Solution of Symmetric RLW Equation by the Meshless Kernel Based Method of Lines

Yılmaz Dereli*

Anadolu University, Science Faculty, Mathematics Department, 26470, Eskişehir, Türkiye
ydereli@anadolu.edu.tr
*Corresponding author / İletişimden sorumlu yazar

Geliş: 4th May (Mayıs) 2016
Kabul: 18th June (Haziran) 2016
DOI: http://dx.doi.org/10.18466/cbujos.92713

Abstract

The Symmetric Regularized Long Wave (SRLW) equation is solved numerically by using the meshless kernel based method of lines. This method is a way of approximating partial differential equations by ordinary differential equations. This used method is trully a meshless method. In computations, Multiquadric and Gaussian radial basis functions and Wendland's compactly supported radial basis functions are used as kernel functions. To test the applicality and reliability of the method different test problems which are single solitary wave motion, the interaction of two positive solitary waves and the clash of waves are studied. For single solitary wave motion whose analytical solution is known the root mean square error norm L_2 and maximum error norm L_∞ are calculated to test the efficiency and accuracy of the method. Also, the numerical values of invariants for all test problems are obtained to show the conservation properties. Simulations of waves for test problems are figured. Obtained numerical results are compared with the numerical results of some earlier papers in the literature. Obtained results are compeletely satisfactory and it is seen that values of invariants are preserved very well during the computations. It is shown that used method is an effective method with high accuracy.

Keywords - Kernel Based, Meshless Method, Method of Lines, Radial Basis Function, Symmetric RLW Equation

Ağsız Çekirdek Tabanlı Çizgiler Metoduyla Simetrik RLW Denkleminin Çözümü

Özet

Simetrik Regularized Long Wave (SRLW) denklemi ağsız çekirdek tabanlı çizgiler metodu ile sayısal olarak çözülür. Bu metot adi diferansiyel denklemlerle kısmi diferansiyel denklemlere yaklaşıldığı bir yöntemdir. Kullanılan bu metot tamamen ağsız bir metottur. Hesaplamalarda Multiquadric ve Gaussian radyal taban fonksiyonlarıyla Wendland'ın kompakt destekli radyal taban fonksiyonları çekirdek fonksiyonları olarak kullanılır. Farklı test problemleri olan tek solitari dalga hareketi, iki pozitif solitari dalganın etkileşimi ve dalgaların çarpışması problemleri metodun uygulanabilirliliğini ve güvenilirliğini test etmek için incelenir. Tek solitari dalga hareketinin analitik çözümü bilindiği için metodun etkinliğini ve doğruluğunu test etmek için ortalama karesel hata normu L_2 ve maksimum hata normu L_∞ hesaplanır. Ayrıca, bütün test problemleri için invaryantların sayısal değerleri elde edilir ve korunum özellikleri gösterilir. Test problemleri için dalgaların simulasyonu şekilde gösterilir. Elde edilen sayısal sonuçlar literatürde ki bazı makalelerin sayısal sonuçları ile karşılaştırılır. Elde edilen sonuçlar tamamen tatmin edicidir ve invaryant değerlerinin hesaplamalar boyunca çok iyi bir şekilde korunduğu görülmektedir. Kullanılan metodun yüksek doğruluklu etkili bir metot olduğu görülür.

Anahtar Kelimeler - Çekirdek Taban, Ağsız Metot, Çizgiler Metodu, Radyal Taban Fonksiyonu, Simetrik RLW denklem

1 Introduction

SRLW equation is written as first order system form

$$\begin{aligned} u_{xxt} - u_t = \rho_x + uu_x \\ \rho_t + u_x = 0 \end{aligned} \quad (1.1)$$

where subscripts t and x denote the partial derivatives with respect to these independent variables. In this system, ρ and u are the dimensionless electron charge density and the fluid velocity, respectively. Where, it is seen that derivative of u with respect to t is first order. Seyler and Fenstermacher [1] introduced this equation for the first time to describe as a model of the propagation of weakly nonlinear ion acoustic and space charge waves. Also the SRLW equation is used in [2,3,4,5]. The physical boundary conditions for the system require $u \to 0$ as $x \to \mp\infty$. To find the numerical solution, boundary conditions for the system (1.1) are given as follows:

$$\begin{gathered} u(a,t) = u(b,t) = 0, \qquad \rho(a,t) = \rho(b,t) = 0, \\ x \in [a,b], t \in [0,T] \end{gathered} \quad (1.2)$$

and initial conditions

$$u(x,0) = f(x), \ \rho(x,0) = g(x) \quad (1.3)$$

The SRLW equation is a symmetric version of the regularized long wave (RLW) equation which was proposed by Peregrine to describe the undular bore development [6]. The SRLW equation is explicitly symmetric in the x and t derivatives and it is very similar to RLW equation. Actually, eliminating ρ from the system (1.1) it is obtained following form SRLW equation:

$$u_{tt} - u_{xx} + \frac{1}{2}(u^2)_{xt} - u_{xxtt} = 0, \qquad x \epsilon \mathbb{R}, t > 0 \quad (1.4)$$

Solving of the SRLW equation has attracted the attention of scientists. Therefore, in the literature various theoretical and numerical studies have been presented. We have listed some of them.

In the reference [2] indicated that interactions of solitary waves were inelastic, thus the solitary wave of the SRLW equation is not soliton. The orbital stability and instability of solitary wave solutions of the generalized SRLW equations was studied in [7]. Also in [8], the influence of the interaction of the nonlinear terms on the orbital stability for the generalized SRLW equation was studied. The existence, uniqueness and regularity of numerical solutions for the periodic initial value problem of the generalized SRLW equations was investigated by spectral method in [9] and the error estimates were obtained.

A Fourier pseudo-spectral method with a restraint operator for the SRLW equation was presented in [10], and proved the stability and the optimal error estimates. In [11], the initial boundary value problem for SRLW equations with non homogenous boundary value was considered. In [12], conservative finite difference methods for the SRLW equation was presented and numerical solutions by using two-level and nonlinear implicit scheme, three-level and linear-implicit scheme and an uncoupled linear-implicit conservative scheme based on the finite difference methods were given.

Radial basis functions collocation method which is a meshless method was applied to the SRLW equation in order to find numerical solutions for different test problems in [13]. Some numerical results for the SRLW equation was presented in [14] by using the trigonometric integrator pseudospectral discretization method.

Application of Exp-function method to the SRLW equation was presented in [15] and generalized solution and periodic solution with some free parameters for the SRLW equation was obtained. Exact solutions of the SRLW equation was obtained in [16]. Analysis of Chebyshev pseduspectral method for multi-dimensional generalized SRLW equations was considered in [17] and fully discrete Chebyshev pseduspectral scheme was constructed.

A linear difference scheme for dissipative SRLW equations with damping term was studied in [18]. Therein a linear three-level implicit finite difference scheme was designed and some numerical results pubslished. Numerical simulation and convergence analysis of a high-order conservative difference Scheme for SRLW Equation was presented in [19]. Crank-Nicolson difference schemes for dissipative SRLW equations with damping terms was studied and nonlinear-implicit finite difference scheme was designed in [20]. Numerical solutions are obtained by using collocation of cubic B-splines finite element in [21].

In this study, the numerical solutions of the SRLW equations are calculated by means of the meshless kernel based method of lines (MKBMOL). In algorithms different kernel functions which are Multiquadric, Gaussian and Wendland's compactly supported radial basis functions are used. Therefore solitary wave solutions will be obtained for the different test problems.

The rest of this paper is organized as follows: in Section 2, we will explain the method. In Section 3, used kernel functions will be given. Then, in Section 4 several obtained numerical results, figures and comparisons are given and finally in Section 5 conclusion is given.

2 Numerical Method

In this study MKBMOL approach is used. This method is a way of approximating partial differential equations by ordinary differential equations. First of all, to solve the given equation a system of ordinary differential equations will be obtained by using the MKBMOL. Therefore there will no time discretization and artificial linearization of the differential equation as different from other numerical methods such as finite element, finite differences and radial basis function collocation method. So the problem will be automatically solved by using any ODE solver. The method uses time dependent coefficients for a linear combination of spatial trial functions [22] as follow:

$$u(x,t)=\sum_{j=1}^{n}\lambda_j(t)v_j(x) \quad (2.1)$$

with smooth functions λ_j on $[0,T]$, $1\le j\le N$. Where $\lambda_j(t)$ is unknown time dependent function to be determined at each time level as a column vector and $v_j(x)$ is an invertible matrix for kernel functions. This function is differentiated with respect to time and space variable easily.

Now, we will show the implementation of the MKBMOL to the SRLW equation. The approximate values of functions $u(x,t)$ and $\rho(x,t)$ in the equations system (1.1) are approached as follows:

$$u(x,t)=\sum_{j=1}^{n}\alpha_j(t)v_j(x)\,,\quad \rho(x,t)=\sum_{j=1}^{n}\beta_j(t)w_j(x) \quad (2.2)$$

Functions $u(x,t)$, $\rho(x,t)$ and their derivative functions are substituted in to the equation system (1.1) following simplified system is obtained:

$$\begin{aligned}&\sum_{j=1}^{n}\alpha_j'(t)v_j''(x)-\sum_{j=1}^{n}\alpha_j'(t)v_j(x)\\&=\sum_{j=1}^{n}\beta_j(t)w_j'(x)+\sum_{j=1}^{n}\alpha_j(t)v_j(x)\sum_{j=1}^{n}\alpha_j(t)v_j'(x),\\&\sum_{j=1}^{n}\beta_j'(t)w(x)+\sum_{j=1}^{n}\alpha_j(t)v_j'(x)=0\end{aligned} \quad (2.3)$$

For brevity, this equations system (2.3) are written in MATLAB notation as follows:

$$\begin{aligned}&V''*\alpha'(t)-V*\alpha'(t)=W'*\beta(t)+\big(V*\alpha(t)\big).*\big(V'*\alpha(t)\big)\,,\\&W*\beta'(t)+V'*\alpha(t)=0\end{aligned} \quad (2.4)$$

where the symbol * means the pointwise product. Also, V,V',V'', W and W′ are invertible matrices consisted of $v_j(x)$, $w_j(x)$ and their derivatives with respect to x. $\alpha(t),\alpha'(t)$, $\beta(t)$ and $\beta'(t)$ are vectors consisted of $\alpha_j(t)$ and its derivatives with respect to t. Therefore this obtained system can be written as follows:

$$\begin{aligned}&\alpha'(t)=(V''-V)^{-1}*\Big(W'*\beta(t)+\big(V*\alpha(t)\big).*\big(V'*\alpha(t)\big)\Big)\\&\beta'(t)=-W^{-1}*\big(V'*\alpha(t)\big)\end{aligned} \quad (2.5)$$

This equations system is solved by using any ODE solver.

3 Kernel Functions

In our algorithms as kernel functions we used Multiquadric [23], Gaussian which are globally supported and Wendland's compactly supported radial basis functions [24]. Definitions of mentioned kernel functions as follows:

$$\begin{aligned}&MQ:\ \phi(r)=\sqrt{(\varepsilon r)^2+1}\\&G:\ \phi(r)=\exp\left(-\frac{r^2}{\varepsilon^2}\right)\\&W:\phi(r)=(1-r)_+^n p(r)\end{aligned}$$

where $r=|x-x_j|$ is the Euclidean distance between collocation points x and x_j and ε is shape parameter. To obtain the optimal value of shape parameter we calculated the condition number of the kernel matrix as in [25]. In the Wendland's compactly supported radial basis functions p is a prescribed polynomial and following form of polynomial is used in our algorithms:

$$\phi_{7,5}(r) = (1-r)_+^{12}(9 + 108r + 566r^2 + 1644r^3 + 2697r^4 + 2048r^5)$$

4 Numerical Examples and Comparisons

In this section, some numerical examples will be presented for the SRLW equation by using MKBMOL. An efficient numerical scheme must keep the the conservation properties. Therefore it will be demostrated that the used method is conservative. In order to study this property we will evaluate the numerical values of invariants. The four invariants and some numerical results have been obtained in [1]. Described the four invariants for the equation as follows:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u(x,t)dx \\ I_2 &= \int_{-\infty}^{\infty} \rho(x,t)dx \\ I_3 &= \int_{-\infty}^{\infty} \left(u^2(x,t) + u_x^2(x,t) + \rho^2(x,t)\right)dx \\ I_4 &= \int_{-\infty}^{\infty} \left(u(x,t)\rho(x,t) + \frac{1}{6}u^3(x,t)\right)dx \end{aligned} \quad (4.1)$$

The single solitary wave motion of the SRLW equation has a known analytical solution, therefore to measure the accuracy of the method the difference between the analytic and numerical solutions will be computed. These computations will be done by using the following error norms:

$$L_2 = \sqrt{h\sum_{j=0}^{N}\left|u_j^{exact} - u_j^{num.}\right|^2} \quad (4.2)$$

$$L_\infty = \max_{0\le j\le N}\left|u_j^{exact} - u_j^{num.}\right| \quad (4.3)$$

The accuracy and performance of the method has been tested by following test problems.

4.1 Single solitary wave motion

The solitary wave solutions of the SRLW equations are defined as follows [1]:

$$\begin{aligned} u(x,t) &= \frac{3(c^2-1)}{c}\sec h^2\left(\sqrt{\frac{c^2-1}{4c^2}}(x-ct)\right) \\ \rho(x,t) &= \frac{3(c^2-1)}{c^2}\sec h^2\left(\sqrt{\frac{c^2-1}{4c^2}}(x-ct)\right) \end{aligned} \quad (4.4)$$

where c is the velocity and $c^2 > 1$, therefore the SRLW equation has the bidirectional propagation as depends upon sign of the its velocity. The simulation is carried out over the domain $-20 \le x \le 80$ in the time period $0 \le t \le 40$ with time step $\Delta t = 0.05$, space step $h = 0.5$ for value $c = \sqrt{2}$. A comparison with the earlier results of the computed values is shown in Tables 1 and 2. The motion of the single solitary wave at some times is depicted in Figure 1. It is seen that solitary wave moves to the right almost with unchanged in form. As seen from the Table 1, the invariants remained as unchanged at acceptable rate while time increases for all kernel functions. It is seen that the computed error norms in Table 2 are very acceptable when compared with other results in the literature. Comparisons showed that the method is very reliable for all kernel functions.

Table 1: Invariant values for a single wave motion

Method	I_1	I_2	I_3	I_4
analytic	12.000000	8.4852755	27.1529000	16.800000
G	11.9999928	8.4852763	27.1529004	16.799999
W	11.9999928	8.4852763	27.1529004	16.800000
MQ	11.9999928	8.4852277	27.1529003	16.799999
[13]	11.997488	8.487162	27.144679	16.794166
[14]	12.0000166	8.4852811	27.1529533	16.8000374

Table 2: Computed Error norms for a single wave motion

Method	L_2	L_∞
G	0.00000273	0.00000114
W	0.00000111	0.00000032
MQ	0.00538742	0.00115856
[13]	0.001770	0.000964

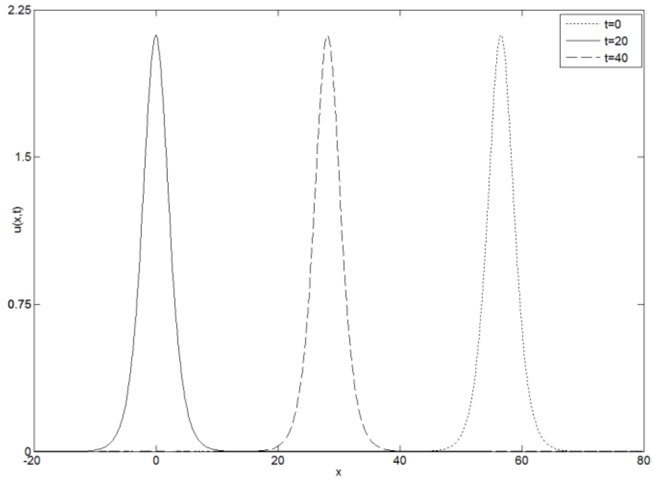


Figure 1: Motion of the single solitary wave

4.2 Interaction of Two Solitary Waves

Secondly, let us consider the initial condition

$$u(x,0) = u_1(x - x_0, 0) + u_2(x + x_0, 0)$$
$$u_1(x - x_0, 0) = \frac{3(c_1^2 - 1)}{c_1} \sec h^2 \left(\sqrt{\frac{c_1^2 - 1}{4c_1^2}} (x - x_0) \right)$$
$$\rho(x + x_0, 0) = \frac{3(c_2^2 - 1)}{c_2} \sec h^2 \left(\sqrt{\frac{c_2^2 - 1}{4c_2^2}} (x - x_0) \right) \quad (4.5)$$

and boundary conditions $u(-30,t) = u(120,t) = 0$. The initial condition for the $\rho(x,0)$ is calculated in a similar manner the initial value of $u(x,0)$. At this test problem, the interaction of two positive solitary waves is observed when time is increasing. The numerical constans are chosen as $\Delta t = 0.05$, $h = 0.5$, $c_1 = 2$, $c_2 = 6$ and $x_0 = 12$ and the program is run until the time $t \leq 16$. Initially the larger wave was placed on the left side of smaller one. Amplitudes of the larger wave and smaller wave were 17.5 and 4.5 and coordinates of peak positions were $x = -12$ and $x = 12$, respectively.
After, two solitary waves move to right with velocities depend upon their magnitudes. While time increasing the larger wave catches up the smaller one and has passed through. The distances between waves will be become longer as time increases because of their magnitudes. At the end of the running time, waves regained their original amplitudes.

Simulations of two solitary waves profiles are depicted in Figure 2. The computed values of invariants for all kernel functions are given in Table 3. For initial time, analytical values of invariants are given in the same table. Evaluated values of invariants at the end time are consistent with analytical values at the initial time. From this calculated results show that invariants are satisfactorily preserved.

Table 3: Invariant values for a single wave motion

Method	I_1	I_2	I_3	I_4
analytic	91.777566	22.2244642	1099.60235	2139.21715
G	91.7775660	22.2244642	1099.60235	2139.21754
W	91.7775660	22.2244642	1099.59270	2339.21754
MQ	91.7775660	22.2244642	1099.55932	2139.21715
[13]	89.7537827	22.2229073	1043.77440	1974.03372

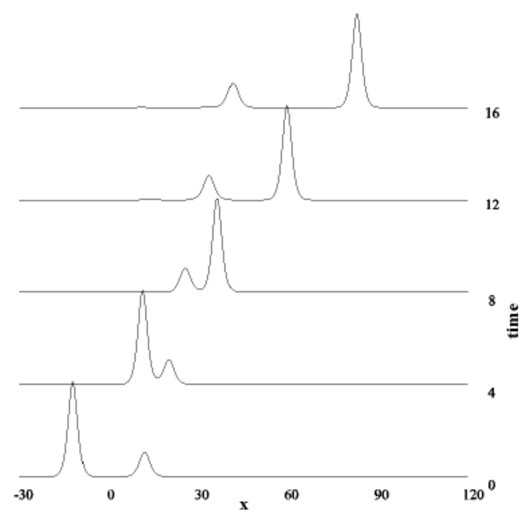


Figure 2: Simulation of interaction ot twowaves

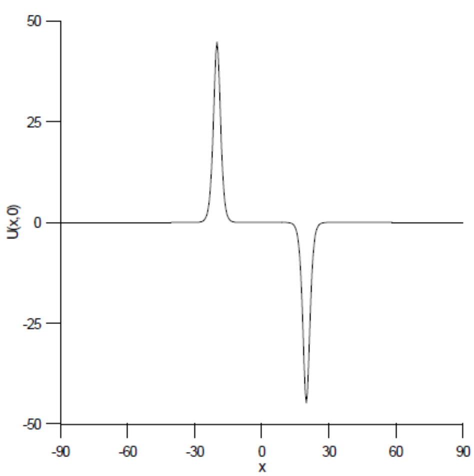
4.3 The clash of two solitary waves

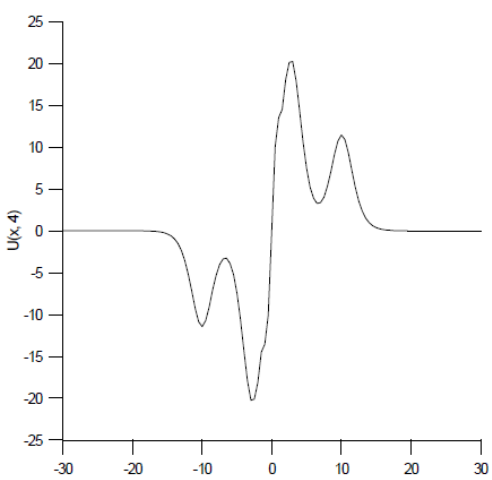
As a final test problem, we consider the clash of two solitary waves. Where two solitary waves are of exactly the same form but different signs move towards each other. For these problem we choose the solution domain as $-90 \leq x \leq 90$ and calculations will done up to $t = 12$ with time step $\Delta t = 0.05$ and space step $h = 0.5$. Initially, amplitude of the wave with positive is 44.8 and located at $x = -20$ and other one is located to $x = 20$ with amplitude -44.8. The clash of waves occurs time increases and new wave pairs areoccurred at the opposite directions.

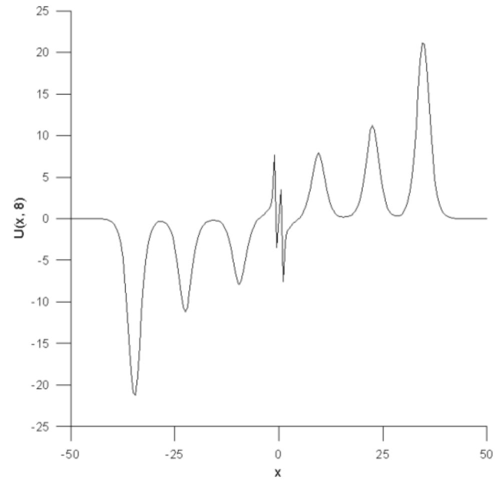
The program is run up to time $t = 12$ and three wave pairs which were the same form but different signs were observed. In Figure 3, profiles of the clash of waves are plotted. Computed invariants values are given in Table 4. Also at the same table analytical values of invariant were given for the initial time. The invariants for different types of kernel functions are satisfactorily preserved.

Table 4: Invariants for the clash of two solitary waves

Method	I_1	I_2	I_3	I_4
analytic	0	23.9466073	12911.84035	0
G	-6.113e-13	23.9466073	12911.84035	-5.264e-10
W	-3.467e-06	23.9466073	12911.84028	-4.516e-04
MQ	-3.180e-07	23.9466073	12911.83302	-9.587e-06
[13]	-0.0000087	23.9766076	12911.83791	-0.0073622







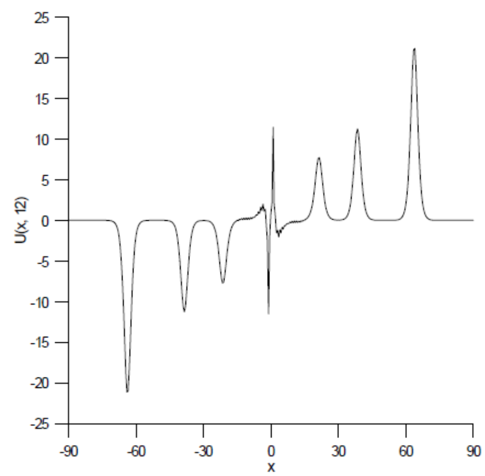


Figure 3: Simulation of the clash of two solitary waves

5 Conclusion

In this work, the meshless kernel based method of lines was used for the solution of the SRLW equation. Multiqaudric, Gaussian and Wendland's compactly supported radial basis functions were used as kernel functions in the algorithms. The efficiency of the method tested by three different test problems which were single solitary wave motion, the interaction of two positive solitary waves and the clash of waves were studied. The accuracy of the method was examined by the error norms for the single solitary wave motion. It was seen that computed error norms were very acceptable and reasonably small. The numerical values of invariants were calculated for all test problems. At the end of running time invariants remained almost unchanged. The numerical method successfully provides very accurate solutions. The numerical results showed that the meshless kernel based method of lines is very effective and can be applied to these kinds of nonlinear partial differential equations systems.

6 Acknowledgements

This work is supported by the Scientific Research Projects of Anadolu University (No: 1505F314) .

7 References

[1] Seyler, C.E.; Fenstermacher, D.L. A Symmetric Regularized-Long-Wave Equation. Physics of Fluids, 1984; 27(1), 1-15.

[2] Bogolubsky, J.L. Some Examples of Inelastic Soliton Interaction. Computer Physics Communications. 1977; 13(3), 149-155.

[3] Carlson, P.A. New Similarity Reductions and Painleve Analysis for the Symmetric Regularized Long Wave and Modified Benjamin-Bona-Mahoney Equations. Journal of Physics A: Mathematical and General. 1989; 22(18), 3821-3848.

[4] Rosenau, P. Evolution and Breaking of Ion-Acoustic Waves. Physics of Fluids. 1988; 31(6), 1317-1319.

[5] Shivamoggi, B.K. A Symmetric Regularized Long Wave Equation for Shallow Water Waves. Physics of Fluids. 1986; 29(3), 890-891.

[6] Peregrine, D.H. Calculations of the Development of an Undular Bore. Journal of Fluid Mechanics. 1966; 25(2), 321-330.

[7] Chen, L. Stability and Instability of Solitary Waves for Generalized Symmetric Regularized Long Wave Equation. Physica D. 1998; 118(1-2), 53-68.

[8] Zhang, W.; Chen, X.; Li, Z.; Zhang, H. Orbital Stability of Solitary Waves for Generalized Symmetric Regularized-Long-Wave Equations with Two Nonlinear Terms. Journal of Applied Mathematics. 2014; 2014, 1-16.

[9] Guo, B. The Spectral Method for Symmetric Regularized Wave Equations. Journal of Computational Mathematics. 1987; 5(4), 297-306.

[10] Zheng, J.D.; Zhang, R.F.; Guo, B.Y. The Fourier Pseudo-spectral Method for SRLW Equation. Applied Mathematics and Mechanics. 1989; 10(9), 801-810.

[11] Miao, C. The Initial Boundary Value Problem for Symmetric Long Wave Equations with Non-homogeneous Boundary Value. Northeastern Mathematics Journal. 1994; 10(4), 463-472.

[12] Wang, T.; Zhang, L.; Chen, F. Conservative Schemes for the Symmetric Regularized Long Wave Equations. Applied Mathematics and Computation. 2007; 190, 1063-1080.

[13] Kaplan, A.G.; Dereli, Y. Numerical Solutions of the Symmetric Regularized Long Wave Equation Using Radial Basis Functions. Computer Modeling in Engineering and Sciences. 2012, 84(4), 423-438.

[14] Dong, X. Numerical Solutions of The Symmetric Regularized-Long-Wave Equation by Trigonometric Integrator Pseudospectral Discretization. http://arxiv.org/abs/1109.0764.

[15] Xu, F. Application of Exp-function method to Symmetric Regularized Long Wave (SRLW) equation. Physics Letter A. 2008; 372(3), 252-257.

[16] Mhlanga, I. E.; Khalique, C. M. Exact Solutions of the Symmetric Regularized Long Wave Equation and the Klein-Gordon-Zakharov Equations. Abstract and Applied Analysis. 2014; 2014, 1-7.

[17] Shang, Y.; Guo, B. Analysis of Chebyshev Pseduspectral Method for Multi-Dimensional Generalized SRLW Equations. Applied Mathematics and Mechanics. 2003; 24(10), 1168-1183.

[18] Hu, J.; Xu, Y.; Hu, B. A Linear Difference Scheme for Dissipative Symmetric Regularized Long Wave Equations with Damping Term. Boundary Value Problems. 2010; 2010, 1-16.

[19] Hu, J.; Zheng, K.; Zheng, M. Numerical Simulation and Convergence Analysis of a High-order Conservative Difference Scheme for SRLW Equation. Applied Mathematical Modelling. 2014; 38, 5573-5581.

[20] Hu, J.; Hu, B.; Xu, Y. C-N Difference Schemes for Dissipative Symmetric Regularized Long Wave Equations with Damping Term. Mathematical Problems in Engineering. 2011; 2011, 1-16.

[21] Mittal, R.C.; Tripathi, A. Numerical Solutions of Symmetric Regularized Long Wave Equations Using Collocation of Cubic B-splines Finite Element. International Journal for Computational Methods in Engineering Science and Mechanics. 2015; 16, 142-150.

[22] Schaback, R. The Meshless Kernel-Based Method of Lines for Solving Nonlinear EvolutionEquations. Preprint, Göttingen, 2008.

[23] Hardy, R.L. Multiquadric Equations of Topography and Other Irregular Surfaces. Journal of Geophysical Research. 1971; 76(8), 1905-1915.

[24] Wendland, H. Piecewise Polynomial Positive Definite and Compactly Supported Radial Functions of Minimal Degree. Advances in Computational Mathematics. 1995; 4(1), 389-396.

[25] Schaback, R. Error Estimates and Condition Numbers or Radial Basis Function Interpolation. Advances in Computational Mathematics. 1995; 3(3), 251-264.