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SPINORS ON KAHLER–NORDEN MANIFOLDS

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It is known that the complex spin group $\mathrm{Spin}(n, \mathbb{C})$ is the universal covering group of complex orthogonal group $SO(n, \mathbb{C})$. In this work we construct a new kind of spinors on some classes of Kahler–Norden manifolds. The structure group of such a Kahler–Norden manifold is $SO(n, \mathbb{C})$ and has a lifting to $\mathrm{Spin}(n, \mathbb{C})$. We prove that the Levi-Civita connection on M is an $SO(n, \mathbb{C})$-connection. By using the spinor representation of the group $\mathrm{Spin}(n, \mathbb{C})$, we define the spinor bundle S on M. Then we define covariant derivative operator ∇ on S and study some properties of ∇. Lastly we define Dirac operator on S.

Keywords: Spinor; Norden metric; anti-Kahler; complex orthogonal group; spin structure; complex spin group.

1. Introduction

Manifolds equipped with additional geometric structures occur in many cases in differential geometry. For example Riemannian, semi-Riemannian, almost complex structures and spin structures are important geometric structures on manifolds. On an n-dimensional manifold M, the existence of additional structures is related to the reduction of structure group from $GL(n, \mathbb{R})$ to a subgroup $G \subset GL(n, \mathbb{R})$. The most important structure groups are $O(n), SO(n), O(p, q), SO(p, q), GL(n, \mathbb{C}), U(n), SU(n)$. The groups $O(n, \mathbb{C})$ and $SO(n, \mathbb{C})$ are subgroups of $GL(2n, \mathbb{R})$ and they have escaped from our attention in general.

A Kahler manifold can be defined as a triple (M, J, g) where M is a smooth manifold, J an almost complex structure on M, g a Riemannian metric on M with the hermitian property $g(JX, JY) = g(X, Y)$ for any $X, Y \in \mathcal{X}(M)$ and J is parallel with respect to the Levi-Civita connection ∇^g, that is $\nabla^g J = 0$. Kahler manifolds are being studied widely in differential geometry ([9]). In this work we consider slightly different family of almost complex manifolds, namely, Kahler–Norden manifolds. By a Kahler–Norden manifold we mean a triple (M, J, g) which consists of a smooth manifold M, an almost complex structure J on M, and a semi-Riemannian metric g on M with the anti-hermitian property $g(JX, JY) = -g(X, Y)$ for any $X, Y \in \mathcal{X}(M)$ and J is parallel with respect to the Levi-Civita connection ∇^g, that is $\nabla^g J = 0$. Note that a Kahler–Norden manifold M must be even dimensional, say $2n$, and the anti-hermitian property $g(JX, JY) = -g(X, Y)$ implies that the signature of g is of type (n, n). This kind of manifolds have been also studied under the names: almost complex manifolds with Norden (or $B-$) metric, anti-kahlerian manifolds (see [1, 6, 12]). The structure group of a Kahler–Norden manifold is the complex orthogonal group $O(n, \mathbb{C})$.

2. The Spinor Representation

For more detailed explanation of the following facts see e.g. [2, 4, 10]. Spinors are geometric objects on manifolds like tensors and have various applications in mathematics and mathematical physics. In the classical theory, for the construction of spinor one use the spinor representation of the spin group $\mathrm{Spin}(n)$. In this work we construct similar objects on a certain class of Kahler–Norden manifolds. To achieve this we use the spinor representation of the complex spin group $\mathrm{Spin}(n, \mathbb{C})$ which is comes from the representation of complex Clifford algebra $\mathbb{C}l_n$. The complex Clifford algebra $\mathbb{C}l_n$ and its representation are well described in literature [10]. If $n = 2k$ is even then $\mathbb{C}l_{2k} \approxeq \mathbb{C}(2^k)$, if $n = 2k + 1$ is odd then $\mathbb{C}l_{2k+1} \approxeq \mathbb{C}(2^k) \oplus \mathbb{C}(2^k)$. When $n = 2k$ or $n = 2k + 1$ the vector space $\mathbb{C}^{2^k}$ is called vector space of complex n-spinors and denoted by Δ_n. Using this notation we can write $\mathbb{C}l_{2k} \approxeq \mathrm{End}(\Delta_n)$ and $\mathbb{C}l_{2k+1} \approxeq \mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$.

Denote by κ_n the so-called spinor representation of the Clifford algebra $\mathbb{C}l_n$. In case of even dimension $n = 2k$, κ_n is the isomorphism from $\mathbb{C}l_n$ to $\mathrm{End}(\Delta_n)$. If $n = 2k + 1$ is odd, then κ_n is the composition of the isomorphism from $\mathbb{C}l_n$ to $\mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$ with the projection from $\mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$ onto first component $\mathrm{End}(\Delta_n)$. Thus Δ_n become a module over the complex Clifford algebra $\mathbb{C}l_n$.

It is clear that $SO(n) \subset SO(n, \mathbb{C})$ and $\mathrm{Spin}(n) \subset \mathrm{Spin}(n, \mathbb{C})$. It is known that all of these groups are connected. The fundamental groups of the orthogonal groups $SO(n)$ and $SO(n, \mathbb{C})$ are the same, namely $\pi_1(SO(n)) = \pi_1(SO(n, \mathbb{C})) = \mathbb{Z}_2$ $(n \geq 2)$ and the fundamental groups of both real and complex spin groups is trivial. The maps $Ad : \mathrm{Spin}(n) \to SO(n)$ by $Ad_g(v) = gvg^*$ for $g \in \mathrm{Spin}(n)$, $v \in \mathbb{R}^n$ and $\mathbb{A}d : \mathrm{Spin}(n, \mathbb{C}) \to SO(n, \mathbb{C})$ by $\mathbb{A}d_g(v) = gvg^*$ for $g \in \mathrm{Spin}(n, \mathbb{C})$, $v \in \mathbb{C}^n$ are onto group homomorphisms with kernel $\{\pm 1\}$. Thus $\mathrm{Spin}(n)$ is the universal covering group of $SO(n)$ and $\mathrm{Spin}(n, \mathbb{C})$ is the universal covering group of $SO(n, \mathbb{C})$. In this work we mainly deal with the groups $SO(n, \mathbb{C})$ and $\mathrm{Spin}(n, \mathbb{C})$.

The restriction of κ_n to $\mathrm{Spin}(n) \subset \mathbb{C}l_n$ gives a group homomorphism $\kappa = \kappa_n : \mathrm{Spin}(n) \to \mathrm{Aut}(\Delta_n)$ called spinor representation of $\mathrm{Spin}(n)$. Similarly the restriction of κ_n to $\mathrm{Spin}(n, \mathbb{C}) \subset \mathbb{C}l_n$ gives a group homomorphism $\kappa = \kappa_n : \mathrm{Spin}(n, \mathbb{C}) \to \mathrm{Aut}(\Delta_n)$ called spinor representation of $\mathrm{Spin}(n, \mathbb{C})$. Some properties of the spinor representation of $\mathrm{Spin}(n, \mathbb{C})$ are as follows: (see [4])

- (i) If $n = 2k + 1$ is odd then κ is irreducible.
- (ii) If $n = 2k$ is even then the spinor space Δ_{2k} decomposes into two subspaces $\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^-$ and $\dim \Delta_{2k}^+ = \dim \Delta_{2k}^- = 2^{k-1}$. From this decomposition one get new representations $\kappa^+ : \mathrm{Spin}(2k, \mathbb{C}) \to \mathrm{End}(\Delta_{2k}^+)$ and $\kappa^- : \mathrm{Spin}(2k, \mathbb{C}) \to \mathrm{End}(\Delta_{2k}^-)$. Both of these representations are irreducible.
- (iii) If $n = 2k + 1$ is odd and $k \equiv 0, 3 \pmod 4$ then there is a non-degenerate symmetric bilinear form on Δ_{2k+1} and the spinor representation κ takes value in the complex orthogonal group $SO(2^k, \mathbb{C})$ that is $\kappa : \mathrm{Spin}(2k + 1, \mathbb{C}) \to SO(2^k, \mathbb{C})$.

(iv) If $n = 2k+1$ is odd and $k \equiv 1, 2 \pmod 4$ then there is a non-degenerate skew-symmetric bilinear form on Δ_{2k+1} and the spinor representation κ takes value in the complex symplectic group $Sp(2^k, \mathbb{C})$ that is $\kappa : \mathrm{Spin}(2k+1, \mathbb{C}) \to Sp(2^k, \mathbb{C})$.

(v) If $n = 2k$ is even and $k \equiv 0 \pmod 4$ then there is a non-degenerate symmetric bilinear form on Δ_{2k}^{+} and the spinor representation κ^{+} takes value in the complex orthogonal group $SO(2^{k-1}, \mathbb{C})$ that is $\kappa : \mathrm{Spin}(2k, \mathbb{C}) \to SO(2^{k-1}, \mathbb{C})$.

(vi) If $n = 2k$ is even and $k \equiv 2 \pmod 4$ then there is a non-degenerate skew-symmetric bilinear form on Δ_{2k}^{+} and the spinor representation κ^{+} takes value in the complex symplectic group $Sp(2^{k-1}, \mathbb{C})$ that is $\kappa : \mathrm{Spin}(2k, \mathbb{C}) \to Sp(2^{k-1}, \mathbb{C})$.

The Lie algebra $\mathfrak{spin}(n, \mathbb{C})$ of the complex spin group $\mathrm{Spin}(n, \mathbb{C})$ lives in $\mathbb{C}l_n$ and it is very similar to the Lie algebra of the real spin group $\mathrm{Spin}(n)$, and given by

$$\mathfrak{spin}(n, \mathbb{C}) = \mathrm{Lin}\{e_i e_j : 1 \leq i < j \leq n\}.$$

Then the differential of the map $\mathbb{A}d : \mathrm{Spin}(n, \mathbb{C}) \to SO(n, \mathbb{C})$ is a map $\mathbb{A}d_* : \mathfrak{spin}(n, \mathbb{C}) \to \mathfrak{so}(n, \mathbb{C})$ defined by $\mathbb{A}d_*(e_i e_j) = 2E_{ij}$, where E_{ij} are basis for $\mathfrak{so}(n, \mathbb{C})$.

3. Kahler–Norden Spin Manifolds

In this work we consider $2n$-dimensional manifold M with structure group $SO(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$. Since $SO(n, \mathbb{C}) \subset O(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap O(n, n)$, the manifold M can be endowed with a complex structure J and a semi-Riemannian metric g with signature (n, n). It can be also checked that

$$g(JX, JY) = -g(X, Y)$$

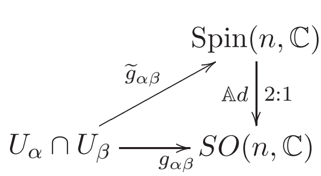
for all vector fields X, Y on M. Additionally if the condition $\nabla^g J = 0$ holds, that is J is parallel with respect to Levi-Civita connection ∇^g, then M is a Kahler–Norden manifold and we denote it by (M, J, g). More informations about these manifolds can be found in ([11, 12]).

Theorem 1. *Let (M, J, g) be a Kahler–Norden manifold. The Levi-Civita connection ∇^g is an $SO(n, \mathbb{C})$-connection. That is, the local connection forms ω_α take their values in the Lie algebra $\mathfrak{so}(n, \mathbb{C})$.*

Spin manifolds constitute an important class of manifolds ([10]). In the present paper we consider a similar but different class of manifolds. Let M be a $2n$-dimensional differentiable manifold with structure group $SO(n, \mathbb{C})$, then there is an open covering $\{U_\alpha\}_{\alpha \in A}$ of M and transition functions $g_{\alpha\beta} : V_\alpha \cap V_\beta \to SO(n, \mathbb{C})$ for TM. If there exists another collection of transition functions

$$\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(n, \mathbb{C})$$

such that following diagram commutes



that is $\mathbb{A}d \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \widetilde{g}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is satisfied then M is called a Kahler–Norden spin manifold. Then one can construct a principal $\mathrm{Spin}(n,\mathbb{C})-$bundle $P_{\mathrm{Spin}(n,\mathbb{C})}$ on M and a $2-1$ bundle map $\Lambda : P_{\mathrm{Spin}(n,\mathbb{C})} \to P_{SO(n,\mathbb{C})}$.

We stated in Theorem 1 that the connection form ω_α of ∇^g takes value in the Lie algebra $\mathfrak{so}(n,\mathbb{C})$. If U_β is another coordinate trivializing neighborhood for TM with $U_\alpha \cap U_\beta \neq \emptyset$ then following relation holds between the connection forms ω_α and ω_β:

$$\omega_\beta = g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d g_{\alpha\beta}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n,\mathbb{C})$ is transition function. These $\mathfrak{so}(n,\mathbb{C})$ valued 1-forms determine a connection 1-form ω on the principal bundle $P_{SO(n,\mathbb{C})}$ with values in $\mathfrak{so}(n,\mathbb{C})$. Now we define a connection 1-form Z on the principal bundle $P_{\mathrm{Spin}(n,\mathbb{C})}$ with values in the Lie algebra $\mathfrak{spin}(n,\mathbb{C})$ by using following diagram

$$\begin{array}{ccc} TP_{\mathrm{Spin}(n,\mathbb{C})} & \xrightarrow{\ Z\ } & \mathfrak{spin}(n,\mathbb{C}) \\ \downarrow{\scriptstyle \Lambda_*} & & \downarrow{\scriptstyle \lambda_*} \\ TP_{SO(n,\mathbb{C})} & \xrightarrow[\ \omega\ ]{} & \mathfrak{so}(n,\mathbb{C}) \end{array}$$

Note that the equality

$$\omega \circ \Lambda_* = \mathbb{A}d_* \circ Z$$

holds.

4. Spinor Bundle on Kahler–Norden Manifolds

The spinor bundle S on a $2n$-dimensional Kahler–Norden spin manifold M is defined as the associated vector bundle

$$S = P_{\mathrm{Spin}(n,\mathbb{C})} \times_\kappa \Delta_n$$

where $\kappa : \mathrm{Spin}(n,\mathbb{C}) \to \mathrm{Aut}(\Delta_n)$ is the spinor representation of $\mathrm{Spin}(n,\mathbb{C})$. In case $n = 2k$, this vector bundle splits into the sum of two subbundles S^+, S^-:

$$S = S^+ \oplus S^-, \quad S^\pm = P_{\mathrm{Spin}(n,\mathbb{C})} \times_{\kappa^\pm} \Delta_n^\pm$$

The composite map $\rho \circ \mathbb{A}d : \mathrm{Spin}(n,\mathbb{C}) \to \mathrm{Aut}(\mathbb{R}^{2n})$ is a representation of $\mathrm{Spin}(n,\mathbb{C})$ on $\mathbb{R}^{2n}$ and gives

$$P_{\mathrm{Spin}(n,\mathbb{C})} \times_{\rho\circ\mathbb{A}d} \mathbb{R}^{2n} \simeq TM.$$

Such interpretations of tangent bundle enable us to product the elements of spinor bundle with tangent vectors by the formula

$$[p,v] \cdot [p,\phi] = [p,\kappa(v)\phi]$$

where $p \in P_{\mathrm{Spin}(n,\mathbb{C})}$, $v \in \mathbb{R}^{2n}$, $\phi \in \Delta_n$. Since the spinor representation is $\mathrm{Spin}(n,\mathbb{C})$-equivariant, the definition of product is independent from the representatives. This product is bilinear, so we extend it the tensor product space

$$TM \otimes S \to S,$$
$$[p, v] \otimes [p, \phi] = [p, \kappa(v)\phi],$$

we denote it as a map $\mu : TM \otimes S \to S$ and call it Clifford multiplication.

We want to define a covariant derivative operator on the spinor bundle S. A section $\Phi \in \Gamma(S)$ is called a spinor field on M. Since S is an associated vector bundle, any spinor field Φ can be identified with the mapping $\hat{\Phi} : P_{\mathrm{Spin}(n,\mathbb{C})} \to \Delta_n$ obeying the transformation rule $\hat{\Phi}(pg) = \kappa(g^{-1})\hat{\Phi}(p)$. Such maps are called equivariant.

The connection 1-form Z on the principal bundle $P_{\mathrm{Spin}(n,\mathbb{C})}$ determines a covariant derivative operator ∇ on the spinor bundle $S = P_{\mathrm{Spin}(n,\mathbb{C})} \times_\kappa \Delta_n$. Let $X \in \mathcal{X}(M)$. The operator

$$\nabla_X : \Gamma(S) \to \Gamma(S),$$

given by $(\nabla_X \hat{\Phi})(p) = (d\hat{\Phi})_p(X_p^*)$ is a covariant derivative on the spinor bundle, where $\hat{\Phi} : P_{\mathrm{Spin}(n,\mathbb{C})} \to \Delta_n$ is an equivariant map associated to the spinor field $\Phi \in \Gamma(S)$, $p \in P_{\mathrm{Spin}(n,\mathbb{C})}$ and X_p^* is the horizontal lift of X_p in $Hor_p(P_{\mathrm{Spin}(n,\mathbb{C})})$ (see [3,8]). It can be expressed as

$$\nabla_X \hat{\Phi} = d\hat{\Phi}(X) + \kappa_{1*}(Z(X^*))\hat{\Phi}$$

where $\kappa_{1*} : \mathfrak{spin}(n, \mathbb{C}) \to \mathrm{End}(\Delta_n)$ is the derivative of κ at identity $1 \in \mathrm{Spin}(n, \mathbb{C})$. It can be also shown that $\kappa_{1*}(e_i e_j) = \kappa(e_i e_j)$.

We can write the covariant derivative operator ∇ locally as follows: Let $s : U \to P_{SO(n,\mathbb{C})}$ be a local section of the frame bundle $P_{SO(n,\mathbb{C})}$. s consists of orthonormal frame $s = \{e_1, e_2, \ldots, e_n, Je_1, Je_2, \ldots, Je_n\}$ of vector fields defined on the open set $U_\alpha \subset M$. We know that the local connection form ω_α is given by the formula

$$\omega_\alpha(X) = \sum_{i<j}(w_{ij}(X) - i\widetilde{w}_{ij}(X))E_{ij}$$

where w_{ij} and $\widetilde{w}_{ij}$ denote the forms defining the Levi-Civita connection, $w_{ij}(X) = g(\nabla_X e_j, e_i)$, $\widetilde{w}_{ij}(X) = -g(\nabla_X e_j, Je_i)$, and $E_{ij} \in \mathfrak{so}(n, \mathbb{C})$ are the standard basis matrices of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$. Let $\widetilde{s} : U_\alpha \to P_{\mathrm{Spin}(n,\mathbb{C})}$ be a lift of s to the 2-fold covering $\Lambda : P_{\mathrm{Spin}(n,\mathbb{C})} \to P_{SO(n,\mathbb{C})}$. Then the local connection forms of Z are given by

$$Z_\alpha(X) = \frac{1}{2}\sum_{i<j}(w_{ij}(X) - i\widetilde{w}_{ij}(X))e_i e_j$$

and

$$\kappa_*(Z_\alpha(X)) = \frac{1}{2}\sum_{i<j}(w_{ij}(X) - i\widetilde{w}_{ij}(X))\kappa_*(e_i e_j)$$

$$\kappa_*(Z_\alpha(X)) = \frac{1}{2}\sum_{i<j}(w_{ij}(X) - i\widetilde{w}_{ij}(X))\kappa(e_i e_j).$$

Let $\Phi \in \Gamma(S)$ be a spinor field and $\hat{\Phi} : P_{\mathrm{Spin}(n,\mathbb{C})} \to \Delta_n$ be associated equivariant map, and consider the composition $\Phi_s = \hat{\Phi} \circ \widetilde{s} : U_\alpha \to \Delta_n$, then we can write $\Phi(x) = [\widetilde{s}(x), \Phi_s(x)]$ for each $x \in U_\alpha$. Hence we can express the covariant derivative of spinors by the formula

$$\nabla_X \Phi = \left[\widetilde{s}, d\Phi_s(X) + \frac{1}{2} \sum_{i<j} (w_{ij}(X) - i\widetilde{w}_{ij}(X)) \kappa(e_i e_j) \Phi_s \right]$$

such expression will be useful for our computations (see [7]).

In the classical theory of spin manifolds, there is a hermitian inner product on the spinor bundle, but in our case there is no such inner product. Instead, one can define some special forms on the spinor bundle S.

Theorem 2. *Let M be an $2n$-dimensional Kahler–Norden spin manifold and S be spinor bundle on M.*

(i) *If $n = 2k+1$ is odd and $k \equiv 1, 2$ (Mod 4), then there is a non-degenerate skew-symmetric bilinear form F, so called symplectic form, on the spinor bundle S with values in $\mathbb{C}$.*
(ii) *If $n = 2k$ is even and $k \equiv 0$ (Mod 4), then there is a non-degenerate symmetric bilinear form B on the spinor bundle S^+ with values in $\mathbb{C}$.*
(iii) *If $n = 2k$ is even and $k \equiv 2$ (Mod 4), then there is a non-degenerate skew-symmetric bilinear form F, so called symplectic form, on S^+ with values in $\mathbb{C}$.*

Proof. Recall $S = P_{\mathrm{Spin}(n,\mathbb{C})} \times_\kappa \Delta_n$ and $S^+ = P_{\mathrm{Spin}(n,\mathbb{C})} \times_{\kappa^+} \Delta_n^+$.

(i) For $[p, \psi_1], [p, \psi_2] \in S$, we set

$$F([p, \psi_1], [p, \psi_2]) = \varepsilon(\psi_1, \psi_2)$$

where ε is the standard symplectic form on Δ_n. Since the spinor representation κ of $\mathrm{Spin}(n, \mathbb{C})$ is symplectic, above equation defines a symplectic form on S.
(ii) For $[p, \psi_1], [p, \psi_2] \in S^+$, we set

$$B([p, \psi_1], [p, \psi_2]) = b(\psi_1, \psi_2)$$

where b is the standard symmetric bilinear form on Δ_n. Since the spinor representation κ^+ of $\mathrm{Spin}(n, \mathbb{C})$ is orthogonal, above equation defines a symmetric bilinear form on S^+.
(iii) Similar to the case (i). □

The following theorem states that the forms F and B are compatible with the connection ∇.

Theorem 3.

(i) *For any $X \in \Gamma(TM)$ and $\Phi, \Psi \in \Gamma(S)$,*

$$X(F(\Phi, \Psi)) = F(\nabla_X \Phi, \Psi) + F(\Phi, \nabla_X \Psi).$$

(ii) *For any $X \in \Gamma(TM)$ and $\Phi, \Psi \in \Gamma(S^+)$,*

$$X(B(\Phi, \Psi)) = B(\nabla_X \Phi, \Psi) + B(\Phi, \nabla_X \Psi).$$

Proof. (i) With respect to a local section $s : U_\alpha \to P_{\mathrm{Spin}(n,\mathbb{C})}$, we have

$$\begin{aligned}
&F(\nabla_X\Phi, \Psi) + F(\Phi, \nabla_X\Psi)\\
&\quad= \varepsilon\left(d\Phi_s(X) + \frac{1}{2}\sum_{i<j}(\omega_{ij}(X) - i\widetilde{w}_{ij}(X))\kappa(e_ie_j)\Phi_s, \Psi\right)\\
&\qquad+ \varepsilon\left(\Phi_s, d\Psi_s(X) + \frac{1}{2}\sum_{i<j}(\omega_{ij}(X) - i\widetilde{w}_{ij}(X))\kappa(e_ie_j)\Psi_s)\right)\\
&\quad= \varepsilon(d\Phi_s(X), \Psi_s) + \frac{1}{2}\sum_{i<j}(\omega_{ij}(X) - i\widetilde{w}_{ij}(X))\varepsilon(\kappa(e_ie_j)\Phi_s, \Psi_s)\\
&\qquad+ \varepsilon(\Phi_s, d\Psi_s(X)) + \frac{1}{2}\sum_{i<j}(\omega_{ij}(X) - i\widetilde{w}_{ij}(X))\varepsilon(\Phi_s, \kappa(e_ie_j)\Psi_s))\\
&\quad= \varepsilon(d\Phi_s(X), \Psi_s) + \varepsilon(\Phi_s, d\Psi_s(X))\\
&\quad= X\varepsilon(\Phi_s, \Psi_s).
\end{aligned}$$

(ii) With respect to a local section $s : U_\alpha \to P_{\mathrm{Spin}(n,\mathbb{C})}$, we have

$$\begin{aligned}
&B(\nabla_X\Phi, \Psi) + B(\Phi, \nabla_X\Psi)\\
&\quad= b\left(d\Phi_s(X) + \frac{1}{2}\sum_{i<j}(\omega_{ij}(X) - i\widetilde{w}_{ij}(X))\kappa^+(e_ie_j)\Phi_s, \Psi_s\right)\\
&\qquad+ b\left(\Phi_s, d\Psi_s(X) + \frac{1}{2}\sum_{i<j}(\omega_{ij}(X) - i\widetilde{w}_{ij}(X))\kappa^+(e_ie_j)\Psi_s)\right)\\
&\quad= b(d\Phi_s(X), \Psi_s) + \frac{1}{2}\sum_{i<j}(\omega_{ij}(X) - i\widetilde{w}_{ij}(X))b(\kappa^+(e_ie_j)\Phi_s, \Psi_s)\\
&\qquad+ b(\Phi_s, d\Psi_s(X)) + \frac{1}{2}\sum_{i<j}(\omega_{ij}(X) - i\widetilde{w}_{ij}(X))b(\Phi_s, \kappa^+(e_ie_j)\Psi_s))\\
&\quad= b(d\Phi_s(X), \Psi_s) + b(\Phi_s, d\Psi_s(X))\\
&\quad= Xb(\Phi_s, \Psi_s).
\end{aligned}$$

□

Now we want to define the Dirac operator on the spinor bundle S. The connection ∇ on S can be thought as a map linear map

$$\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$$

satisfies the following Leibnitz rule:

$$\nabla(f\Phi) = (df) \otimes \Phi + f\nabla\Phi.$$

In a local orthonormal frame $\{e_1, e_2, \ldots, e_n, Je_1, Je_2, \ldots, Je_n\}$ it can be written in the following form:

$$\nabla\Phi = \sum_{i=1}^{n}(e_i^* \otimes \nabla_{e_i}\Phi + (Je_i)^* \otimes \nabla_{Je_i}\Phi).$$

Note that the Clifford multiplication $\mu : TM \otimes S \to S$ induces a map

$$\mu : \Gamma(TM \otimes S) \to \Gamma(S)$$

this means we can product a spinor field with a vector field.

Definition 1 (Dirac Operator). The composition

$$D = \mu \circ \nabla : \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \overset{g}{\cong} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S)$$

is called Dirac operator on Kahler–Norden spin manifold M.

Obviously $D : \Gamma(S) \to \Gamma(S)$ is first order differential operator. With respect to local orthonormal frame $\{e_1, e_2, \ldots, e_n, Je_1, Je_2, \ldots, Je_n\}$

$$D\Phi = \sum_{i=1}^{n}(e_i \cdot \nabla_{e_i}\Phi - (Je_i) \cdot \nabla_{Je_i}\Phi).$$

The investigations of the main properties of such a Dirac operator will be a subject of an another paper.

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