Turk J Phys
30 (2006) , 411 – 417.

Degenerate Spin Structures and the Lévy-Leblond Equation

T. DERELİ[1], Ş. KOÇAK[2], M. LİMONCU[2]
[1]*Department of Physics, Koç University, 34450 Sarıyer-İstanbul, TURKEY*
e-mail: tdereli@ku.edu.tr
[2]*Department of Mathematics, Anadolu University, 26470 Eskişehir-TURKEY*
e-mail: skocak@anadolu.edu.tr , mlimoncu@anadolu.edu.tr

Received 14.04.2006

Abstract

Newton-Cartan manifolds and the Galilei group are defined by the use of co-rank one degenerate metric tensor. Newton-Cartan connection is lifted to the degenerate spinor bundle over a Newton-Cartan 4-manifold by the aid of degenerate spin group. Levy-Leblond equation is constructed with the lifted connection.

1. Motivation

Degenerate spin structures in general is a subject that received little attention up to now. This provides by itself sufficient reason to study degenerate Clifford algebras and related structures in differential geometry.

From physics point of view too this is an interesting but a neglected subject. It is well-known that Newton's non-relativistic theory of gravitation may be given a locally Galilei covariant formulation over a 4-dimensional space-time equipped with independent space and time metrics that are both degenerate. A Newton-Cartan manifold is a space-time that admits a linear connection compatible with both space and time metrics.

In what follows, the degenerate Clifford algebra $\mathcal{C}\ell_{1,0,3}$ is defined over a 4-dimensional Newton-Cartan manifold. The corresponding degenerate spin group $SPIN(1,0,3)$ is defined. The Newton-Cartan connection is lifted to the degenerate spinor bundle. This allows one to write down the Lèvy-Leblond equation satisfied by a non-relativistic spin-1/2 electron directly, thus coupling it to gravity.

2. Galilei Group and Degenerate Spin Group

Let $\langle , \rangle$ be a symmetric bilinear form on $\mathbb{R}^n$ and let us consider the subspace W of $\mathbb{R}^n$

$$W = \{w \in \mathbb{R}^n \mid \langle w, x\rangle = 0 \text{ for all } x \in \mathbb{R}^n\}.$$

W is called the radical and the dimension of W is called the co-rank of $\langle , \rangle$. If co-rank is zero/non-zero, then $\langle , \rangle$ is called non-degenerate/degenerate. If $W' \subset \mathbb{R}^n$ is any complementary subspace to W, then the restriction of $\langle , \rangle$ to W' is non-degenerate [1]. If the co-rank of $\langle , \rangle$ is r and the the restriction of $\langle , \rangle$ to W' has signature (p, q) (in the sense that W' can be decomposed as $W'' \oplus W'''$, where the dimension of W''

respectively W''' are p resp. q and the restriction of $\langle , \rangle$ to W'' resp. W''' is negative resp. positive definite), then the pair $(\mathbb{R}^n, \langle , \rangle)$ is said to be of type $\mathbb{R}^{r,p,q}$. In this terminology, $\mathbb{R}^{0,1,3}$ is called relativistic space-time and $\mathbb{R}^{1,0,3}$ is called non-relativistic space-time.

We will consider throughout the non-relativistic $\mathbb{R}^{1,0,3}$. Let the radical be spanned by the vector f and let $\{e_1, e_2, e_3\}$ be an orthonormal basis for a complementary W'. Then, obviously, the set $\{f, e_1, e_2, e_3\}$ becomes a basis for $\mathbb{R}^{1,0,3}$.

In a non-relativistic space-time, the set of linear-automorphisms φ from $\mathbb{R}^{1,0,3}$ to $\mathbb{R}^{1,0,3}$ is called the Galilei group if $\langle \varphi(v), \varphi(w) \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{R}^{1,0,3}$, $det(\varphi) = 1$ and $\varphi|_W = id$. Thus the Galilei group can be written as

$$SO(1,0,3) = \{\varphi \in Aut(\mathbb{R}^{1,0,3}) \mid \langle \varphi(v), \varphi(w) \rangle = \langle v, w \rangle,\ det(\varphi) = 1,\ \varphi|_W = id\}.$$

Using the matrix of φ denoted by Φ with respect to the basis $\{f, e_1, e_2, e_3\}$, we can write

$$SO(1,0,3) = \{\Phi \in Mat(4 \times 4) \mid \Phi^t G \Phi = G,\ det(\Phi) = 1,\ \Phi|_{sp\{f\}} = I\}$$

where $W = sp\{f\} = \{\lambda f \mid \lambda \in \mathbb{R}\}$ and

$$G = \begin{pmatrix} 0 & 0 \\ 0 & I_{3\times 3} \end{pmatrix}.$$

From the above equations we obtain that $SO(1,0,3)$ can be parameterized as

$$SO(1,0,3) = \{\begin{pmatrix} 1 & A \\ 0 & R \end{pmatrix} \mid R \in SO(3) \quad A \in Mat(1 \times 3)\}.$$

Moreover $SO(1,0,3)$ is also isomorphic to the semi-direct product of $SO(3)$ and $Mat(3 \times 1)$

$$SO(1,0,3) \simeq SO(3) \ltimes_{id} Mat(3 \times 1).$$

Let $\mathcal{C}\ell_{1,0,3}$ be the degenerate Clifford algebra of $\mathbb{R}^{1,0,3}$. Degenerate spin group is defined to be a special subset of $\mathcal{C}\ell_{1,0,3}$

$$SPIN(1,0,3) = \{s(1 + vf) \mid s \in SPIN(3), v \in W' = sp\{e_1, e_2, e_3\}\}$$

where $SPIN(3)$ is taken as the spin group lying in the Clifford algebra on W', $\mathcal{C}\ell(W') \subset \mathcal{C}\ell_{1,0,3}$. We can regard $SPIN(3)$ as a subspace of $SPIN(1,0,3)$ taking $v = \mathbf{0}$. (For the general $SPIN(r,p,q)$ see [2])

$SPIN(1,0,3)$ is indeed a group under the Clifford multiplication. Let us first see that the Clifford multiplication on $SPIN(1,0,3)$ is a binary-operation:

$$\begin{aligned} s(1+vf)s'(1+v'f) &= (s+svf)(s'+s'v'f) \\ &= ss' + ss'v'f + svfs' + svfs'v'f \\ &= ss' + ss'v'f + svs'f - svf^2s'v' \\ &= ss' + ss'v'f + svs'f \\ &= ss' + ss'v'f + ss's'^{-1}vs'f \end{aligned}$$

by $f^2 = 0$. Using the 2:1 group homomorphism $\rho : SPIN(3) \to SO(3)$ which is defined by $s \mapsto \rho(s)(v) = svs^{-1}$, we can write

$$\begin{aligned} s(1+vf)s'(1+v'f) &= ss' + ss'v'f + ss'\rho(s'^{-1})(v)f \\ &= ss'(1 + (v' + \rho(s'^{-1})(v))f). \end{aligned}$$

Since the Clifford algebra is an associative algebra, the Clifford multiplication is also associative on $SPIN(1,0,3)$.

Obviously, $1 \in \mathcal{C}\ell_{1,0,3}$ belongs to $SPIN(1,0,3)$.
Inverse of any element of $SPIN(1,0,3)$ is given by

$$(s(1+vf))^{-1} = s^{-1}(1-\rho(s)(v)f).$$

Degenerate spin group can also be interpreted as a semi-direct product [3],[4]

$$SPIN(1,0,3) \simeq SPIN(3) \ltimes_\rho \mathbb{R}^3$$

where $\rho : SPIN(3) \to SO(3)$ is the 2:1 group homomorphism.

There is a second 2:1 group homomorphism given by

$$\begin{array}{rccc} \rho' : & SPIN(1,0,3) & \longrightarrow & SO(1,0,3) \\ & \mathfrak{s} & \longmapsto & \rho'(\mathfrak{s})(x) = \mathfrak{s}x\mathfrak{s}^{-1} \end{array}$$

or equally,

$$\begin{array}{rccc} \rho' : & SPIN(1,0,3) & \longrightarrow & SO(1,0,3) \\ & \mathfrak{s} = s(1+vf) & \longmapsto & \begin{pmatrix} 1 & 2v \\ 0 & \rho(s) \end{pmatrix} \end{array}$$

where the vector $v = v_1e_1 + v_2e_2 + v_3e_3$ is taken to be a row vector.

Lie algebra of $SO(1,0,3)$ is

$$so(1,0,3) = \{\phi \in Mat(4\times 4) \mid (G\phi)^t + G\phi = 0\}$$

that is parameterized as

$$so(1,0,3) = \{\begin{pmatrix} 0 & a \\ 0 & r \end{pmatrix} \mid r \in so(3) \quad a \in Mat(1\times 3)\}.$$

After this we will consider the following basis for $so(1,0,3)$

$$E_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} E_{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} E_{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} E_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} E_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Lie algebra of $SPIN(1,0,3)$ is given by the span of the following elements of $\mathcal{C}\ell_{1,0,3}$

$$spin(1,0,3) = \{a \mid a \in sp\{fe_1, fe_2, fe_3, e_1e_2, e_1e_3, e_2e_3\}\} \subset \mathcal{C}\ell_{1,0,3}.$$

Differential map $(d\rho')_{1_{SPIN(1,0,3)}} : spin(1,0,3) \longrightarrow so(1,0,3)$ is an isomorphism between Lie algebras $spin(1,0,3)$ and $so(1,0,3)$. Their basis elements correspond as follows:

$$(d\rho')_{1_{SPIN(1,0,3)}}(e_ie_j) = 2E_{ij}$$
$$(d\rho')_{1_{SPIN(1,0,3)}}(fe_i) = -2E_{0i}$$

3. Newton-Cartan Manifold, Degenerate Spin Manifold and Degenerate Spinor Bundle

Let g be a co-rank one degenerate symmetric metric tensor field on smooth 4-manifold M, and let τ be a smooth 1-form such that $\tau(V) = 1$ where the non-zero smooth vector field V satisfies $g(X, V) = 0$ for all vector fields X. Then we obtain the following tensor fields ([5]):

$\bar{g} = \tau \otimes \tau + g$ is a non-degenerate symmetric (0,2)-tensor field. In fact, $\bar{g}$ is obviously symmetric. For non-degeneracy, let us consider kernel of $\mathbb{R}$-linear map $\tau_p : T_pM \to \mathbb{R}$ for all $p \in M$. As $ker(\tau_p) = \{U_p \in T_pM | \tau_p(U_p) = 0\}$ is a subspace of T_pM and $dim(T_pM) = dim(ker(\tau_p)) + dim(\mathbb{R})$, the dimension of $ker(\tau_p)$ is found to be 3. From $\tau(V) = 1$ we see that V_p is not element of $ker(\tau_p)$ because $\tau_p(V_p) = 1$. Then choosing a basis $\{(U_1)_p, (U_2)_p, (U_3)_p\}$ of $ker(\tau_p)$, the set $\{V_p, (U_1)_p, (U_2)_p, (U_3)_p\}$ becomes a basis of T_pM. On the other hand g_p restricted to $ker(\tau_p)$ is non-degenerate because of $rank(g) = 3$. Otherwise from the basis $\{V_p, (U_1)_p, (U_2)_p, (U_3)_p\}$ we would find $rank(g) < 3$. The basis $\{(U_1)_p, (U_2)_p, (U_3)_p\}$ of $ker(\tau_p)$ can be chosen orthonormal. Finally if we construct the matrix of $\bar{g}$ with respect to the basis $\{V_p, (U_1)_p, (U_2)_p, (U_3)_p\}$ we see that

$$\begin{aligned} \bar{g}_p(V_p, V_p) &= \tau_p(V_p)\tau_p(V_p) + g_p(V_p, V_p) = 1 \\ \bar{g}_p(V_p, (U_i)_p) &= \tau_p(V_p)\tau_p((U_i)_p) + g_p(V_p, (U_i)_p) = 0 \\ \bar{g}_p((U_i)_p, (U_j)_p) &= \tau_p((U_i)_p)\tau_p((U_j)_p) + g_p((U_i)_p, (U_j)_p) = \delta_{ij} \end{aligned}$$

the matrix is diagonal and $det(\bar{g}_p) = 1 \neq 0$ for all $p \in M$ where $i, j = 1, 2, 3$. Therefore $\bar{g}_p$ is non-degenerate. Note that non-degeneracy (and degeneracy) is independent of the chosen of basis.

Secondly there is another non-degenerate symmetric (2,0)-tensor field $\bar{h}$ given by $\bar{h}(\alpha, \beta) = \bar{g}(\alpha^*, \beta^*)$ for all one-forms α, β, where the star $*$ denotes the metric-dual of one-forms, and we have, locally, the relation $\bar{h}^{\mu\lambda}\bar{g}_{\lambda\nu} = \delta^\mu_\nu$.

Note: For each one -form α there exits a unique vector field α^*, metric-dual of one-form α, such that $\alpha(X) = \bar{g}(\alpha^*, X)$ for all vector fields X. Therefore we have an isomorphism from the set of one-forms to the set of vector fields $\alpha \longmapsto \alpha^*$. As known, the set of one-forms and vector fields are modules over the commutative ring $C^\infty(M, \mathbb{R})$.

$\bar{h}$ is obviously symmetric and non-degenerate because of the above note and non-degeneracy of $\bar{g}$. The local relation is obtained as follows: In any coordinate system, if we take $\alpha = dx^\eta$, then we obtain $(dx^\mu)^* = (\bar{g}^{\,-1})^{\mu\kappa}\partial_\kappa$ by the aid of $\alpha(X) = \bar{g}(\alpha^*, X)$ (where $(\bar{g}^{\,-1})^{\mu\nu}$ is inverse matrix of $\bar{g}_{\mu\nu}$). Thus we have

$$\begin{aligned} \bar{h}(dx^\mu, dx^\nu) &= \bar{g}((dx^\mu)^*, (dx^\nu)^*) \\ \bar{h}^{\mu\nu} &= (\bar{g}^{\,-1})^{\mu\kappa}(\bar{g}^{\,-1})^{\nu\sigma}\bar{g}_{\kappa\sigma} = (\bar{g}^{\,-1})^{\nu\mu} \end{aligned}$$

and from the symmetry of $\bar{g}^{\,-1}$

$$\bar{h}^{\mu\nu} = (\bar{g}^{\,-1})^{\mu\nu}.$$

Therefore the expression $\bar{h}^{\mu\lambda}\bar{g}_{\lambda\nu}$ equals to δ^μ_ν.

Finally, we note that $h = -V \otimes V + \bar{h}$ is a co-rank one degenerate symmetric (2,0)-tensor field such that $h^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu - V^\mu\tau_\nu$.

M is called a Newton-Cartan manifold if M can be furnished by tensor fields g and τ satisfying the above properties. A linear connection on M compatible with g and τ will be called a Newton-Cartan connection [5].

Let M be a Newton-Cartan manifold, and let us consider the principal bundle $P_{SO(1,0,3)}$ on M. Then M will be called a degenerate spin manifold if there exists a principal $SPIN(1, 0, 3)$ bundle $P_{SPIN(1,0,3)}$ on M satisfying the following property: Transition maps $\varphi_{\alpha\beta}$ and $\widetilde{\varphi}_{\alpha\beta}$ of the principal bundles $P_{SO(1,0,3)}$ and $P_{SPIN(1,0,3)}$ can be chosen on a joint covering $\{U_\alpha\}$, such that the following diagram commutes:

$$\begin{tikzcd} & SPIN(1,0,3) \arrow[d, "\rho'"', "2:1"] \\ U_\alpha \cap U_\beta \arrow[ur, "\widetilde{\varphi}_{\alpha\beta}"] \arrow[r, "\varphi_{\alpha\beta}"'] & SO(1,0,3) \end{tikzcd}$$

An algebra homomorphism $\theta : \mathcal{C}\ell_{1,0,3} \longrightarrow End(\mathbb{C}^4)$, i.e. a matrix representation of the degenerate Clifford algebra $\mathcal{C}\ell_{1,0,3}$, gives a group homomorphism $\theta : SPIN(1,0,3) \longrightarrow Aut(\mathbb{C}^4)$. By the aid of this group homomorphism we can construct the associated vector bundle $P_{SPIN(1,0,3)} \times_\theta \mathbb{C}^4$ which is called a degenerate spinor bundle.

4. Lifting the Connection

Let ∇ be a linear connection on M such that $\nabla_Z g = 0$ for all vector fields Z, i.e., g-compatible linear connection. We define the connection 1-form on $P_{SO(1,0,3)}$ by

$$\mathcal{A}_\alpha(W) = e^a(\nabla_W X_b) = \sum_{c=0}^{3} W^c \Gamma^a_{cb}$$

or

$$\mathcal{A}_\alpha = \sum_{c=0}^{3} \Gamma^a_{cb} e^c = \omega^a{}_b$$

on $U_\alpha \subset M$. Here $\{e^a\}$, $\{X_a\}$ are local coframe and frame field, $W \in \Gamma(TU_\alpha)$ and $\mathcal{A}_\alpha(W)(x) \in so(1,0,3)$ for all $x \in U_\alpha$. Using the basis of $so(1,0,3)$, we can write

$$\mathcal{A}_\alpha(W) = \sum_{i=1}^{3} \omega^0{}_i(W)\, E_{0i} + \sum_{i<j=1}^{3} \omega^i_j(W)\, E_{ij}.$$

Since M is a spin manifold, $\mathcal{A}_\alpha$ can be lifted to $P_{SPIN(1,0,3)}$ as

$$\widetilde{\mathcal{A}}_\alpha(W) = (d\rho')^{-1}_{1_{SPIN(1,0,3)}}(\mathcal{A}_\alpha(W)).$$

From this expression we obtain

$$\widetilde{\mathcal{A}}_\alpha(W) \quad = \quad -\frac{1}{2}\sum_{i=1}^{3} \omega^0{}_i(W) f e_i + \frac{1}{2}\sum_{i<j=1}^{3} \omega^i_j(W) e_i e_j$$

Secondly $\widetilde{\mathcal{A}}_\alpha$ is lifted to the degenerate spinor bundle $P_{SPIN(1,0,3)} \times_\theta \mathbb{C}^4$ as

$$\overline{\mathcal{A}}_\alpha(W) = (d\theta)_{1_{SPIN(1,0,3)}}(\widetilde{\mathcal{A}}_\alpha(W))$$

by the differential map $(d\theta)_{1_{SPIN(1,0,3)}} : spin(1,0,3) \rightarrow gl(4,\mathbb{C})$ which is given by

$$(d\theta)_{1_{SPIN(1,0,3)}}(e_i e_j) = \theta(e_i)\theta(e_j)$$
$$(d\theta)_{1_{SPIN(1,0,3)}}(f e_i) = \theta(f)\theta(e_i).$$

Thus one gets

$$\overline{\mathcal{A}}_\alpha(W) = -\frac{1}{2}\sum_{i=1}^{3} \omega^0{}_i(W)\theta(f)\theta(e_i) + \frac{1}{2}\sum_{i<j=1}^{3} \omega^i_j(W)\theta(e_i)\theta(e_j).$$

Using $\overline{\mathcal{A}}_\alpha$, the following connection is defined on the degenerate spinor bundle:

$$(\nabla\psi_\alpha)(W) = \nabla_W\psi_\alpha = (d\psi_\alpha)(W) + (\overline{\mathcal{A}}_\alpha(W))(\psi_\alpha)$$

and

$$\begin{aligned}(\nabla\psi_\alpha)(W) = \nabla_W\psi_\alpha = (d\psi_\alpha)(W) &- \frac{1}{2}\sum_{i=1}^{3}{\omega^0}_i(W)\theta(f)\theta(e_i)(\psi_\alpha)\\ &+\frac{1}{2}\sum_{i<j=1}^{3}\omega^i_j(W)\theta(e_i)\theta(e_j)(\psi_\alpha).\end{aligned}$$

5. Lèvy-Leblond Equation

By the use of the equivalent bundles $P_{SPIN(1,0,3)} \times_\sigma \mathcal{C}\ell_{1,0,3} \cong \mathcal{C}\ell(TM)$, where $\sigma : SPIN(1,0,3) \longrightarrow Aut(\mathcal{C}\ell_{1,0,3})$ is the group homomorphism given by $\mathfrak{s} \longmapsto \sigma(\mathfrak{s})(c) := \mathfrak{s}c\mathfrak{s}^{-1}$, the algebra homomorphism $\theta : \mathcal{C}\ell_{1,0,3} \longrightarrow End(\mathbb{C}^4)$ can be extended as follows:

$$\begin{array}{rccc}\theta: & P_{SPIN(1,0,3)} \times_\sigma \mathcal{C}\ell_{1,0,3} \cong \mathcal{C}\ell(TM) & \longrightarrow & End(P_{SPIN(1,0,3)} \times_\theta \mathbb{C}^4)\\ & [u,c] & \longmapsto & \theta([u,c])([u,v]) := [u,\theta(c)(v)]\end{array}$$

Using this extension we define the operator

$$D : \Gamma(P_{SPIN(1,0,3)} \times_\theta \mathbb{C}^4) \longrightarrow \Gamma(P_{SPIN(1,0,3)} \times_\theta \mathbb{C}^4)$$

by

$$D\psi = \sum_{a,b=0}^{3}\bar{h}(e^a,e^b)\theta(X_a)((\nabla\psi)(X_b)) = \sum_{a,b=0}^{3}\bar{h}(e^a,e^b)\theta(X_a)(\nabla_{X_b}\psi),$$

where $\bar{h}$ is the non-degenerate contravariant tensor field on the Newton-Cartan manifold. This operator can be written as

$$D\psi = \sum_{a=0}^{3}\gamma^a(\nabla\psi)(X_a) = \sum_{a=0}^{3}\gamma^a\nabla_{X_a}\psi$$

with the notation

$$\bar{h}(e^a,e^b) = \bar{h}^{ab} \qquad \theta(X_a) = \gamma_a \qquad \bar{h}^{ab}\gamma_b = \gamma^a.$$

Using the expression of $\nabla_{X_a}\psi$, we find

$$\begin{aligned}D\psi = \sum_{a=0}^{3}\gamma^a((d\psi_\alpha)(X_a) &- \frac{1}{2}\sum_{i=1}^{3}{\omega^0}_i(X_a)\theta(f)\theta(e_i)(\psi_\alpha)\\ &+\frac{1}{2}\sum_{i<j=1}^{3}\omega^i_j(X_a)\theta(e_i)\theta(e_j)(\psi_\alpha))\end{aligned}$$

or

$$\begin{aligned}D\psi = \sum_{a=0}^{3}\gamma^a((d\psi_\alpha)(X_a) &- \frac{1}{2}\sum_{i=1}^{3}\Gamma^0_{ai}\theta(f)\theta(e_i)(\psi_\alpha)\\ &+\frac{1}{2}\sum_{i<j=1}^{3}\Gamma^i_{aj}\theta(e_i)\theta(e_j)(\psi_\alpha)).\end{aligned}$$

Under the following representation of $\mathcal{C}\ell_{1,0,3}$,

$$\theta(f) = \gamma_0 = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \qquad \theta(e_i) = \gamma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

where σ_i are the Pauli spin matrices, we obtain the equation

$$D\psi + 2mi\gamma_0^t\psi = 0,$$

which is called the Lèvy-Leblond equation [6],[3]. In the flat case ($X_a = \partial_a$, $e^a = dx^a$, $\Gamma^c_{ab} = 0$) it becomes the original Lèvy-Leblond equation

$$\gamma^a\partial_a\psi + 2mi\gamma_0^t\psi = 0$$

and from this one gets by iteration the Schrödinger equation

$$-\frac{1}{2m}\triangle = i\partial_t\psi.$$

References

[1] K.L. Duggal and A. Bejancu, Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publishers, 1996.

[2] A. Crumeyrolle, Orthogonal and Symplectic Clifford Algebras, Kluwer Academic Publishers, 1990

[3] H. P. Künzle and C. Duval, Dirac field on Newtonian space-time, Ann. Inst. Henri Poincarè, **41(4)**, (1984), 363.

[4] M.Limoncu, Newton-Levy-Leblond Equations, Ph. D. Thesis (In Turkish), Anadolu University, Eskisehir, Turkey (2005).

[5] T. Dereli, Ş. Koçak and M. Limoncu, Newton-Cartan Connections with Torsion, *ARI Bull. Istan. Tech. Univ.,* **54(1)**, (2004) 1.

[6] J.M. Lèvy-Leblond, Nonrelativistic Particles and Wave Equation, Commun. *Math. Phys.,* **6**, (1967), 286.