Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 812 – 818

doi:10.3906/mat-1303-34

Seiberg–Witten-like equations on 5-dimensional contact metric manifolds**

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Received: 18.03.2013 • **Accepted:** 26.02.2014 • **Published Online:** 01.07.2014 • **Printed:** 31.07.2014

Abstract: In this paper, we write Seiberg–Witten-like equations on contact metric manifolds of dimension 5. Since any contact metric manifold has a Spin^c-structure, we use the generalized Tanaka–Webster connection on a Spin^c spinor bundle of a contact metric manifold to define the Dirac-type operators and write the Dirac equation. The self-duality of 2-forms needed for the curvature equation is defined by using the contact structure. These equations admit a nontrivial solution on 5-dimensional strictly pseudoconvex CR manifolds whose contact distribution has a negative constant scalar curvature.

Key words: Seiberg–Witten equations, spinor, Dirac operator, contact metric manifold, self-duality

1. Introduction

Seiberg–Witten equations were defined on 4-dimensional Riemannian manifolds by Witten in [14]. The solution space of these equations gives differential topological invariants for 4-manifolds [1, 11]. Some generalizations were given later on higher dimensional manifolds [4, 7, 10].

Seiberg–Witten equations consist of 2 equations. The first is the Dirac equation, which is meaningful for the manifolds having Spin^c−structure. The second is the curvature equation, which couples the self-dual part of a connection 2-form with a spinor field. In order to be able to write down the curvature equation, the notion of the self-duality of a 2-form is needed. This notion is meaningful for 4-dimensional Riemannian manifolds. On the other hand, there are similar self-duality notions for some higher dimensional manifolds [5, 13]. In the present paper, we propose Seiberg–Witten-like equations for 5-dimensional contact metric manifolds by using the Spin^c-structure and the notion of self-duality given in [12] and [3], respectively.

The paper is organized as follows. We begin with a section introducing some basic facts concerning contact metric manifolds. In the following section, we study self-dual 2-forms on 5-dimensional contact metric manifolds. In Section 4, we discuss the Spin^c-structures and Dirac-type operators associated to the generalized Tanaka–Webster connection. In the final section we propose the Dirac and curvature equations and hence write Seiberg–Witten-like equations on contact metric manifolds of dimension 5. Finally, we obtain a special solution for these equations on the 5-dimensional strictly pseudoconvex CR manifolds whose contact distribution has a negative constant scalar curvature.

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2010 *AMS Mathematics Subject Classification:* 15A66, 58Jxx.
** This paper was supported by Project 1105F099 of the Scientific Research Foundation of Anadolu University.

2. Contact metric manifolds

A contact form on a smooth manifold M of dimension $(2n+1)$ is a 1-form η such that $\eta\wedge(d\eta)^n \neq 0$ everywhere on M. The contact form η induces a hyperplane subbundle H of the tangent bundle TM given by $H = Ker\,\eta$. The Reeb vector field associated to η is the vector field ξ uniquely determined by $\eta(\xi) = 1$ and $d\eta(\xi, .) = 0$. Then (M, η) is called a contact manifold.

Note that given $H = Ker\,\eta$ and ξ such that $\eta(\xi) = 1$, we can split the tangent bundle into $TM = H \oplus \mathbb{R}\xi$. If X is any vector field on M, then X decomposes as $X = X_H + f\xi$ for any $f \in C^\infty(M, \mathbb{R})$. X_H is called the horizontal part of X.

If (M, η) is a contact manifold, then the pair $(H, d\eta|_H)$ is a symplectic vector bundle. We fix an almost complex structure J_H on H compatible with $d\eta|_H$, i.e. $d\eta|_H(J_H(X), J_H(Y)) = d\eta|_H(X, Y)$. We can extend J_H to an endomorphism J of the tangent bundle TM by setting $J\xi = 0$. The relation $J^2 = -Id + \eta\otimes\xi$ then holds. With this in mind, g_η, given by

$$g_\eta(X, Y) = d\eta(X, JY) + \eta(X)\eta(Y),$$

defines a Riemannian metric on TM. The metric g_η is called a Webster metric and is said to be associated to η. Moreover, the following relations hold:

$$g_\eta(\xi, X) = \eta(X), \quad g_\eta(JX, Y) = d\eta(X, Y), \quad g_\eta(JX, JY) = g_\eta(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$. We call $(M, g_\eta, \eta, \xi, J)$ a contact metric manifold. For detailed information, see [2, 12].

The generalized Tanaka–Webster connection ∇ is a well-known connection on the contact metric manifold $(M, g_\eta, \eta, \xi, J)$. This connection satisfies the conditions $\nabla\eta = 0$ and $\nabla g_\eta = 0$. Moreover, if J is integrable, i.e. $\nabla J = 0$, then the contact metric manifold $(M, g_\eta, \eta, \xi, J)$ is called a strictly pseudoconvex CR manifold [12].

3. Self-dual 2-forms on 5-dimensional contact metric manifolds

Let $(M, g_\eta, \eta, \xi, J)$ be a 5-dimensional contact metric manifold. The p-form α is called a horizontal p-form if $i(\xi)\alpha = 0$ where i is contraction operator. For any 2-form $\alpha \in \Omega^2(M)$ we have the splitting $\alpha = \alpha_H + \alpha_\xi$ where $\alpha_H = \alpha\circ\Pi$, $\Pi : TM \to H$ is the canonical projection and $\alpha_\xi = \eta\wedge i(\xi)\alpha$. The decomposition of $\Omega^2(M)$ is then given by

$$\Omega^2(M) = \Omega^2_H(M) \oplus \eta\wedge\Omega^1_H(M), \tag{1}$$

where $\Omega^2_H(M)$ and $\Omega^1_H(M)$ are the bundles of horizontal forms. Moreover, any horizontal 2-form can be split into its self-dual and anti-self dual parts as follows.

Let $*$ be the Hodge-star operator acting on the cotangent bundle T^*M. We can define the operator

$$\star : \Omega^2(M) \to \Omega^2(M), \quad \star(\beta) := *(\eta\wedge\beta).$$

We can restrict the operator $\star$ to the space of horizontal 2-forms $\Omega^2_H(M)$:

$$\star_H : \Omega^2_H(M) \to \Omega^2_H(M), \quad \star_H(\beta) := *(\eta\wedge\beta).$$

This operator satisfies $\star_H^2 = id$. Then we have the following orthogonal decomposition:

$$\Omega^2_H(M) = \Omega^2_H(M)^+ \oplus \Omega^2_H(M)^-, \tag{2}$$

where $\Omega^2_H(M)^{\pm}$ is the eigenspace associated to eigenvalue ± 1 of the operator $\star_H$. The eigenspace $\Omega^2_H(M)^+$ is called as the space of self-dual 2-forms. In a similar way, the eigenspace $\Omega^2_H(M)^-$ is called the space of anti-self-dual 2-forms (see [3, 8]). From equalities (1) and (2), we have

$$\Omega^2(M) = \Omega^2_H(M)^+ \oplus \Omega^2_H(M)^- \oplus \eta \wedge \Omega^1_H(M).$$

Hence, any 2-form α can be written as $\alpha = \alpha^+_H + \alpha^-_H + \eta \wedge \beta$ where β is a 1-form on H. The self-dual part of α is defined as the self-dual part of α_H, i.e. $\alpha^+ := \alpha^+_H$.

Locally, we can specify the self-dual and anti-self-dual 2-forms. For this, choose a local orthonormal frame field $\{e_1, e_2 = J(e_1), e_3, e_4 = J(e_3), \xi\}$ and denote by $\{e^1, e^2, e^3, e^4, \eta\}$ the dual basis. From (2), the 2-form $d\eta$ has the form $d\eta = e^1 \wedge e^2 + e^3 \wedge e^4$. The forms $e^1 \wedge e^2 + e^3 \wedge e^4$, $e^1 \wedge e^3 - e^2 \wedge e^4$ and $e^1 \wedge e^4 + e^2 \wedge e^3$ are an orthonormal basis for $\Omega^2_H(M)^+$. An orthonormal basis for $\Omega^2_H(M)^-$ is given by the forms $e^1 \wedge e^2 - e^3 \wedge e^4$, $e^1 \wedge e^3 + e^2 \wedge e^4$, and $e^1 \wedge e^4 - e^2 \wedge e^3$.

4. Dirac operators on contact metric manifolds

In this section we will describe Dirac operators on contact metric manifolds. For this, we need a Spin^c-structure. Any contact metric manifold admits a canonical Spin^c-structure. Then we have a Spin^c-bundle $P_{Spin^c(2n)}$, an S^1-bundle P_{S^1}, and the canonical line bundle $\mathcal{L}$. The spinor bundle S can be identified with the bundle $\wedge^{0,*}_H M$ of the $(0,*)$ forms. For the definitions and more details about these notions, we refer to [12]. For our purpose, we use the following representation of the complex Clifford algebra $\mathbb{C}l_5$:

$$\kappa(e_1) = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \ \kappa(e_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\kappa(e_3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \ \kappa(e_4) = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\kappa(e_5) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \ \kappa(d\eta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}.$$

Let $(M, g_\eta, \eta, \xi, J)$ be a contact metric manifold equipped with a $Spin^c$-structure. Each unitary connection A on $\mathcal{L}$ induces a spinorial connection ∇^A on S with the generalized Tanaka–Webster connection ∇. The Kohn–Dirac operator D^A_H is defined as follows:

$$D^A_H = \sum_{i=1}^{2n} \kappa(e_i)(\nabla^A_{e_i}),$$

where $\{e_i\}$ is a local orthonormal frame of H. The Dirac operator D_A is defined by

$$D_A = D^A_H + \xi \cdot \nabla^A_\xi$$

(see also [12]).

5. Seiberg–Witten-like equations on 5-dimensional contact metric manifolds

In [6], Seiberg–Witten-like equations on 5-dimensional Euclidean space $\mathbb{R}^5$ were written. In this section, we will write Seiberg–Witten-like equations on 5-dimensional contact metric manifolds and give a solution to these equations on strictly pseudoconvex CR manifolds.

For a spinor ψ we define a 2-form $\sigma(\psi)$ by the following formula:

$$\sigma(\psi)(X,Y) = < X \cdot Y \cdot \psi, \psi > + g_\eta(X,Y)|\psi|^2,$$

where $X, Y \in \Gamma(TM)$ and $<,>$ is the Hermitian inner product on the spinor space S. Note that $\sigma(\psi)$ is an imaginary valued 2-form. The restriction of $\sigma(\psi)$ to H is denoted by $\sigma_H(\psi)$.

Definition 1 *Let $(M, g_\eta, \eta, \xi, J)$ be a contact metric 5-manifold. Fix a Spin^c-structure and a connection A in the $U(1)$-principal bundle associated with the Spin^c-structure. For any $\psi \in \Gamma(S)$ Seiberg–Witten equations are defined by*

$$\begin{aligned} D_A(\psi) &= 0, \\ F_A^+ &= -\frac{1}{4}\sigma(\psi)^+, \end{aligned} \tag{3}$$

where F_A^+ is the self-dual part of the curvature F_A and $\sigma(\psi)^+$ is the self-dual part of the 2-form $\sigma(\psi)$.

Now we give a solution for Seiberg–Witten equations in dimension 5. To do this, we follow the method given in [9]. From now on we suppose that $(M, g_\eta, \eta, \xi, J)$ is a strictly pseudoconvex CR manifold.

Let (M, g_η) be a contact metric manifold endowed with Spin^c-structure. The spinor bundle is then $S = \wedge_H^{0,*}(M)$. Namely,

$$S = \wedge_H^{0,2}(M) \oplus \wedge_H^{0,1}(M) \oplus \wedge_H^{0,0}(M),$$

where $\wedge_H^{0,2}(M)$ is the eigenspace corresponding to the eigenvalue $2i$ of the mapping $\kappa(d\eta) : S \to S$ and has dimension 1, $\wedge_H^{0,1}(M)$ is the eigenspace corresponding to the eigenvalue 0 of the mapping $\kappa(d\eta) : S \to S$ and has dimension 2, and $\wedge_H^{0,0}(M)$ is the eigenspace corresponding to the eigenvalue $-2i$ of the mapping $\kappa(d\eta) : S \to S$ and has dimension 1.

If $\psi_0 \in \wedge_H^{0,0}(M)$, then ψ_0 denotes the spinor corresponding to the constant function 1 in the chosen coordinates

$$\psi_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Moreover, we have $d\eta \cdot \psi_0 = -2i\psi_0$. By using the expression of $\sigma_H(\psi)$ in the local coordinates, we obtain the following identity:

$$\sigma_H(\psi_0) = -id\eta. \tag{4}$$

5.1. Some identities

In this part, we collect some identities needed for the special solution of Seiberg–Witten equations.

When M is a strictly pseudoconvex CR manifold, M also has a complex CR structure [2]. Let $\{Z_1, \ldots, Z_n\}$ be a local unitary frame of $T^{1,0}$ over $U \subset M$ where $Z_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - \sqrt{-1}Je_\alpha)$, $1 \leq \alpha \leq n$. Let us denote by $\omega := (\omega_{\alpha\beta})$ the matrix of the connection form of ∇ with respect to the frame. Then we can write the following:

$$\nabla Z_\alpha = \sum_\beta \omega_{\alpha\beta} Z_\beta.$$

$\{Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, \xi\}$ is a local frame of the complexified tangent bundle $TM^{\mathbb{C}}$ over U.
Let $\{\theta^1, \ldots, \theta^n, \overline{\theta}^1, \ldots, \overline{\theta}^n, \eta\}$ be the corresponding dual basis. Thus,

$$\zeta = \overline{\theta}^1 \wedge \ldots \wedge \overline{\theta}^n : U \to \Lambda_H^{0,n}(M)$$

is a local section in determinant line bundle $\Lambda_H^{0,n}(M)$. The Webster connection ∇ defines a covariant derivative in the canonical line bundle $\Lambda_H^{0,n}(M)$ such that

$$\nabla(\overline{\theta}^1 \wedge \ldots \wedge \overline{\theta}^n) = -Tr(\omega)\overline{\theta}^1 \wedge \ldots \wedge \overline{\theta}^n.$$

Since ∇ is a metric with respect to g_η, the trace $Tr(\omega)$ is purely imaginary. Therefore, this connection ∇ in $\Lambda_H^{0,n}(M)$ induces a connection on the associated S^1-principal bundle P_{S^1}. Let us denote this connection by A. Then,

$$\zeta^* A = -Tr\ \overline{\omega} = Tr\omega$$

is a local connection form on S^1-bundle P_{S^1}. Let F_A be the curvature form of the connection A. The curvature form F_A is a 2-form on M with values in $\mathrm{i}\mathbb{R}$. Over $U \subset M$ we have

$$F_A = dA = Trd\omega. \tag{5}$$

Moreover,

$$Ric(X, Y) = Tr(d\omega) - Tr(\omega \wedge \omega) = Trd\omega. \tag{6}$$

From (5) and (6) it follows that

$$F_A = Ric. \tag{7}$$

Here we follow the similar procedures given in [2].

In the following, the Ricci form ρ_H is defined by

$$\rho_H(X, Y) = Ric(X, J_H Y) = g_\eta(X, J_H RicY)$$

for any $X, Y \in \Gamma(H)$. In the case of a strictly pseudoconvex CR manifold, the almost complex structure J_H is complex. Therefore, we have the equation

$$Ric(X, Y) = i\rho_H(X, Y) \tag{8}$$

for any $X, Y \in \Gamma(H)$.

Proposition 2 *Let ρ_H be a Ricci form on H and s_H be a scalar curvature of the subbundle H. Then the following identity holds:*

$$\rho_H^+ = -\frac{s_H}{4} d\eta, \tag{9}$$

where ρ_H^+ is a the self-dual part of the Ricci form ρ_H.

Proof In local coordinates the almost complex structure J is given as follows.

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $J \circ Ric = Ric \circ J$, we obtain the reduced form of the Ric as follows.

$$Ric = \begin{pmatrix} R_{11} & 0 & R_{13} & R_{14} & 0 \\ 0 & R_{11} & -R_{14} & R_{13} & 0 \\ R_{13} & -R_{14} & R_{33} & 0 & 0 \\ R_{14} & R_{13} & 0 & R_{33} & 0 \\ 0 & 0 & 0 & 0 & R_{55} \end{pmatrix}$$

The Ricci form ρ_H can be written in the following way:

$$\rho_H = -R_{11} e^1 \wedge e^2 - R_{33} e^3 \wedge e^4 - R_{13}(e^1 \wedge e^4 - e^2 \wedge e^3) + R_{14}(e^1 \wedge e^3 + e^2 \wedge e^4).$$

Since the 2-forms $e^1 \wedge e^4 - e^2 \wedge e^3$ and $e^1 \wedge e^3 + e^2 \wedge e^4$ are anti-self-dual 2-forms, the self-dual part of ρ_H is given by

$$\rho_H^+ = \frac{-R_{11} - R_{33}}{2} d\eta = -\frac{R_{11} + R_{22} + R_{33} + R_{44}}{4} d\eta = -\frac{s_H}{4} d\eta,$$

where s_H is the restricted scalar curvature to H. □

5.2. A special solution to 5-dimensional Seiberg–Witten equations

Let $(M, g_\eta, \eta, \xi, J)$ be a strictly pseudoconvex contact manifold of dimension 5. Suppose that the scalar curvature s_H of the subbundle H is negative and constant. Then let $\psi = \sqrt{-s_H}\psi_0$. In this case, $\psi \in \wedge_H^{0,0}(M)$. From (4) we have

$$\sigma_H(\psi) = i s_H d\eta. \tag{10}$$

By using (7),(8), (9), and (10) we obtain

$$F_A^+ = Ric^+ = i\rho_H^+ = -i\frac{s_H}{4} d\eta = -\frac{1}{4}\sigma_H(\psi). \tag{11}$$

Note that since $d\eta$ is a self-dual 2-form, $\sigma_H(\psi)$ is also i.e., $\sigma_H(\psi)^+ = \sigma_H(\psi)$. Because of $\sigma(\psi)^+ = \sigma_H(\psi)^+$ and with identity (11), we get

$$F_A^+ = -\frac{1}{4}\sigma(\psi)^+.$$

One can show that $\nabla^A_{e_i}\psi_0 = 0$. Therefore, we deduce that

$$D^A_H\psi = 0.$$

Moreover,

$$D_A\psi = 0.$$

The pair $(A, \psi = \sqrt{-s_H}\psi_0)$ is a solution of Seiberg–Witten-like equations in (3).

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