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Geodesics in $(\mathbb{R}^n, d_1)$

Mehmet KILIÇ *[1]

[1]Anadolu Üniversitesi, Fen Fakültesi, Matematik Bölümü, 26470, Eskişehir

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Abstract: The notion of geodesic, which may be regarded as an extension of the line segment in Euclidean geometry to the space we study in, has an important place in many branches of geometry, such as Riemannian geometry, Metric geometry, to name but a few. In this article, the concept of geodesic in a metric space will be introduced, then geodesics in the space $(\mathbb{R}^n, d_1)$ will be characterized. Furthermore, some examples will be presented to demonstrate the effectiveness of the main result.

$(\mathbb{R}^n, d_1)$'de Jeodezikler

Anahtar Kelimeler
Metrik uzay,
Yol,
Jeodezik,
Kuadrant

Özet: Öklid geometrisindeki bir doğru parçasının içinde çalıştığımız uzaya genelleştirilmesi olarak görülebilecek olan jeodezik kavramı, geometrinin bir çok dalında (Riemann Geometrisi, Metrik geometri vb.) önemli bir yere sahiptir. Bu çalışmada bir metrik uzay içinde jeodezik kavramının nasıl tanımlandığı açıklandıktan sonra $(\mathbb{R}^n, d_1)$ içindeki jeodezikler karakterize edilecektir. Ayrıca asıl sonucun etkisini göstermek için bir takım örnekler sunulacaktır.

1. Introduction

Let (X,d) be a metric space, $\alpha : [a,b] \to X$ be a path and $a = t_0 < t_1 < \cdots < t_n = b$ is an arbitrary partition of the interval $[a,b]$. Then, the length of α is defined as

$$\sup_{\mathscr{P}} \left\{ \sum_{i=1}^{n} d(\alpha(t_{i-1}), \alpha(t_i)) \right\}$$

over all partitions $P = \{t_0 = a, t_1, \ldots, t_n = b\}$ of $[a,b]$ and it is denoted by $L(\alpha)$. If α satisfies $L(\alpha) = d(x,y)$, where $\alpha(a) = x$, $\alpha(b) = y$, then α is called a geodesic [1],[2]. For a given metric space and any two points in it, there may not exist any geodesics between these two points. For example, $S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ with the induced standard metric and for $p = (1,0)$ and $q = (-1,0)$, there is no path connecting these points, whose length is less than π, but the distance between p and q is equal to 2 according to the standard metric. If there is at least one geodesic between any two points in a metric space, this metric space is called "geodesic space" according to Papadopoulos [1] or "strictly intrinsic space" according to Burago [3]. Hence, S^1 is not a geodesic space with the induced standard metric, but it becomes a geodesic space with the "arc length metric". Since the line between any two points in $\mathbb{R}^n$ is obviously a geodesic according to the metric d_1 where

$$d_1((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = |x_1 - y_1| + \cdots + |x_n - y_n|,$$

$(\mathbb{R}^n, d_1)$ is also a geodesic space. But in that space, there are too many geodesics different from lines [4]. So far, there is no survey about these geodesics. It would therefore be desirable to determine them. In this work, we give a necessary and sufficient condition which identifies the geodesics between any two points of this space.

2. Geodesics in the Space $(\mathbb{R}^n, d_1)$

Definition 1 *For $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ and $\varepsilon_i = \pm$, $i = 1, 2, \ldots, n$, we define*

$$Q_p^{\varepsilon_1 \cdots \varepsilon_n} = \{(q_1, \ldots, q_n) \in \mathbb{R}^n \,|\, \varepsilon_i (q_i - p_i) \geq 0,\ i = 1, 2, \ldots, n\}$$

which we call $\varepsilon_1 \ldots \varepsilon_n$−quadrant of the point p.

Note that the equality

$$Q_p^{\varepsilon_1 \cdots \varepsilon_n} = \{q = (q_1, \ldots, q_n) \in \mathbb{R}^n \,|\, d_1(p,q) = \sum_{i=1}^{n} \varepsilon_i (q_i - p_i)\}$$

obviously holds. There are exactly 2^n quadrants in $\mathbb{R}^n$. All quadrants of a point p in $\mathbb{R}^2$ are shown in Figure 1 and the quadrant Q_O^{+++} of the origin in $\mathbb{R}^3$ is shown in Figure 2.

Theorem 1 *Let $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ be two points, $q \in Q_p^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$ and $\alpha : [a,b] \to \mathbb{R}^n$ be a path such that $\alpha(a) = p$ and $\alpha(b) = q$. Then α is a geodesic in $(\mathbb{R}^n, d_1)$ if and only if $\alpha(t') \in Q_{\alpha(t)}^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$ for all $t, t' \in [a,b]$ such that $t < t'$.*

* Corresponding author: kompaktuzay@gmail.com

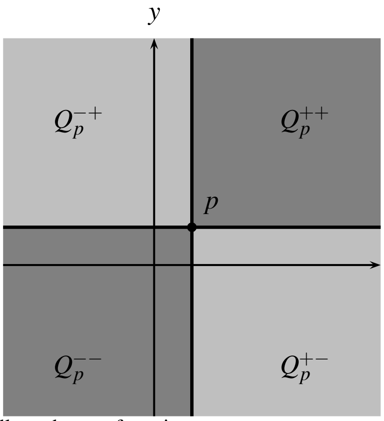

Figure 1. All quadrants of a point p.

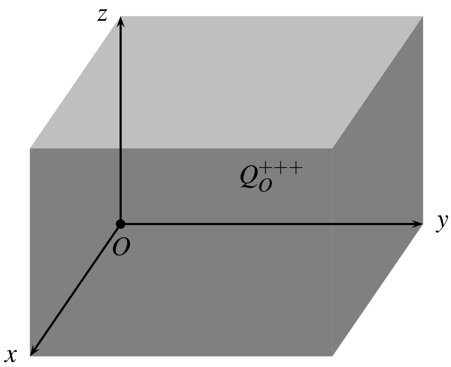

Figure 2. The quadrant Q_O^{+++} of the origin.

Proof. $(\Rightarrow)$ Assume that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a geodesic but $\alpha(t') \notin Q_{\alpha(t)}^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$ for some $t < t'$. Then we have

$$d_1(\alpha(t), \alpha(t')) > \sum_{i=1}^{n} \varepsilon_i(\alpha_i(t') - \alpha_i(t))$$

Since $q \in Q_p^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$, it holds that

$$d_1(p,q) = \sum_{i=1}^{n} \varepsilon_i(q_i - p_i).$$

On the other hand, for the partition $a < t < t' < b$ of $[a,b]$, we get

$$\begin{aligned} & d_1(\alpha(a), \alpha(t)) + d_1(\alpha(t), \alpha(t')) + d_1(\alpha(t'), \alpha(b)) \\ > & \sum_{i=1}^{n} \varepsilon_i(\alpha_i(t) - \alpha_i(a)) + \sum_{i=1}^{n} \varepsilon_i(\alpha_i(t') - \alpha_i(t)) + \\ & \sum_{i=1}^{n} \varepsilon_i(\alpha_i(b) - \alpha_i(t')) \\ = & \sum_{i=1}^{n} \varepsilon_i(\alpha_i(b) - \alpha_i(a)) \\ = & \sum_{i=1}^{n} \varepsilon_i(q_i - p_i) \\ = & \; d_1(p,q). \end{aligned}$$

This inequality implies $L_{d_1}(\alpha) > d_1(p,q)$ which is a contradiction yielding the required result. $(\Leftarrow)$ Let $a = t_0 < t_1 < \cdots < t_m = b$ be an arbitrary partition of $[a,b]$. Since $\alpha(t_j) \in Q_{\alpha(t_{j-1})}^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$ for all $j = 1, 2, \cdots, m$, the equality

$$d_1(\alpha(t_j), \alpha(t_{j-1})) = \sum_{i=1}^{n} \varepsilon_i(\alpha_i(t_j) - \alpha_i(t_{j-1})).$$

is satisfied. Thus, we have

$$\begin{aligned} \sum_{j=1}^{m} d_1(\alpha(t_j), \alpha(t_{j-1})) &= \sum_{j=1}^{m} \sum_{i=1}^{n} \varepsilon_i(\alpha_i(t_j) - \alpha_i(t_{j-1})) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_i(\alpha_i(t_j) - \alpha_i(t_{j-1})) \\ &= \sum_{i=1}^{n} \varepsilon_i(\alpha_i(b) - \alpha_i(a)) \\ &= \sum_{i=1}^{n} \varepsilon_i(q_i - p_i) \\ &= d_1(p,q). \end{aligned}$$

This implies $L_{d_1}(\alpha) = d_1(p,q)$. Hence α is a geodesic. ■

Example 1 *Let the path α be defined as $\alpha : [-1,1] \to (\mathbb{R}^2, d_1)$, $\alpha(t) = (-2t, e^t)$, $p = \alpha(-1) = (2, \frac{1}{e})$ and $q = \alpha(1) = (-2, e)$. It is clear that $q \in Q_p^{-+}$. Furthermore, for $-1 \leq t < t' \leq 1$, the following inequalities hold:*

$$-(-2t' - (-2t)) \geq 0$$

and

$$e^{t'} - e^t \geq 0.$$

This implies that $\alpha(t') \in Q_{\alpha(t)}^{-+}$ for all $-1 \leq t < t' \leq 1$, it follows that α is a geodesic (see Figure 3).

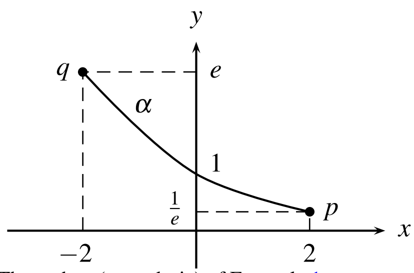

Figure 3. The path α (a geodesic) of Example 1.

Example 2 *Let the path α be defined as $\alpha : [0, \frac{3\pi}{2}] \to (\mathbb{R}^2, d_1)$, $\alpha(t) = (t, \sin t)$, $p = \alpha(0) = (0,0)$ and $q = \alpha(\frac{3\pi}{2}) = (\frac{3\pi}{2}, -1)$. It is obvious that $q \in Q_p^{+-}$. However, $\alpha(t') \notin Q_{\alpha(t)}^{+-}$ for $t = 0$ and $t' = \frac{\pi}{2}$ because $-(\sin \frac{\pi}{2} - \sin 0) < 0$. Therefore, the path α is not a geodesic (see Figure 4).*

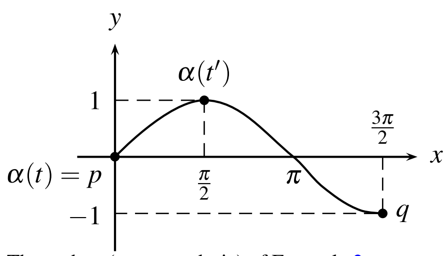

Figure 4. The path α (not a geodesic) of Example 2.

Theorem 1 can be restated as follows: Let $q \in Q_p^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$. Then the path α between p and q is a geodesic if and only if when the quadrants $Q_{(.)}^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$ travel on the image of α, the rest of the path is contained from the quadrant at every point (see Figure 5).

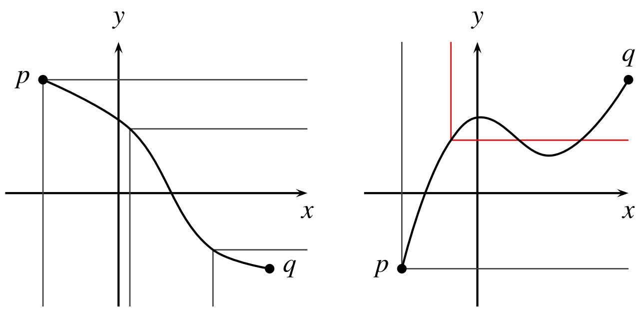

Figure 5. Two paths between p and q in $(\mathbb{R}^2, d_1)$ which one of them (the left) is a geodesic but the other is not a geodesic.

By means of Theorem 1, when the graph of a path is given in $(\mathbb{R}^2, d_1)$ or $(\mathbb{R}^3, d_1)$, we can figure out whether this path is a geodesic or not. Nevertheless, if the equation of a path is given in $(\mathbb{R}^n, d_1)$, to make a decision whether this path is a geodesic still presents some difficulty. Let us investigate Theorem 1 more closely: Let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be two points in $\mathbb{R}^n$, $q \in Q_p^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : [a,b] \to \mathbb{R}^n$ be a path connecting these points ($\alpha(a) = p$, $\alpha(b) = q$). Then, according to Theorem 1, α is a geodesic in $(\mathbb{R}^n, d_1)$ if and only if

$$t < t' \Rightarrow \alpha(t') \in Q_{\alpha(t)}^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$$

for all $t, t' \in [a,b]$. Note that $\alpha(t') \in Q_{\alpha(t)}^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}$ if and only if

$$\varepsilon_i(\alpha_i(t') - \alpha_i(t)) \geq 0$$

for all $i = 1, 2, \ldots, n$ by the definition of a quadrant. Therefore, α is a geodesic in $(\mathbb{R}^n, d_1)$ if and only if

$$t < t' \;\Rightarrow\; \varepsilon_i(\alpha_i(t') - \alpha_i(t)) \geq 0 \tag{1}$$

for all $i = 1, 2, \ldots, n$ and $t, t' \in [a,b]$. Note that, in equation (1), if ε_i is "+", then the component function α_i is non-decreasing and, likewise, if ε_i is "−", then the component function α_i is non-increasing. Thus, we get the following corollary:

Corollary 1 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : [a,b] \to \mathbb{R}^n$ is a path. Then α is a geodesic in $(\mathbb{R}^n, d_1)$ if and only if all component functions α_i of α are non-decreasing or non-increasing on the interval $[a,b]$.*

Note that the path in Example 1 is a geodesic since its first component function is non-increasing and its second component function is non-decreasing. However, the path in Example 2 is not a geodesic because its second component function is neither non-increasing nor non-decreasing on its domain. Finally, we now give an example in $\mathbb{R}^4$:

Example 3 $\alpha : [-1,1] \to \mathbb{R}^4$, $\alpha(t) = (1-2t, 1+t^2, e^t, t^3-1)$ *is not a geodesic since the second component function of α is neither non-decreasing nor non-increasing on the interval* $[-1,1]$.

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