Witten theory in four dimensions [1] yields these invariants essentially by counting the number of solutions of a set of massless, Abelian monopole equations [2,3]. Although the starting point of Seiberg-Witten theory was the determination of diffeomorphism invariants of 4-manifolds, it was later noted that topological quantum field theories also exist in higher dimensions [4-7]. Thus it is of interest

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to consider monopole equations in higher dimensions and generalize the 4-dimensional Seiberg-Witten theory.

The Seiberg-Witten equations can be constructed on any even dimensional manifold (D=2n) with a spin^c-structure [8], but a straightforward generalization yields an over determined set of equations having no non-trivial solutions even locally [9]. In order to obtain a nontrivial generalization, one first needs an appropriate notion of "self-duality" of 2-forms in dimensions greater than four. In a previous paper [10] we reviewed the existing definitions of self-duality and given a (non-linear) eigenvalue criterion for specifying self-dual 2-forms on any even dimensional manifold. Any maximal linear subspace of these self-dual 2-forms allows to define a linear notion of self-duality. In particular, on 8-manifolds with Spin(7) holonomy the choice of a maximal linear subspace is globally meaningful and relates to other definitions given in the literature [11-13]. Eight dimensions is also special because the set of linear Spin(7) self-duality equations can be solved by making use of octonions [14]. The existence of *octonionic instantons* which realise the last Hopf fibration $S^{15} \to S^8$ is closely related with the properties of the octonion algebra [15-17].

In the next section we shall give the set up for the Seiberg-Witten theory, i.e. the spin^c-structures and the Weitzenböck formula in arbitrary dimensions. A detailed exposition of the theory in 4-dimensions will be given in Section 3, with the purpose of pointing out possible generalizations to higher dimensions. We note that any 8-manifold with Spin(7) holonomy is a spin manifold [18,19], hence carries a spin^c-structure. In Section 4, we shall give the monopole equations on 8-manifolds with Spin(7) holonomy [20], and show that these equations minimize a certain energy integral.

2. Definitions and notation

A spin^c-structure on a $2n$-dimensional real inner-product space V is defined as a pair (W, Γ), where W is a 2^n-dimensional complex Hermitian space and $\Gamma : V \to End(W)$ is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v), \qquad \Gamma(v)^2 = -\|v\|^2 \tag{2.1}$$

for $v \in V$. Globalizing this defines the notion of a spin^c-structure $\Gamma : TX \to End(W)$ on a $2n$-dimensional (oriented) manifold X, W being a 2^n-dimensional complex Hermitian vector bundle on X. Such a structure exists if and only if the second Stiefel-Whitney class $w_2(X)$ has an integral lift. Γ extends to an isomorphism between the complex Clifford algebra bundle $C^c(TX)$ and $End(W)$. There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n}e_{2n-1}\cdots e_1)$ where $e_1, e_2, \cdots, e_{2n}$ is any positively oriented local orthonormal frame of TX. The restriction of $\Gamma(v)$ determines a linear map $\gamma(v) : V \to Hom(W^-, W^+)$ which satisfies

$$\gamma(v)^*\gamma(v) = |v|. \tag{2.2}$$

One can as well start with $\gamma(v)$ and define $\Gamma(v)$ as

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix}. \tag{2.3}$$

The extension of Γ to $C_2(X)$ gives, via the identification of $\Lambda^2(T^*X)$ with $C_2(X)$, a map $\rho : \Lambda^2(T^*X) \to End(W)$ given by

$$\rho(\sum_{i<j} \eta_{ij} e_i^* \wedge e_j^*) = \sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j). \tag{2.4}$$

The bundles $W^{\pm}$ are invariant under $\rho(\eta)$ for $\eta \in \Lambda^2(T^*X)$. Denote $\rho^{\pm}(\eta) = \rho(\eta)|_{W^{\pm}}$. Then using (2.3) it can be seen that $\rho^+ : \Lambda^2(T * X) \to End(W^+)$ is given by

$$\rho^+(\sum_{i<j} \eta_{ij} e_i^* \wedge e_j^*) = \sum_{i<j} -\eta_{ij} \gamma(e_i)\gamma^*(e_j). \tag{2.5}$$

The maps ρ and $\rho^{\pm}$ can be extended to complex valued 2-forms in the obvious way.

A Hermitian connection on W, compatible with the Levi-Civita connection of the manifold induces an imaginary valued connection on certain associated line bundle called the *virtual line bundle* $L_\Gamma^{1/2}$. we denote the corresponding connection 1-form by A and its curvature 2-form by $F_A \in \Omega^2(X, i\mathbf{R})$. The Dirac operator corresponding to A

$$D_A : C^\infty(X, W^+) \to C^\infty(X, W^-)$$

is now defined by

$$D_A(\Phi) = \sum_{i=1}^{2n} \Gamma(e_i) \nabla_{A,e_i}(\Phi) \tag{2.6}$$

where $\Phi \in C^\infty(X, W^+)$ and $e_1, e_2, \cdots, e_{2n}$ is any local orthonormal frame.

The Weitzenböck formula below [8] is the key in writing the energy integral for Seiberg-Witten equations.

Weitzenböck Formula: Let $s \to \mathbf{R}$ denote the scalar curvature of X. Then

$$D_A^* D_A \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{4} s\Phi + \rho^+(F_A)\Phi, \tag{2.7a}$$

$$D_A D_A^* \phi = \nabla_A \nabla_A^* \phi + \frac{1}{4} s\phi + \rho^-(F_A)\phi, \tag{2.7b}$$

where $\Phi \in C^\infty(X, W^+)$ and $\phi \in C^\infty(X, W^-)$.

The action integral leading to Seiberg-Witten equations in 4-dimensions is

$$E(A, \Phi) = \int_X \big(\mid \nabla_A \Phi \mid^2 + \frac{s}{4} \mid \Phi \mid^2 + \alpha \mid \Phi \mid^4 + \beta \mid F_A \mid^2 \big) \mathrm{dvol}, \tag{2.8}$$

where $\alpha = \frac{1}{4}$ and $\beta = 1$. Note that this action can be negative for manifolds with negative scalar curvature. The Dirac equation for Φ now comes into play as

$$\mid D_A\Phi \mid^2 = (D_A^* D_A \Phi, \Phi) = \mid \nabla_A \Phi \mid^2 + \frac{1}{4} s \mid \Phi \mid^2 + (\rho^+(F_A)\Phi, \Phi), \tag{2.9}$$

where $(U, V) = \bar{U}^t V$. The energy integral reduces to

$$E(A, \Phi) = \int_X \big(\mid D_A \Phi \mid^2 - (\rho^+(F_A)\Phi, \Phi) + \frac{1}{4} \mid \Phi \mid^4 + \mid F_A \mid^2 \big) \mathrm{dvol}. \tag{2.10}$$

In 4-dimensions, this energy integral will be minimized by solutions of the Seiberg-Witten equations below.
Seiberg-Witten equations: Let $\Gamma : TX \to End(W)$ be a $spin^c$-structure on X, A be the connection on the virtual line bundle and $\Phi \in C^\infty(X, W^+)$. The Seiberg- Witten equations read

$$D_A(\Phi) = 0 \quad , \qquad \rho^+(F_A) = (\Phi\Phi^*)_0 \tag{2.11}$$

where $(\Phi\Phi^*)_0$ denotes the traceless part of the matrix $\Phi\Phi^*$.

A generalization of the Seiberg-Witten equations in the form above leads to trivial solutions [9]. In the next section we will see that in 4-dimensions, $\rho^+(F)$ is automatically equal to $\rho^+(F^+)$, and the trace free part of $\Phi\Phi^*$ corresponds to a projection denoted by $(\Phi\Phi^*)^+$ on a subspace determined by the spin^c structure. Thus rewriting the Seiberg-Witten equation as $\rho^+(F^+) = (\Phi\Phi^*)^+$, leads to a set of elliptic monopole equations, admitting solutions of the 4-dimensional Seiberg-Witten equations [20].

In the next section, we will see that (2.11) arise naturally as minimizers of the action (2.10). In 8-dimensions, the monopole equations given in [20] will be minimizers of the action (2.10) with $\rho^+(F)$ replaced with $\rho^+(F^+)$. This interpretation is however not quite satisfactory, because the action (2.10) can no longer be tied to (2.8) via the Weitzenböck formula.

3. Seiberg-Witten equations on 4-manifolds

On a 4-dimensional manifold, let $\{e_1, e_2, e_3, e_4\}$ be a local basis for the tangent bundle, Φ be a local section of W^+ and let the spin^c structure be given as

$$\gamma(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma(e_2) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma(e_3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma(e_4) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3.1}$$

Using the notation $e_{ij} = e_i^* \wedge e_j^*$, we write $F = F_A$ as

$$\begin{aligned} F = &\frac{1}{2}\big[(F_{12} + F_{34})(e_{12} + e_{34}) + (F_{13} - F_{24})(e_{13} - e_{24}) + (F_{14} + F_{23})(e_{14} + e_{23})\big] \\ &+ \frac{1}{2}\big[(F_{12} - F_{34})(e_{12} - e_{34}) + (F_{13} + F_{24})(e_{13} + e_{24}) + (F_{14} - F_{23})(e_{14} - e_{23})\big] \end{aligned}$$

$$(3.2)$$

Note that $\gamma(e_i)\gamma(e_j) = \gamma(k)$ for any even permutation of $(i = 2, j = 3, k = 4)$ and hence the map ρ^+ vanishes on the anti-selfdual part of F. Using the fact that F_{ij}'s are pure imaginary, we have

$$(\rho^+(F_{ij}e_{ij})\Phi, \Phi) = -F_{ij}(\gamma(e_j)\Phi, \gamma(e_i)^*\Phi). \tag{3.3}$$

We introduce the notation $\Phi^{(i)} = \gamma(e_i)\Phi$ and we write

$$(\rho^+(F)\Phi, \Phi) = -(F_{12}+F_{34})(\Phi^{(2)}, \Phi^{(1)}) - (F_{13}-F_{24})(\Phi^{(3)}, \Phi^{(1)}) - (F_{14}+F_{23})(\Phi^{(4)}, \Phi^{(1)}). \tag{3.4}$$

On the other hand it can be seen that

$$(\Phi^{(2)}, \Phi^{(1)})^2 + (\Phi^{(3)}, \Phi^{(1)})^2 + (\Phi^{(4)}, \Phi^{(1)})^2 = - \mid \Phi \mid^4 \tag{3.5}$$

and

$$\mid F \mid^2 = -(F_{12}+F_{34})^2 - (F_{13}-F_{24})^2 - (F_{14}+F_{23})^2 + 2(F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23}) \tag{3.6}$$

Thus when the first part of the Seiberg-Witten equations is satisfied, i.e. $D_A\Phi = 0$ the energy integral reduces to

$$\begin{aligned} E(A, \Phi) = \int_X & [(F_{12}+F_{34})(\Phi^{(2)}, \Phi^{(1)}) + (F_{13}-F_{24})(\Phi^{(3)}, \Phi^{(1)}) + (F_{14}+F_{23})(\Phi^{(4)}, \Phi^{(1)})] \\ & -\alpha[(\Phi^{(2)}, \Phi^{(1)})^2 + (\Phi^{(3)}, \Phi^{(1)})^2 + (\Phi^{(4)}, \Phi^{(1)})^2] \\ & -\beta[(F_{12}+F_{34})^2 + (F_{13}-F_{24})^2 + (F_{14}+F_{23})^2] \\ & +2\beta[(F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23})] \end{aligned} \tag{3.7}$$

The last term in the action is a topological term proportional to the first Pontrjagin class of the line bundle. Thus the action will be minimized if the sum of the first three term can be made zero. If we define

$$U = ((F_{12}+F_{34}), (F_{13}-F_{24}), (F_{14}+F_{23})), \quad V = ((\Phi^{(2)}, \Phi^{(1)}), (\Phi^{(3)}, \Phi^{(1)}), (\Phi^{(4)}, \Phi^{(1)}) \tag{3.8}$$

the first three terms in the action becomes $(U, V) - \alpha(V, V) - \beta(U, U)$, and the lowest value will be achieved when $U = kV$, which leads to $k - \alpha - \beta k^2 = 0$ for positive real numbers k, α and β. Taking $\beta = 1$, it can be seen that $k = \frac{1}{2}$ and $\alpha = \frac{1}{4}$, we are led to the Seiberg-Witten equations (2.11).

In [20] we have shown that the right hand side of (2.11) has an interpretation in terms of projections determined by the spin^c structure. Namely, given any (global, imaginary-valued) 2-form F, the image under the map ρ^+ of its self-dual part F^+ coincides with the orthogonal projection of $(\Phi\Phi^*)^+$ of $\Phi\Phi^*$ on the subspace generated by those matrices which are the images under the ρ^+ map of the self-dual 2-forms [20]. As a result the Seiberg-Witten in 4-dimensions has the expression given below.

$$
\begin{aligned}
F_{12}+F_{34} &= 1/2(\Phi^{(2)},\Phi^{(1)}) = -i/2(\phi_1\bar{\phi}_1-\phi_2\bar{\phi}_2) = -1/2\Phi^*\gamma(e_2)\Phi,\\
F_{13}-F_{24} &= \frac{1}{2}(\Phi^{(3)},\Phi^{(1)}) = 1/2(\phi_1\bar{\phi}_2-\phi_2\bar{\phi}_1) = -1/2\Phi^*\gamma(e_3)\Phi,\\
F_{14}+F_{23} &= \frac{1}{2}(\Phi^{(4)},\Phi^{(1)}) = -i/2(\phi_1\bar{\phi}_2+\phi_2\bar{\phi}_1) = -1/2\Phi^*\gamma(e_4)\Phi. \qquad (3.9)
\end{aligned}
$$

4. Monopole equations on 8-manifolds and the action integral

Let F be a 2-form on an even dimensional manifold and let $\pm\lambda_i$ be the eigenvalues of the corresponding skew-symmetric matrix. We have defined the notion of "self-duality" of a 2-form as the equality of the λ_i's, as a concept generalizing the self-duality in 4-dimensions [13]. In eight dimensions, these 2-forms constitute a 13-dimensional submanifold in which there are 7-dimensional maximal linear subspaces. The choice of any such subspace gives a set self-dual 2-forms say $\{f_i\}$, $i=1,\dots 7$. Furthermore if the 8-manifold posseses Spin(7) holonomy, the 4-form $\Psi=\sum_i f_i\wedge f_i$ is globally defined [6]. However the requirement that the curvature 2-form F belongs to one of these subspaces is too strong, for it gives 21 equations for the 8 components of the connection [21]. On the other hand determining the the projection of F onto a 7-dimensional linear space via Seiberg-Witten type equations gives 7 differential equations for the components of A. It has been shown that these equations form an elliptic system under the Coulomb gauge [21], hence they are expected to be useful in applying index theorems to define global invariants.

The most widely known of the maximal linear subspaces of strong self-dual 2 forms is the so-called CDNF-plane, corresponding to a set of self-duality equations proposed in [11]. We use however a different maximal linear subspace [10,20] that will lead to monopole equations reducible to 4-dimensions. An orthonormal basis for this maximal linear subspace is

$$
\begin{aligned}
f_1 &= dx_1\wedge dx_5+dx_2\wedge dx_6+dx_3\wedge dx_7+dx_4\wedge dx_8,\\
f_2 &= dx_1\wedge dx_2+dx_3\wedge dx_4-dx_5\wedge dx_6-dx_7\wedge dx_8,\\
f_3 &= dx_1\wedge dx_6-dx_2\wedge dx_5-dx_3\wedge dx_8+dx_4\wedge dx_7,\\
f_4 &= dx_1\wedge dx_3-dx_2\wedge dx_4-dx_5\wedge dx_7+dx_6\wedge dx_8,\\
f_5 &= dx_1\wedge dx_7+dx_2\wedge dx_8-dx_3\wedge dx_5-dx_4\wedge dx_6,\\
f_6 &= dx_1\wedge dx_4+dx_2\wedge dx_3-dx_5\wedge dx_8-dx_6\wedge dx_7,\\
f_7 &= dx_1\wedge dx_8-dx_2\wedge dx_7+dx_3\wedge dx_6-dx_4\wedge dx_5. \qquad (4.1)
\end{aligned}
$$

The self-dual part of $F=\sum_{i<j}F_{ij}e_i\wedge e_j$, F^+ is defined as the projection of F on the linear subspace spanned by the f_i's above, explicitly given as

$$
\begin{aligned}
F^+ = 1/4(F_{15} + F_{26} + F_{37} + F_{48})f_1 \\
+1/4(F_{12} + F_{34} - F_{56} - F_{78})f_2 \\
+1/4(F_{16} - F_{25} - F_{38} + F_{47})f_3 \\
+1/4(F_{13} - F_{24} - F_{57} + F_{68})f_4 \\
+1/4(F_{17} + F_{28} - F_{35} - F_{46})f_5 \\
+1/4(F_{14} + F_{23} - F_{58} - F_{67})f_6 \\
+1/4(F_{18} - F_{27} + F_{36} - F_{45})f_7.
\end{aligned} \tag{4.2}
$$

The *spin*^{c} structure is defined by choosing $\gamma(e_1) = I$, where I is the identity matrix, and $\gamma(e_i)$, $i = 2, \ldots, 8$ as the real skew-symmetric matrix corresponding to f_{i-1}, which can be written down easily. Then $\rho^+(f_i)$'s can be computed from (4.2) and as the resulting matrix is skew-symmetric, they can be identified with a 2-form, denoted also by $\rho^+(_i)$, by abuse of notation.

$$
\begin{aligned}
\rho^+(f_1) = \gamma(e_1)\gamma(e_5) + \gamma(e_2)\gamma(e_6) + \gamma(e_3)\gamma(e_7) + \gamma(e_4)\gamma(e_8) = 2[e_{13} - e_{24} - e_{57} - e_{68}] \\
\rho^+(f_2) = \gamma(e_1)\gamma(e_2) + \gamma(e_3)\gamma(e_4) - \gamma(e_5)\gamma(e_6) - \gamma(e_7)\gamma(e_8) = 2[e_{15} - e_{26} + e_{37} + e_{48}] \\
\rho^+(f_3) = \gamma(e_1)\gamma(e_6) - \gamma(e_2)\gamma(e_5) - \gamma(e_3)\gamma(e_8) + \gamma(e_4)\gamma(e_7) = 2[e_{17} + e_{28} - e_{35} + e_{46}] \\
\rho^+(f_4) = \gamma(e_1)\gamma(e_3) - \gamma(e_2)\gamma(e_4) - \gamma(e_5)\gamma(e_7) + \gamma(e_6)\gamma(e_8) = 2[e_{12} + e_{34} + e_{56} - e_{78}] \\
\rho^+(f_5) = \gamma(e_1)\gamma(e_7) + \gamma(e_2)\gamma(e_8) - \gamma(e_3)\gamma(e_5) - \gamma(e_4)\gamma(e_6) = 2[e_{14} + e_{23} - e_{58} + e_{67}] \\
\rho^+(f_6) = \gamma(e_1)\gamma(e_4) + \gamma(e_2)\gamma(e_3) - \gamma(e_5)\gamma(e_8) - \gamma(e_6)\gamma(e_7) = 2[-e_{16} - e_{25} - e_{38} + e_{47}] \\
\rho^+(f_7) = \gamma(e_1)\gamma(e_8) - \gamma(e_2)\gamma(e_7) + \gamma(e_3)\gamma(e_6) - \gamma(e_4)\gamma(e_5) = 2[e_{18} - e_{27} - e_{36} - e_{45}].
\end{aligned} \tag{4.3}
$$

The monopole equations are now [20]

$$
\rho^+({F_A}^+) = \sum_{i=1}^{7} < \rho^+(f_i), \Phi\Phi^* > \rho^+(f_i)/|\rho^+(f_i)|^2 \tag{4.4}
$$

which are equivalent to the set of equations

$$
\begin{aligned}
H_1 = F_{15} + F_{26} + F_{37} + F_{48} = 1/4(\phi_1\bar\phi_3 - \phi_3\bar\phi_1 - \phi_2\bar\phi_4 + \phi_4\bar\phi_2 - \phi_5\bar\phi_7 + \phi_7\bar\phi_5 - \phi_6\bar\phi_8 + \phi_8\bar\phi_6) = 1/4\varphi_1, \\
H_2 = F_{12} + F_{34} - F_{56} - F_{78} = 1/4(\phi_1\bar\phi_5 - \phi_5\bar\phi_1 - \phi_2\bar\phi_6 + \phi_6\bar\phi_2 + \phi_3\bar\phi_7 - \phi_7\bar\phi_3 + \phi_4\bar\phi_8 - \phi_8\bar\phi_4), = 1/4\varphi_2 \\
H_3 = F_{16} - F_{25} - F_{38} + F_{47} = 1/4(\phi_1\bar\phi_7 - \phi_7\bar\phi_1 + \phi_2\bar\phi_8 - \phi_8\bar\phi_2 - \phi_3\bar\phi_5 + \phi_5\bar\phi_3 + \phi_4\bar\phi_6 - \phi_6\bar\phi_4) = 1/4\varphi_3, \\
H_4 = F_{13} - F_{24} - F_{57} + F_{68} = 1/4(\phi_1\bar\phi_2 - \phi_2\bar\phi_1 + \phi_3\bar\phi_4 - \phi_4\bar\phi_3 + \phi_5\bar\phi_6 - \phi_6\bar\phi_5 - \phi_7\bar\phi_8 + \phi_8\bar\phi_7), = 1/4\varphi_4 \\
H_5 = F_{17} + F_{28} - F_{35} - F_{46} = 1/4(\phi_1\bar\phi_4 - \phi_4\bar\phi_1 + \phi_2\bar\phi_3 - \phi_3\bar\phi_2 - \phi_5\bar\phi_8 + \phi_8\bar\phi_5 + \phi_6\bar\phi_7 - \phi_7\bar\phi_6) = 1/4\varphi_5,
\end{aligned}
$$

$$H_6 = F_{14}+F_{23}-F_{58}-F_{67} = 1/4(-\phi_1\bar{\phi}_6+\phi_6\bar{\phi}_1-\phi_2\bar{\phi}_5+\phi_5\bar{\phi}_2-\phi_3\bar{\phi}_8+\phi_8\bar{\phi}_3+\phi_4\bar{\phi}_7-\phi_7\bar{\phi}_4) = 1/4\varphi_6,$$
$$H_7 = F_{18}-F_{27}+F_{36}-F_{45} = 1/4(\phi_1\bar{\phi}_8-\phi_8\bar{\phi}_1-\phi_2\bar{\phi}_7+\phi_7\bar{\phi}_2-\phi_3\bar{\phi}_6+\phi_6\bar{\phi}_3-\phi_4\bar{\phi}_5+\phi_5\bar{\phi}_4) = 1/4\varphi_7, \tag{4.5}$$

and it can be seen that

$$(\rho^+(F^+)\Phi, \Phi) = -1/2\sum_{i=1}^{7} H_i\varphi_i. \tag{4.6}$$

Let $e_{ijkl} = e_1^* \wedge e_j^* \wedge e_k^* \wedge e_l^*$. Then the self-dual 4-form $\Psi = \sum_{i=1}^{7} f_i \wedge f_i$ is

$$\begin{aligned}\Psi = 6[&e_{1234} - e_{1256} - e_{1278} - e_{1357} + e_{1368} - e_{1458} - e_{1467} \\ &+e_{5678} - e_{3478} - e_{3456} - e_{2468} + e_{2457} - e_{2367} - e_{2358}]\end{aligned} \tag{4.7}$$

and it can be seen that

$$\begin{aligned}*(F^2 \wedge \Psi) &= 3\sum_{i=1}^{7} H_i^2 - 3\sum_{i<j} F_{ij}^2 \\ &= -3(F^+, F^+) + 3(F, F).\end{aligned} \tag{4.8}$$

Finally

$$(\Phi \wedge \bar{\Phi}, \Phi \wedge \bar{\Phi}) = -\sum_{i<j}(\phi_i\bar{\phi}_j - \phi_j\bar{\phi}_i)^2 \tag{3.10}$$

$$\tag{4.9}$$

We modify the energy integral (2.10) as

$$E(A, \Phi) = \int_X -(\rho^+(F^+)\Phi, \Phi) + \alpha(\Phi \wedge \bar{\Phi}, \Phi \wedge \bar{\Phi}) + \beta(F, F)\text{dvol}. \tag{4.10}$$

Setting $H_i = k\varphi_i$, it can be seen that the integrand turns out to be

$$(-1/2k + \alpha + \beta k^2)(\Phi \wedge \bar{\Phi}, \Phi \wedge \bar{\Phi}) + 3 * (F^2 \wedge \Psi) \tag{4.11}$$

where the last term is a topological invariant and (taking $\beta = 1$) the first term is minimized for $k = 1/4$. It can be checked that this corresponds to the monopole equations (4.5).

Thus we have shown that the monopole equations given in [20] are minimizers of the action (4.10). However, as noted earlier, this action fails to relate to an analogue of the action (2.8) involving scalar curvature and the Dirac equation for Φ never comes into play.

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