Mathematical & Computational Applications, Vol. 8, No. 1, pp. 127-134, 2003

ON THE SEMICONTINUITY PROPERTIES OF THE ATTAINABLE SETS OF CONTROL SYSTEMS WITH CONTROLS IN L_p

Kh.G.Guseinov, Orhan Ozer and Emrah Akyar
Department of Mathematics, Faculty of Science
Anadolu University 26470 Eskisehir, Turkey
kguseynov@anadolu.edu.tr, oozer@anadolu.edu.tr, eakyar@anadolu.edu.tr

Abstract- In this article the properties of attainable sets of control systems with integral constraints on control are studied. The admissible control functions are chosen from the ball centered at the origin and of radius μ_0 in L_p, $p>1$. It is proved that the attainable sets of the system at the fixed instant of time are left side lower semicontinuous and right side upper semicontinuous with respect to p.

Keywords- Control System, Attainable Set, Integral Constraint

1. INTRODUCTION

The numerous publications have been devoted to investigation of various properties of attainable sets of control systems (see, e.g. [1-8] and references therein). The properties of control systems with integral constraints on control have been studied in (see, e.g. [5-8] and references therein). In this article the control systems, which are non-linear with respect to the phase state vector, are considered. The admissible control functions are chosen from the ball centered at the origin and of radius μ_0 in L_p, $p>1$. It is proved that the attainable sets of the system are left side lower semicontinuous and right side upper semicontinuous with respect to p.

Consider a control system whose behavior is described by a differential equation

$$\dot{x} = f(t,x) + B(t,x)u, \quad x(t_0) = x_0 \tag{1.1}$$

where $x \in R^n$ is the n-dimensional phase vector of the system, u is the r-dimensional control vector, $t \in [t_0, \theta]$ $(t_0 < \theta < \infty)$, $f(t,x)$ is n-dimensional vector function and $B(t,x)$ is $(n \times r)$-dimensional matrix function.

It is assumed that the realizations $u(t)$, $t \in [t_0, \theta]$ of the control u are restricted by the constraint

$$\int_{t_0}^{\theta} \|u(t)\|^p dt \le \mu_0{}^p, \qquad p > 1 \tag{1.2}$$

where $\mu_0 > 0$, $\|\cdot\|$ denotes the Euclidean norm. It is assumed that the following conditions are satisfied.

A. The functions $f(t,x)$ and $B(t,x)$ are continuous on (t,x) and locally Lipschitz with respect to x i.e., for any bounded subset $D \subset [t_0,\theta] \times R^n$ there exist Lipschitz constants $L_i = L_i(D) \ge 0$, $(i=1,2)$ such that

$$\|f(t,x^*) - f(t,x_*)\| \le L_1 \|x^* - x_*\|, \qquad \|B(t,x^*) - B(t,x_*)\| \le L_2 \|x^* - x_*\|$$

for any $(t,x^*),(t,x_*)\in D$.

B. There exist constants $\gamma_1>0$ and $\gamma_2>0$ such that

$$\|f(t,x)\|\le\gamma_1(1+\|x\|),\qquad \|B(t,x)\|\le\gamma_2(1+\|x\|)$$

for every $(t,x)\in[t_0,\theta]\times R^n$.

Every function $u(\cdot)\in L_p[t_0,\theta]$, $(1<p<\infty)$ satisfying the inequality (1.2) is said to be an admissible control function. The set of all $u(\cdot)$ admissible control functions is denoted by the symbol U_p.

Let $u_*(\cdot)\in U_p$. The absolutely continuous function $x_*(t):[t_0,\theta]\to R^n$, which satisfies the equation $\dot{x}_*(t)=f(t,x_*(t))+B(t,x_*(t))u_*(t)$ a.e. in $[t_0,\theta]$ and the initial condition $x_*(t_0)=x_0$ is said to be solution of the system (1.1) generated by the admissible control function $u_*(\cdot)$. By $X_p(t_0,x_0)$ we denote the set of all solutions of the system (1.1) generated by all admissible control functions $u(\cdot)\in U_p$. We set

$$X_p(t;t_0,x_0)=\{x(t)\in R^n : x(\cdot)\in X_p(t_0,x_0)\}$$

$$Z_p(t_0,x_0)=\{(t,x(t))\in[t_0,\theta]\times R^n : x(\cdot)\in X_p(t_0,x_0)\}$$

$X_p(t;t_0,x_0)$ is called the attainable set of the system (1.1) with constraints (1.2) at the instant of time t. It means that the set $X_p(t;t_0,x_0)$ consists of all $x\in R^n$, at which the solutions of the system (1.1), which are generated by all possible controls $u(\cdot)\in U_p$, arrive at the instant of time $t\in[t_0,\theta]$. The set $Z_p(t_0,x_0)$ is called the integral funnel of the system (1.1) with constraints (1.2).

It is known (see, [6]) that the attainable sets $X_p(t;t_0,x_0)$ are compact for every $t\in[t_0,\theta]$ and there exists a cylinder

$$D=\{(t,x)\in[t_0,\theta]\times R^n : \|x\|\le r\} \tag{1.3}$$

such that $Z_p(t_0,x_0)\subset D$ for any $p\in(1,\infty)$. Here $r>0$ is a constant depending on x_0, μ_0 and γ_1 and γ_2 defined in condition B. From now on D will denote the cylinder (1.3).

2. SEMICONTINUITY OF ATTAINABLE SETS

Suppose $H>0$. The symbol U_p^H denotes the set of all admissible control functions $u(\cdot)\in U_p$ satisfying the geometric constraint $\|u(t)\|\le H$ for every $t\in[t_0,\theta]$ that is

$$U_p^H=\{u(\cdot)\in U_p : \|u(t)\|\le H \text{ for all } t\in[t_0,\theta]\}$$

By $X_p^H(t_0,x_0)$ we denote the set of all solutions of the system (1.1) generated by all possible control functions $u(\cdot)\in U_p^H$. We set

$$X_p^H(t;t_0,x_0)=\{x(t)\in R^n : x(\cdot)\in X_p^H(t_0,x_0)\}$$

We denote by

$$\alpha(E,F)=\inf\{r>0 : E\subset F+rB,\quad F\subset E+rB\}$$

the Hausdorff distance between the sets $E,F \subset R^n$ where B is unit ball in R^n.

Following proposition has proved in [6].

Proposition 2.1. The inequality

$$\alpha(X_p(t;t_0,x_0), X_p^H(t;t_0,x_0)) \le \frac{K}{H^{p-1}}$$

holds for all $t\in[t_0,\theta]$. Here $K = 2K_1\mu_0^p(1+c_*e^{c_*})$, $c_* = L_1(\theta-t_0)+L_2\mu_0(\theta-t_0)^{\frac{p-1}{p}}$, $K_1 = \max_{(t,x)\in D}\|B(t,x)\|$.

Corollary 2.1. Let $p_*>1$. Then for any $\varepsilon>0$ there exist $\tau(\varepsilon)\in(0,p_*-1)$, $H_*>0$ such that for all $H\ge H_*$ the inequality

$$\alpha(X_p(t;t_0,x_0), X_p^H(t;t_0,x_0)) \le \varepsilon$$

holds for all $p\in(1+\tau(\varepsilon),2p_*)$ and $t\in[t_0,\theta]$.

Proposition 2.2. Let $p_*>1$, $H_1>\mu_0$, $H_2>\mu_0$ and $H_2>H_1$. Then for every $\varepsilon\in(0,\mu_0)$ there exists $\delta_1=\delta_1(\varepsilon)\in(0,p_*-1)$ such that for any $p\in(p_*-\delta_1,p_*)$ the inclusion

$$X_{p_*}^{H_1}(t;t_0,x_0) \subset X_p^{H_2}(t;t_0,x_0)+\varepsilon K_* B$$

holds for all $t\in[t_0,\theta]$. Here $K_*\ge 0$ is constant.

Proof. We set

$$p_1 = \max\left\{1, \quad \frac{p_*}{1+\log_{\frac{H_1}{\mu_0}}\dfrac{H_2}{H_1}}\right\}. \tag{2.1}$$

It is obvious that $p_1\in[1,p_*)$. Let $\varepsilon\in(0,\mu_0)$ be a fixed number. One can to show that there exists $h(\varepsilon)>0$ such that

$$-\frac{\varepsilon}{H_1} \le 1-\left(\frac{H_1}{\mu_0}\right)^{\frac{p_*-p}{p}}, \qquad 1-\left(\frac{\varepsilon}{\mu_0}\right)^{\frac{p_*-p}{p}} \le \frac{\varepsilon}{H_1} \tag{2.2}$$

for every $p\in(p_*-h(\varepsilon),p_*)$. Now we set

$$\delta_1(\varepsilon) = \min\{h(\varepsilon), p_*-p_1\}. \tag{2.3}$$

Let $x(\cdot)\in X_{p_*}^{H_1}(t_0,x_0)$ be an arbitrary solution of system (1.1). Then there exists $u(\cdot)\in U_{p_*}^{H_1}$ such that

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau,x(\tau)) + B(\tau,x(\tau))u(\tau)d\tau$$

holds for all $t\in[t_0,\theta]$.

Let us choose an arbitrary $p\in(p_*-\delta_1(\varepsilon),p_*)$ and define control function

$$u_*(t)=u(t)\|u(t)\|^{\frac{p_*-p}{p}}\mu_0^{\frac{p-p_*}{p}}\,,\ t\in[t_0,\theta] \tag{2.4}$$

By virtue of (2.1) and (2.3) one can to prove that $u_*(\cdot)\in U_p^{H_2}$. For the solution $x_*(\cdot)\in X_p^{H_2}(t_0,x_0)$ which is generated by $u_*(\cdot)\in U_p^{H_2}$,

$$x_*(t)=x_0+\int_{t_0}^{t}f(\tau,x_*(\tau))+B(\tau,x_*(\tau))u_*(\tau)d\tau$$

holds for all $t\in[t_0,\theta]$. By the condition A, the solutions $x(\cdot)$ and $x_*(\cdot)$ satisfy the inequality

$$\|x(t)-x_*(t)\|\le\int_{t_0}^{t}(L_1+L_2\|u(\tau)\|)\|x(\tau)-x_*(\tau)\|d\tau+K_1\int_{t_0}^{t}\|u(\tau)-u_*(\tau)\|d\tau \tag{2.5}$$

for every $t\in[t_0,\theta]$ where $K_1=\max_{(t,x)\in D}\|B(t,x)\|$. By (2.4) we have

$$\int_{t_0}^{t}\|u(\tau)-u_*(\tau)\|d\tau=\int_{t_0}^{t}\|u(\tau)\|\cdot\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right|d\tau$$

For $\varepsilon\in(0,\mu_0)$ we denote

$$A_t^{\varepsilon}=\{\tau\in[t_0,t]:0\le\|u(\tau)\|\le\varepsilon\},\qquad B_t^{\varepsilon}=\{\tau\in[t_0,t]:\varepsilon<\|u(\tau)\|\le H_1\}.$$

Then we obtain

$$\begin{aligned}\int_{t_0}^{t}\|u(\tau)\|\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right|d\tau=&\int_{A_t^{\varepsilon}}\|u(\tau)\|\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right|d\tau\\&+\int_{B_t^{\varepsilon}}\|u(\tau)\|\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right|d\tau\end{aligned} \tag{2.6}$$

It is not difficult to verify that

$$\|u(\tau)\|\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right|\le\varepsilon$$

for all $\tau\in A_t^{\varepsilon}$. Thus,

$$\int_{A_t^{\varepsilon}}\|u(\tau)\|\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right|d\tau\le\varepsilon(\theta-t_0). \tag{2.7}$$

Since $\varepsilon<\|u(\tau)\|\le H_1$ for every $\tau\in B_t^{\varepsilon}$, then

$$1-\left(\frac{H_1}{\mu_0}\right)^{\frac{p_*-p}{p}}\le1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\le1-\left(\frac{\varepsilon}{\mu_0}\right)^{\frac{p_*-p}{p}}$$

for every $\tau \in B_t^\varepsilon$. Since $p \in (p_* - \delta_1(\varepsilon), p_*)$, then it follows from (2.2) and (2.3) that

$$\left| 1 - \left(\frac{\|u(\tau)\|}{\mu_0} \right)^{\frac{p_* - p}{p}} \right| \le \frac{\varepsilon}{H_1}$$

is satisfied for all $\tau \in B_t^\varepsilon$. Thus we obtain that

$$\|u(\tau)\| \left| 1 - \left(\frac{\|u(\tau)\|}{\mu_0} \right)^{\frac{p_* - p}{p}} \right| \le \varepsilon$$

holds for all $\tau \in B_t^\varepsilon$ and consequently, the inequality

$$\int_{B_t^\varepsilon} \|u(\tau)\| \left| 1 - \left(\frac{\|u(\tau)\|}{\mu_0} \right)^{\frac{p_* - p}{p}} \right| d\tau \le \varepsilon(\theta - t_0) \tag{2.8}$$

is verified. By virtue of (2.6)-(2.8) we have that

$$\int_{t_0}^{t} \|u(\tau)\| \left| 1 - \left(\frac{\|u(\tau)\|}{\mu_0} \right)^{\frac{p_* - p}{p}} \right| d\tau \le 2\varepsilon(\theta - t_0)$$

is verified for every $t \in [t_0, \theta]$. It follows from here and (2.5) that

$$\|x(t) - x_*(t)\| \le \int_{t_0}^{t} (L_1 + L_2 \|u(\tau)\|) \|x(\tau) - x_*(\tau)\| d\tau + 2K_1 \varepsilon(\theta - t_0)$$

holds for every $t \in [t_0, \theta]$. Using Gronwall lemma and Hölder inequality (see, e.g. [9]) we obtain that

$$\|x(t) - x_*(t)\| \le 2K_1 \varepsilon(\theta - t_0)(1 + c_1 e^{c_1})$$

is true for every $t \in [t_0, \theta]$ where $c_1 = L_1(\theta - t_0) + L_2 \mu_0 (\theta - t_0)^{\frac{2p_* - 1}{2p_*}}$. Since $p \in (p_* - \delta_1(\varepsilon), p_*)$ is arbitrary chosen, setting $K_* = 2K_1(\theta - t_0)(1 + c_1 e^{c_1})$ we obtain the proposition.

By the virtue of proposition 2.1 and proposition 2.2 we obtain following theorem.

Theorem 2.1. Let $p_* > 1$. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $p \in (p_* - \delta, p_*)$ the inclusion

$$X_{p_*}(t; t_0, x_0) \subset X_p(t; t_0, x_0) + \varepsilon B$$

holds for all $t \in [t_0, \theta]$.

Proof. By corollary 2.1 we have that for any $\varepsilon > 0$ there exist $\tau(\varepsilon) \in (0, p_* - 1)$, $H_* > \mu_0$ such that for all $H \ge H_*$ the inclusions

$$X_p(t; t_0, x_0) \subset X_p^H(t; t_0, x_0) + \frac{\varepsilon}{3} B \tag{2.9}$$

$$X_p^H(t;t_0,x_0) \subset X_p(t;t_0,x_0) + \frac{\varepsilon}{3}B \tag{2.10}$$

holds for all $p \in (1+\tau(\varepsilon), 2p_*)$ and $t \in [t_0, \theta]$.

Let $H_1 > H_* > \mu_0$, $H_2 > H_* > \mu_0$ and $H_2 > H_1$. Then by proposition 2.2 for every $\varepsilon \in (0, \mu_0)$ there exists $\delta_* = \delta_*(\varepsilon) > 0$ such that for any $p \in (p_* - \delta_*, p_*)$

$$X_{p_*}^{H_1}(t;t_0,x_0) \subseteq X_p^{H_2}(t;t_0,x_0) + \frac{\varepsilon}{3}B \tag{2.11}$$

holds for all $t \in [t_0, \theta]$. Let $\delta(\varepsilon) = \min\{\delta_*(\varepsilon), p_* - 1 - \tau(\varepsilon)\}$ It follows from (2.9)-(2.11) that for every $p \in (p_* - \delta(\varepsilon), p_*)$

$$X_{p_*}(t;t_0,x_0) \subset X_{p_*}^{H_1}(t;t_0,x_0) + \frac{\varepsilon}{3}B \subset X_p^{H_2}(t;t_0,x_0) + \frac{2\varepsilon}{3}B \subset X_p(t;t_0,x_0) + \varepsilon B$$

holds for any $t \in [t_0, \theta]$. This completes the proof.

It follows from theorem 2.1 that the set valued map $p \to X_p(t;t_0 x_0)$, $p \in (1, \infty)$, is left side lower semicontinuous with respect to p.

Proposition 2.3. Let $p_* > 1$, $H_1 > \mu_0$, $H_2 > \mu_0$ and $H_1 > H_2$. Then for every $\varepsilon \in (0, \mu_0)$ there exists $\delta_2 = \delta_2(\varepsilon) \in (0, p_*)$ such that for any $p \in (p_*, p_* + \delta_2)$ the inclusion

$$X_p^{H_2}(t;t_0,x_0) \subset X_{p_*}^{H_1}(t;t_0,x_0) + K_*\varepsilon B$$

holds for all $t \in [t_0, \theta]$. Here $K_* \geq 0$ is defined in proposition 2.2.

Proof. Proof is similar to proof of proposition 2.2. Let

$$p_2 = \min\left\{2p_*,\ p_*\left(1 + \log_{\frac{H_2}{\mu_0}} \frac{H_1}{H_2}\right)\right\} \tag{2.12}$$

It is possible to verify that $p_2 \in (p_*, 2p_*]$. Let $\varepsilon \in (0, \mu_0)$ be a fixed number. It is possible to verify that that there exists $h_*(\varepsilon) > 0$ such that

$$-\frac{\varepsilon}{H_2} \leq 1 - \left(\frac{H_2}{\mu_0}\right)^{\frac{p-p_*}{p_*}}, \qquad 1 - \left(\frac{\varepsilon}{\mu_0}\right)^{\frac{p-p_*}{p_*}} \leq \frac{\varepsilon}{H_2} \tag{2.13}$$

for every $p \in (p_*, p_* + h_*(\varepsilon))$. Now we set

$$\delta_2(\varepsilon) = \min\{h_*(\varepsilon), p_2 - p_*\}. \tag{2.14}$$

Let us choose arbitrary $p \in (p_*, p_* + \delta_2(\varepsilon))$ and let $x(\cdot) \in X_p^{H_2}(t_0, x_0)$ be any solution of system (1.1). Then there exist $u(\cdot) \in U_p^{H_2}$ such that

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) + B(\tau, x(\tau))u(\tau)d\tau$$

holds for all $t \in [t_0, \theta]$. We define a control function

$$u_*(t) = u(t)\|u(t)\|^{\frac{p-p_*}{p_*}} \mu_0^{\frac{p_*-p}{p_*}}, \quad t \in [t_0, \theta] \tag{2.15}$$

By (2.12) and (2.14), it is not difficult to show that $u_*(\cdot)\in U_{p_*}^{H_1}$. For the solution $x_*(\cdot)\in X_{p_*}^{H_1}(t_0,x_0)$ which is generated by $u_*(\cdot)\in U_{p_*}^{H_1}$,

$$x_*(t)=x_0+\int_{t_0}^{t} f(\tau,x_*(\tau))+B(\tau,x_*(\tau))u_*(\tau)d\tau$$

holds for all $t\in[t_0,\theta]$. By the condition A, the solutions $x(\cdot)$ and $x_*(\cdot)$ satisfy the inequality

$$\|x(t)-x_*(t)\|\le\int_{t_0}^{t}(L_1+L_2\|u(\tau)\|)\|x(\tau)-x_*(\tau)\|d\tau+K_1\int_{t_0}^{t}\|u(\tau)-u_*(\tau)\|d\tau \qquad (2.16)$$

for every $t\in[t_0,\theta]$ where $K_1=\max_{(t,x)\in D}\|B(t,x)\|$. By (2.15) we have

$$\int_{t_0}^{t}\|u(\tau)-u_*(\tau)\|d\tau=\int_{t_0}^{t}\|u(\tau)\|\cdot\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p-p_*}{p_*}}\right|d\tau \qquad (2.17)$$

For $\varepsilon\in(0,\mu_0)$ we denote

$$A_t^{\varepsilon}=\{\tau\in[t_0,t]:0\le\|u(\tau)\|\le\varepsilon\},\quad B_t^{\varepsilon}=\{\tau\in[t_0,t]:\varepsilon<\|u(\tau)\|\le H_2\}. \qquad (2.18)$$

Taking into consideration (2.13) and (2.14), one can similarly to proof of proposition 2.2 to prove that the inequalities

$$\int_{A_t^{\varepsilon}}\|u(\tau)\|\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p-p_*}{p_*}}\right|d\tau\le\varepsilon(\theta-t_0),\qquad \int_{B_t^{\varepsilon}}\|u(\tau)\|\left|1-\left(\frac{\|u(\tau)\|}{\mu_0}\right)^{\frac{p-p_*}{p_*}}\right|d\tau\le\varepsilon(\theta-t_0).$$

hold. It follows from here (2.16)-(2.18) that

$$\|x(t)-x_*(t)\|\le\int_{t_0}^{t}(L_1+L_2\|u(\tau)\|)\|x(\tau)-x_*(\tau)\|d\tau+2K_1\varepsilon(\theta-t_0)$$

holds for every $t\in[t_0,\theta]$. Using Gronwall lemma and Hölder inequality (see, e.g. [9]) we obtain that

$$\|x(t)-x_*(t)\|\le 2K_1\varepsilon(\theta-t_0)(1+c_1e^{c_1})$$

is valid for every $t\in[t_0,\theta]$ where $c_1=L_1(\theta-t_0)+L_2\mu_0(\theta-t_0)^{\frac{2p_*-1}{2p_*}}$. Since $p\in(p_*,p_*+\delta_2(\varepsilon))$ is arbitrary chosen, setting $K_*=2K_1(\theta-t_0)(1+c_1e^{c_1})$ we obtain the proposition.

By the virtue of proposition 2.1 and proposition 2.3 similarly to theorem 2.1 the following theorem is obtained.

Theorem 2.2. Let $p_*>1$. Then for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that for any $p\in(p_*,p_*+\delta)$ the inclusion

$$X_p(t;t_0,x_0)\subset X_{p_*}(t;t_0,x_0)+\varepsilon B$$

holds for all $t\in[t_0,\theta]$.

It follows from theorem 2.2 that the set valued map $p \to X_p(t;t_0x_0)$, $p \in (1,\infty)$, is right side upper semicontinuous with respect to p.

3. CONCLUSON

The obtained results can be used in numerical computations of the attainable sets of nonlinear control systems with integral constraints on control.

Acknowledgement- This research was supported by Anadolu University Research Foundation.

REFERENCES

[1] Kh.G.Guseinov, A.N.Moiseyev and V.N.Ushakov, The Approximation of the Reachable Domains of Control Systems, *J. Appl. Math. Mechs.*, **62,** 169-175, 1998.

[2] A.B.Kurzhanski and I.Valyi, *Ellipsoidal Calculus for Estimation and Control,* Birkhauser, Boston, 1996.

[3] K.Deimling, *Multivalued Differential Equations,* Walter de Gruyter, Berlin, 1992.

[4] V.I.Blagodatskikh and A.F.Filippov, Differential Inclusions and Optimal Control, *Proc. of the Steklov Inst. of Math.,* **169,** 194-252, 1985.

[5] Kh.G.Guseinov, A.A.Neznakhin and V.N.Ushakov, Approximate Construction of Attainability Sets of Control Systems with Integral Constraints on Control, *J. Appl. Math. Mechs.*, **63,** 557-567, 1999.

[6] Kh.G.Guseinov, Orhan Özer and Emrah Akyar, Numerical Method for Approximation of Reachable Sets of Control Systems with Integral Constraints, *Abstracts International Conference on Applicable General Topology*, 1-2, Ankara, Turkey, 2001.

[7] A.G.Chentsov, Asymptotic Attainability Accompanying Perturbation of Integral Constraints, Kibernetika i Sist. Analiz, 1, 87-99, 1995.

[8] M.Motta and F.Rampazzo, Multivalued Dynamics on a Closed Domain with Absorbing Boundary. Applications to Optimal Control Problems with Integral Constraints, *Nonlinear Anal., Ser. A: Theory Methods,* **41**, no. 5-6, 631-647, 2000.

[9] J.Warga, *Optimal Control of Differential and Functional Equations,* Academic Press, New York, 1972.