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Canonical bases for real representations of Clifford algebras

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Abstract

The well-known classification of the Clifford algebras $Cl(r, s)$ leads to canonical forms of complex and real representations which are essentially unique by virtue of the Wedderburn theorem. For $s \geqslant 1$ representations of $Cl(r, s)$ on R^{2N} are obtained from representations on R^N by adding two new generators while in passing from a representation of $Cl(p, 0)$ on R^N to a representation of $Cl(r, 0)$ on R^{2N} the number of generators that can be added is either 1, 2 or 4, according as the Clifford algebra represented on R^N is of real, complex or quaternionic type. We have expressed canonical forms of these representations in terms of the complex and quaternionic structures in the half dimension and we obtained algorithms for transforming any given representation of $Cl(r, s)$ to a canonical form. Our algorithm for the transformation of the representations of $Cl(8d + c, 0)$, $c \leqslant 7$ to canonical forms is based on finding an abelian subalgebra of $Cl(8d + c, 0)$ and its invariant subspace. Computer programs for determining explicitly the change of basis matrix for the transformation to canonical forms are given for lower dimensions. The construction of the change of basis matrices uniquely up to the commutant provides a constructive proof of the uniqueness properties of the representations and may have applications in computer graphics and robotics.

Keywords: Clifford algebras; Representation; Canonical forms

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1. Introduction

The classification and representation of Clifford algebras is well-known: Any real Clifford algebra $Cl(r,s)$ is isomorphic to one of the matrix algebras $K(2^n) \cong \mathrm{End}(K^{2^n})$ or $K(2^n) \oplus K(2^n)$, where K is either the reals R, the complex numbers C or the quaternions H [1]. The proof of the classification theorem is constructive and can be used directly to build real representations in the case $s \geqslant 1$ while the problem is more tricky for $Cl(r,0)$. The construction of complex representations is also straightforward.

The structure of the representations is based on the Wedderburn theorem which states that the matrix algebras $K(N)$ have a unique representation on the vector space K^N [2]. It follows that the Clifford algebra $Cl(r,s)$ has exactly one or two representations on K^{2^n}, according as it is isomorphic to $K(2^n)$ or to $K(2^n) \oplus K(2^n)$. The uniqueness of the representation over K means that any two representation can be transformed to each other by conjugation with a unique matrix P with entries in K.

The main result of our paper, given in Section 4, is the construction of an orthonormal basis for the representation space with respect to which the matrices in any given representation have desired canonical forms. We use here the word "canonical" for representations expressible as homogeneous tensor products of the standard Pauli matrices (Eq. (2.6)). Computer programs leading to the corresponding change of basis matrix P in lower dimensions are presented in Appendix A.

In Section 2 we give a concise overview of the classification and representation of Clifford algebras, as the proof of the classification theorem is the key in understanding the construction of canonical representations. As an immediate corollary of the classification theorem, we give the formula for the algebraic type of a real representation of $Cl(r,s)$ in terms of r and s (Proposition 2.8), previously obtained using the representation theory of finite groups [3,4].

Note that if a Clifford algebra $Cl(r,s)$ is isomorphic to a matrix algebra over complex or quaternionic numbers and we use a real representation, then the matrices in the complex or quaternionic subalgebras will commute with all matrices of the representation. Such matrices are called the "commutant" [4].

In Section 3 we obtain the relations between maximal Clifford algebras that can be represented on R^{2N} and the structure of the commutant in the half dimension. The classification theorem states that the generators of $Cl(p,q)$ can be expressed as a tensor product of the generators of $Cl(r,s)$ with $p+q=r+s+2$ and the generators of $Cl(1,1)$, $Cl(0,2)$ or $Cl(2,0)$ (Lemmas 2.3 and 2.4). As $Cl(1,1)$ and $Cl(0,2)$ have 2-dimensional real representations, the representations of Clifford algebras $Cl(r,s)$ with $s \geqslant 1$ follow directly from the classification theorem. The difficulty with the representations of $Cl(r,0)$ comes from the fact that irreducible representations of $Cl(2,0)$ are 4-dimensional, hence the addition of 2 or more generators as one doubles the dimension is a non-trivial problem. In the representations of $Cl(r,0)$ the number of generators to be added as we double the dimension is 1, 2 or 4, depending on whether the representation in the half dimension is real, complex or quaternionic. It turns out that the possibility of adding more than one generator is due to existence of this non-trivial commutant. On the other hand, the existence of a non-trivial commutant is tied to the structure of the maximal Clifford algebra $Cl(r,s)$ that can be represented in the half dimension. Namely, the extendibility of a representation of $Cl(8d,0)$ to a representation of $Cl(8d,1)$ at the same dimension leads to the complex structure for the representations of $Cl(8d+1,0)$. Similarly the extendibility of $Cl(8d+1,0)$ to $Cl(8d+1,2)$ leads to the quaternionic structure for $Cl(8d+3,0)$. The interrelations between these structures are displayed in Table 2.

In Section 4 we study the problem of transforming a given real irreducible representation of a Clifford algebra to a canonical form. As noted above, the Wedderburn theorem implies that if A_i's and $\widetilde{A}_i$'s are arbitrary representations of $Cl(r, s)$, there is a matrix P unique up to the commutant, such that $A_i P = P\widetilde{A}_i$ for $i = 1, \ldots, r + s$, but the determination of such a matrix P is non-trivial especially for the representations of $Cl(r, 0)$. We describe below the difficulties involved and outline our solution.

Recall that as $Cl(1, 1)$ and $Cl(0, 2)$ have 2-dimensional representations, one can easily construct representations on $2N$ dimensions with tensor products using representations on N dimensions. The converse problem is to "recognize" the generators of $Cl(1, 1)$ or $Cl(0, 2)$ and express the remaining elements as tensor products. This is easy because any two anti-commuting elements in the representation with squares $\pm I$ as appropriate (Lemma 4.1), can be put to canonical forms and any matrix anti-commuting with the two anti-commuting elements has a block-diagonal structure (Proposition 4.2).

For representations of $Cl(r, 0)$ it is easy to put one generator to a canonical form, but this does not lead to block-diagonalization. Hence the converse problem is non-trivial even for the case when only a single generator is added in passing to the double dimension. For transforming the generators of $Cl(r, 0)$ to canonical forms we shall use an algorithm mimicking the situation for $Cl(3, 0)$. If A_1, A_2 and $A_3 = A_1A_2$ belong to a representation of $Cl(3, 0)$ on R^4, they can be transformed to "canonical" quaternionic structures by choosing a basis $\{X, -A_1X, -A_2X, -A_1A_2X\}$ where X is an *arbitrary* unit vector (Proposition 4.5). This construction cannot be used for $Cl(r, 0)$ with $r \geqslant 3$, because the products of the images of the generators are in general linearly independent matrices. To apply this procedure to higher dimensions, we find a special vector X and a subalgebra $\mathscr{A}$ generated by a certain subset of the generators such that the action of $\mathscr{A}$ on X leads to the required basis.

To illustrate the procedure, consider a representation of $Cl(6, 0) \cong R(8)$ on R^8. The images of the standard generators A_i, $i = 1, \ldots, 6$ generate the matrix algebra $R(8)$ in which the diagonal matrices constitute an 8-dimensional maximal abelian subalgebra that we denote by $\mathscr{D}$. We aim to express this abelian subalgebra in terms of the generators of the representation and in Proposition 4.7 we show that $\{A_1A_2A_3, A_1A_4A_5, A_2A_4A_6\}$ consisting of matrices with squares I is a generating set. As a set of commuting diagonalizable matrices, they are simultaneously diagonalizable and in addition, they have a unique common eigenvector X corresponding to the eigenvalue 1. As $\mathscr{D}$ acts as identity on the one dimensional subspace spanned by X, only three of the A_i's $i = 1, \ldots, 6$ are independent and the action of the subalgebra $\mathscr{A}$ generated by $\{A_1, A_2, A_4\}$ on X generates the required basis. The construction for higher dimensional real representations is similar, but for complex and quaternionic representations the vector X is not unique. The uniqueness of the vector X up to the commutant leads to an alternative proof of the uniqueness of the representations up to the commutant.

With continuing interest in the relations to group representations [5], representations of Clifford algebras are now finding applications in the field of robotics and computer graphics [6–8]. In these approaches, the motions of rigid bodies in 3-space are modelled with Clifford algebras of various types related to quaternions. These Clifford algebras are representable on R^8 and the translation of the data from one coordinate system to another is a basic problem for which the transformation algorithms given in Appendix A are expected to be useful.

2. Classification and representation of Clifford algebras

In this section we give an overview of the classification and representation of Clifford algebras, based on the presentation in [1]. In Section 2.1 we introduce the notation and give basic definitions.

We have also included a section on the classification and representation of complex Clifford algebras for completeness. In Section 2.3 we give the classification of real Clifford algebras and we conclude with the determination of the algebraic type of a representation of $Cl(r,s)$ in terms of $r-s$ and $r+s$. Propositions 2.7 and 2.8 provide an alternative derivation of some of the results given in [4].

2.1. Basic definitions

Let V be a vector space over the field k and q be a quadratic form on V. The *Clifford algebra* $Cl(V,q)$ associated to V and q is an associative algebra with identity 1, generated by the vector space V and by the identity, subject to the relations $v\cdot v=-q(v)1$ for any vector v in V. The map $\alpha(v)=-v$ for $v\in V$ extends to an involution of the Clifford algebra $Cl(V,q)$ and its ± 1 eigenvalues are called respectively *even* and *odd* parts. Furthermore, the Clifford algebra and the exterior algebra of V are isomorphic as vector spaces. The *order* of a Clifford algebra element is defined as its order as an exterior algebra element.

The real Clifford algebras associated to $V=R^{r+s}$ and to the quadratic form $q(x)=x_1^2+\cdots+x_r^2-x_{r+1}^2-\cdots-x_{r+s}^2$, is denoted by $Cl(r,s)$. For $V=C^n$, as all non-degenerate quadratic forms over C^n are equivalent, $q(z)$ is necessarily $q(z)=z_1^2+\cdots+z_n^2$. The corresponding complex Clifford algebra is denoted by $Cl_c(n)$ [1]. If $\{e_1,e_2,\ldots,e_n\}$ is an orthonormal basis for V, the real Clifford algebra $Cl(r,s)$ is generated by the $\{e_i\}$'s, subject to the relations,

$$e_i^2=-1,\ \ i=1,\ldots,r;\quad e_{r+i}^2=1,\ \ i=1,\ldots,s;\quad e_ie_j+e_je_i=0,\ \ i\neq j. \tag{2.1}$$

Similarly, the generators of the complex Clifford algebra $Cl_c(n)$ satisfy

$$e_i^2=-1,\ \ i=1,\ldots,n,\quad e_ie_j+e_je_i=0,\ \ i\neq j. \tag{2.2}$$

Both $Cl(r,s)$ with $r+s=n$ and $Cl_c(n)$ are 2^n-dimensional vector spaces spanned by the set

$$\{1,e_1,e_2,\ldots,e_{r+s},e_1e_2,\ldots,e_{r+s-1}e_{r+s},e_1e_2e_3,\ldots,e_1e_2e_3,\ldots,e_{r+s}\}. \tag{2.3}$$

If K is a division algebra containing the field k, a K-*representation* of the Clifford algebra $Cl(V,q)$ on the finite dimensional K-vector space W, is a k-algebra homomorphism

$$\rho: Cl(V,q)\to \mathrm{Hom}_K(W,W). \tag{2.4}$$

A representation is called *reducible* if W can be written as a non-trivial direct sum of ρ invariant subspaces. A representation which is not reducible is called *irreducible*. It is known that every K-representation ρ of a Clifford algebra $Cl(V,q)$ can be decomposed into a direct sum $\rho=\rho_1\oplus\cdots\oplus\rho_m$ of irreducible representations.

Two representations $\rho_j: Cl(V,q)\to\mathrm{Hom}_K(W_j,W_j)$ for $j=1,2$ are said to be *equivalent* if there exists a K-linear isomorphism $F: W_1\to W_2$ such that $F\circ\rho_1(\varphi)\circ F^{-1}=\rho_2(\varphi)$ for all $\varphi\in Cl(V,q)$. In particular, for $W_1=W_2$ if two representations are equivalent, $\rho_2(\phi)$ is obtained from $\rho_1(\phi)$ for each ϕ in $Cl(V,q)$, by conjugation with the same matrix.

The algebra of linear endomorphisms of K^N, $\mathrm{End}(K^N)$, is denoted by $K(N)$. It is known that these matrix algebras are simple and have a unique representation up to equivalence [2]. This result known as the Wedderburn theorem also determines the structure of the representations of Clifford algebras.

Proposition 2.1 (Wedderburn theorem). *Let $K=R$, C or H and consider the ring $K(N)$ of $N\times N$ matrices as an algebra over R. Then the natural representation ρ of $K(N)$ on the vec-*

tor space K^N is, up to equivalence, the only irreducible representation of $K(N)$. The algebra $K(N) \oplus K(N)$ has exactly two irreducible representations given by

$$\rho_1(\phi_1, \phi_2) = \rho(\phi_1), \quad \rho_2(\phi_1, \phi_2) = \rho(\phi_2), \tag{2.5}$$

where ρ is the natural representation.

We give below certain isomorphisms that are needed in proofs.

Proposition 2.2 (Proposition 4.2 in [1]). *There are isomorphisms*

$$\begin{aligned}
&R(n) \otimes_R R(m) \cong R(nm), \quad R(n) \otimes_R K \cong K(n), \quad K = C, H, \\
&C \otimes_R C \cong C \oplus C, \quad C \otimes_R H \cong C(2), \quad H \otimes_R H \cong R(4), \\
&(K(n) \oplus K(n)) \otimes_K K(m) \cong K(nm) \oplus K(nm), \quad K = R, C.
\end{aligned}$$

Finally we present our notation. In 2-dimensions the standard Pauli matrices are denoted as

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.6}$$

and their multiplication rules are

$$\sigma\tau = \epsilon, \quad \sigma\epsilon = \tau, \quad \epsilon\tau = \sigma. \tag{2.7}$$

The tensor products are expressed as

$$\sigma \otimes a = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \epsilon \otimes b = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad \tau \otimes c = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}. \tag{2.8}$$

For simplicity of notation, in tensor products, identity matrices of any size will be denoted by 1 *unless the distinction is important. Also by abuse of language, the images of the generators of a Clifford algebra under a representation are called the generators of the representation.*

2.2. Complex Clifford algebras

We give the classification and representation of complex Clifford algebras.

Lemma 2.3. *For all $n \geqslant 0$, there are isomorphisms*

$$Cl_c(n+2) \cong Cl_c(2) \otimes_C Cl_c(n), \tag{2.9a}$$

$$Cl_c(2n) \cong C(2^n), \quad Cl_c(2n+1) \cong C(2^n) \oplus C(2^n). \tag{2.9b}$$

Proof. Let e_j, $j = 1, \ldots, n$ be the generators of $Cl_c(n)$, and let e'_1 and e'_2 be the generators of $Cl_c(2)$. Then

$$\tilde{e}_j = \mathrm{i}e'_1e'_2 \otimes e_j, \quad j = 1, \ldots, n, \quad \tilde{e}_{n+1} = e'_1 \otimes 1, \quad \tilde{e}_{n+2} = e'_2 \otimes 1 \tag{2.10}$$

is a set of generators for $Cl_c(n+2)$, hence Eq. (2.9a) is proved. Eq. (2.9b) follows from Eq. (2.9a) and from the fact that $Cl_c(1) \cong C \oplus C$ and $Cl_c(2) \cong C(2)$, using the isomorphisms in Proposition 2.2. □

As there is a 2-periodicity, the construction of the representations is straightforward. Starting with a representation of $Cl_c(3)$ on C^2 as

$$\rho(e_1) = \mathrm{i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad \rho(e_3) = \mathrm{i}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.11}$$

and given any representation of $Cl(n)$ on C^N, an irreducible representation of $Cl_c(n+2)$ can be obtained by replacing the generators with their representations in the proof of Lemma 2.3. Namely, if $\rho(e_j) = a_j,\ j = 1, \ldots, n$ is a representation on R^N, then

$$\rho(\widetilde{e}_j) = \mathrm{i}\sigma \otimes a_j = \begin{pmatrix}a_j & 0\\ 0 & -a_j\end{pmatrix}, \quad \rho(\widetilde{e}_{n+1}) = \epsilon \otimes 1 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

$$\rho(\widetilde{e}_{n+2}) = \mathrm{i}\tau \otimes 1 = \mathrm{i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{2.12}$$

gives a representation on R^{2N}.

If $Cl(r, s)$ is a Clifford algebra isomorphic to $C(2^n)$, it will have an irreducible representation on $R^{2^{n+1}}$. These representations can be obtained from a complex representation on C^{2^n} once the complex structure is known. This provides an alternative method for the construction of representations of $Cl(8d+1, 0)$, because the product of the generators is an odd Clifford algebra element which is central, hence it is the complex structure J (see Proposition 4.4).

2.3. *Real Clifford algebras*

The crucial step in the classification of real Clifford algebras is the isomorphism theorem below.

Lemma 2.4 (Theorem 4.1 in [1]). *There are isomorphisms*

$$Cl(1,1) \otimes Cl(r,s) \cong Cl(r+1, s+1), \tag{2.13a}$$

$$Cl(0,2) \otimes Cl(r,s) \cong Cl(s, r+2), \tag{2.13b}$$

$$Cl(2,0) \otimes Cl(r,s) \cong Cl(s+2, r) \tag{2.13c}$$

for all $r,\ s \geqslant 0$.

Proof. The generators of $Cl(r, s)$ with squares -1 are denoted by a_i and the ones with square $+1$ by b_i. Let (a'_1, b'_1), (a'_1, a'_2), (b'_1, b'_2) be the generators of $Cl(1, 1)$, $Cl(2, 0)$ and $Cl(0, 2)$, respectively. Note that as $a_i^2 = -1$, $b_i^2 = 1$ and as they form an anti-commuting set, $(a'_1 b'_1)^2 = 1$ while, $(a'_1 a'_2)^2 = (b'_1 b'_2)^2 = -1$. Thus there will be a twisting whenever a Clifford algebra element of $Cl(r, s)$ is tensored with an element of $Cl(2, 0)$ or $Cl(0, 2)$. Thus

$$(a'_1 b'_1 \otimes a_i)^2 = -1, \quad (a'_1 b'_1 \otimes b_i)^2 = 1,$$

$$(a'_1 a'_2 \otimes a_i)^2 = 1, \quad (a'_1 a'_2 \otimes b_i)^2 = -1,$$

$$(b'_1 b'_2 \otimes a_i)^2 = 1, \quad (b'_1 b'_2 \otimes b_i)^2 = -1. \tag{2.14}$$

Hence the generators of $Cl(r+1, s+1)$ can be obtained easily as

$$\begin{aligned} \widetilde{a}_i &= a_1' b_1' \otimes a_i, \quad i = 1, \ldots, r, \quad \widetilde{a}_{r+1} = a_1' \otimes 1, \\ \widetilde{b}_i &= a_1' b_1' \otimes b_i, \quad i = 1, \ldots, s, \quad \widetilde{b}_{s+1} = b_1' \otimes 1. \end{aligned} \tag{2.15}$$

A similar construction works for $Cl(s, r+2)$ with a twisting. The standard generators are obtained as

$$\begin{aligned} \widetilde{a}_i &= b_1' b_2' \otimes b_i, \quad i = 1, \ldots, s, \\ \widetilde{b}_i &= b_1' b_2' \otimes a_i, \quad i = 1, \ldots, r, \quad \widetilde{b}_{r+1} = b_1' \otimes 1, \quad \widetilde{b}_{r+2} = b_2' \otimes 1. \end{aligned} \tag{2.16}$$

Finally the generators of $Cl(s+2, r)$ are also obtained with a twist as

$$\begin{aligned} \widetilde{a}_i &= a_1' a_2' \otimes b_i, \quad i = 1, \ldots, s, \quad \widetilde{a}_{s+1} = a_1' \otimes 1, \quad \widetilde{a}_{s+2} = a_2' \otimes 1, \\ \widetilde{b}_i &= b_1' b_2' \otimes a_i, \quad i = 1, \ldots, r. \end{aligned} \tag{2.17}$$

and the proof is completed. □

The proofs of the isomorphisms (2.13a) and (2.13b) lead directly to the construction of real representations of $Cl(r, s)$ for $s \neq 0$. However the construction of the real representations of $Cl(r, 0)$ is not a direct consequence of the proof of (2.13c), because the generators of $Cl(2, 0)$ are represented by 4×4 matrices.

The Clifford algebras $Cl(n, 0)$ and $Cl(0, n)$ for $n \leqslant 8$ as given in [1, Section 1, Table 1] in proving Proposition 2.6, and can be read off from Eqs. (2.19a–h) we do not list them here.

Iterating the isomorphisms in Lemma 2.4, we can obtain the "periodicity isomorphisms" (Theorem 4.3, in [1]) as follows.

Proposition 2.5. *There are isomorphisms*

$$Cl(0, n+8) \cong Cl(0, n) \otimes Cl(0, 8), \tag{2.18a}$$

$$Cl(n+8, 0) \cong Cl(n, 0) \otimes Cl(8, 0), \tag{2.18b}$$

$$Cl(r+8d, s) \cong Cl(r, s+8d) \cong Cl(r, s) \otimes R(2^{4d}), \quad r, s \leqslant 7, \tag{2.18c}$$

$$Cl(n+r, n+s) \cong Cl(r, s) \otimes R(2^n), \quad r, s \leqslant 7. \tag{2.18d}$$

We can then obtain the classification of $Cl(r, s)$ as

Proposition 2.6. *There are isomorphisms*

$$Cl(n, n) \cong R(2^n), \tag{2.19a}$$

$$Cl(n, n+1) \cong R(2^n) \oplus R(2^n), \qquad Cl(n+1, n) \cong C(2^n), \tag{2.19b}$$

$$Cl(n, n+2) \cong R(2^{n+1}), \qquad Cl(n+2, n) \cong H(2^n), \tag{2.19c}$$

$$Cl(n, n+3) \cong C(2^{n+1}), \qquad Cl(n+3, n) \cong H(2^n) \oplus H(2^n), \tag{2.19d}$$

$$Cl(n, n+4) \cong H(2^{n+1}), \qquad Cl(n+4, n) \cong H(2^{n+1}), \tag{2.19e}$$

$$Cl(n, n+5) \cong H(2^{n+1}) \oplus H(2^{n+1}), \qquad Cl(n+5, n) \cong C(2^{n+2}), \tag{2.19f}$$

$$Cl(n, n+6) \cong H(2^{n+2}), \qquad Cl(n+6, n) \cong R(2^{n+3}), \tag{2.19g}$$

$$Cl(n, n+7) \cong C(2^{n+3}), \qquad Cl(n+7, n) \cong R(2^{n+3}) \oplus R(2^{n+3}). \tag{2.19h}$$

Proof. From Eq. (2.13a) of Lemma 2.4, $Cl(n,n)\cong Cl(n-1,n-1)\otimes Cl(1,1)$. Then (2.19a) can be proved by induction using the isomorphism $Cl(1,1)\cong R(2)$. The proofs of (2.19b–h) are similar. □

The isomorphisms (2.19a–h) can be rearranged in the format below.

Proposition 2.7. *The Clifford algebras $Cl(r,s)$ are isomorphic to either of the matrix algebras $R(2^n)$, $R(2^n)\oplus R(2^n)$, $C(2^n)$, $H(2^n)$ or $H(2^n)\oplus H(2^n)$ according to the values of r and s as given below*:

$$R(2^n): \quad s+r=2n, \quad s-r=0,2 \pmod 8, \tag{2.20a}$$
$$R(2^n)\oplus R(2^n): \quad s+r=2n+1, \quad s-r=1 \pmod 8, \tag{2.20b}$$
$$C(2^n): \quad s+r=2n+1, \quad s-r=3,7 \pmod 8, \tag{2.20c}$$
$$H(2^n): \quad s+r=2n+2, \quad s-r=4,6 \pmod 8, \tag{2.20d}$$
$$H(2^n)\oplus H(2^n): \quad s+r=2n+3, \quad s-r=5 \pmod 8. \tag{2.20e}$$

Recall that a Clifford algebra isomorphic to $K(n)$ or $K(n)\oplus K(n)$ is called respectively of *real*, *complex* or *quaternionic type*, according as $K=R$, $K=C$ or $K=H$. The discussion above, together with Proposition 2.5, leads to the classification of the type of the representation according to the values of $s-r \pmod 8$. A proof of this theorem is given in [3, 4], using finite group representations. The result below follows immediately from the classification theorem.

Proposition 2.8. *The Clifford algebras $Cl(r,s)$ are of real, complex or quaternionic types, respectively according as $s-r=0,1,2$, $s-r=3,7$ or $s-r=4,5,6 \pmod 8$.*

Remark 2.9. The maximal number of linearly independent vector fields on the sphere S^{N-1} is known as the *Radon–Hurwitz number* $k(N)$ computed as follows. If $N=(2a+1)2^{4d+c}$, $c=0,1,2,3$, then $k(N)=8d+2^c-1$. By Proposition 7.1 in [1], representations of $Cl(r,0)$ on R^N give linearly independent vector fields on S^{N-1}, hence the irreducible representations of $Cl(k(N),0)$ are N-dimensional.

3. Canonical forms of representation

In this section we obtain canonical expressions for the representations of $Cl(r,s)$, in the sense of homogeneous tensor products of two dimensional representations of the standard generators. We prefer to work with a form where the generators coming from the half dimension are represented by block diagonal matrices while it would as well be possible to represent them with off diagonal blocks.

The constructions for $s\geqslant 1$ given in Section 3.2 are straightforward, while the constructions for $Cl(r,0)$ are non-trivial and closely related to the commutants of representations in half dimensions.

3.1. Preliminaries

A representation of the Clifford algebra $Cl(r,s)$ on R^N determines an $r+s$-dimensional subspace in $\mathrm{End}(R^N)\cong R(N)$. The images of the standard generators are linear transformations with square $\pm I$. We first note that without loss of generality one can represent the standard generators with symmetric or skew-symmetric matrices [1, Proposition 5.16].

Proposition 3.1. *Let A_i, $i = 1, \dots, r+s$ be an anti-commuting set of endomorphisms of R^N satisfying $A_i^2 = \epsilon I$ where $\epsilon = \pm 1$. There is an inner product on R^N with respect to which A_i's are skew symmetric or symmetric, according as $\epsilon = -1$ or $\epsilon = 1$.*

Proof. Let (X, Y) denote the standard inner product on R^N and define a new inner product

$$\begin{aligned}\langle X, Y\rangle = (X, Y) &+ \sum_i (A_i X, A_i Y) + \sum_{i<j} (A_i A_j X, A_i A_j Y) \\ &+ \sum_{i<j<k} (A_i A_j A_k X, A_i A_j A_k Y) + \cdots \\ &+ \sum_{i_1<\dots<i_{r+s}} (A_{i_1} \dots A_{i_{r+s}} X, A_{i_1} \dots A_{i_{r+s}} Y).\end{aligned}$$

It can be checked that if $A_i^2 = \epsilon I$, then $\langle A_i X, Y\rangle - \epsilon\langle X, A_i Y\rangle = 0$, hence A_i is symmetric or skew-symmetric and the proposition is proved. □

Let $S^{(N)}$ be the set of matrices in $R(N)$ with minimal polynomial $A^2 + \lambda I = 0$, where λ can be positive, negative or zero. If λ is positive, as complex eigenvalues occur in conjugate pairs, the eigenspaces of A have equal dimension and A is trace zero. However, for λ zero or negative (or for complex representations) this is no longer true. Nevertheless we show that if at least two such matrices lie in the same linear subspace their eigenspaces have the same dimension and they are trace zero.

Lemma 3.2. *Let A and B be $2n \times 2n$ matrices satisfying $A^2 + \lambda I = 0$, $B^2 + \mu I = 0$, $\lambda \neq 0$, $\mu \neq 0$, and $AB + BA = 0$. Then the eigenspaces of A and B have equal dimension.*

Proof. We may assume that A is in Jordan canonical form over C, i.e. $A = \sqrt{-\lambda}\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ with $p + q = 2n$, where p and q may be unequal. Multiple by B the equation $AB + BA = 0$ and take its trace to get $0 = 2\,\mathrm{tr}(AB^2) = -2\mu\,\mathrm{tr}(A)$. Here we used the fact that $\mathrm{tr}(BAB) = \mathrm{tr}(AB^2)$. This shows that $\mathrm{tr}(A) = 0$ and hence, $p = q$. Similar argument on B shows Lemma 3.2. □

An important implication of this result is the following.

Corollary 3.3. *Let A_i $i = 1, \dots, n$, $n \geqslant 2$ be a set of anti-commuting matrices with $A_i^2 = \pm I$. Then $\mathrm{tr}(A_i) = 0$ for $i = 1, \dots, n$.*

The Clifford algebras that can be represented at the same dimension are shown in Table 1, which is the classification table in [1], where we joined the cells to display the Clifford algebras that are represented at the same dimension. Given an irreducible representation of $Cl(r, s)$ on R^N, if no new generator can be added without increasing the dimension, the representation is called *maximal*. For example the representations of $Cl(7, 0)$, $Cl(3, 4)$ and $Cl(4, 1)$ on R^8 are all maximal. If $Cl(p, q)$ is not maximal, then the set of generators that can be added to get a representation on the same dimension is called *complementary generators*. Note that the set of complementary generators is not unique. For example starting with a representation of $Cl(2,0)$ on R^4, one can either add one complementary generator with negative square to get $Cl(3,0)$, or three complementary generators with positive squares to get $Cl(2,3)$.

Table 1
The list of Clifford algebras that can be represented on the same dimension

R	**R(2)**	**R(4)**		**R(8)**				**R(16)**
Cl(0,0) R(1)	Cl(1,0) C(1)	Cl(2,0) H(1)	Cl(3,0) H(1) $\oplus$ H(1)	Cl(4,0) H(2)	Cl(5,0) C(4)	Cl(6,0) R(8)	Cl(7,0) R(8) $\oplus$ R(8)	Cl(8,0) R(16)
Cl(0,1) R(1) $\oplus$ R(1)	Cl(1,1) R(2)	Cl(2,1) C(2)	Cl(3,1) H(2)	Cl(4,1) H(2) $\oplus$ H(2)	Cl(5,1) H(4)	Cl(6,1) C(8)	Cl(7,1) R(16)	Cl(8,1) R(16) $\oplus$ R(16)
Cl(0,2) R(2)	Cl(1,2) R(2) $\oplus$ R(2)	Cl(2,2) R(4)	Cl(3,2) C(4)	Cl(4,2) H(4)	Cl(5,2) H(4) $\oplus$ H(4)	Cl(6,2) H(8)	Cl(7,2) C(16)	Cl(8,2) R(32)
Cl(0,3) C(2)	Cl(1,3) R(4)	Cl(2,3) R(4) $\oplus$ R(4)	Cl(3,3) R(8)	Cl(4,3) C(8)	Cl(5,3) H(8)	Cl(6,3) H(8) $\oplus$ H(8)	Cl(7,3) H(16)	Cl(8,3) C(32)
Cl(0,4) H(2)	Cl(1,4) C(4)	Cl(2,4) R(8)	Cl(3,4) R(8) $\oplus$ R(8)	Cl(4,4) R(16)	Cl(5,4) C(16)	Cl(6,4) H(16)	Cl(7,4) H(16) $\oplus$ H(16)	Cl(8,4) H(32)
Cl(0,5) H(2) $\oplus$ H(2)	Cl(1,5) H(4)	Cl(2,5) C(8)	Cl(3,5) R(16)	Cl(4,5) R(16) $\oplus$ R(16)	Cl(5,5) R(32)	Cl(6,5) C(32)	Cl(7,5) H(32)	Cl(8,5) H(32) $\oplus$ H(32)
Cl(0,6) H(4)	Cl(1,6) H(4) $\oplus$ H(4)	Cl(2,6) H(8)	Cl(3,6) C(16)	Cl(4,6) R(32)	Cl(5,6) R(32) $\oplus$ R(32)	Cl(6,6) R(64)	Cl(7,6) C(64)	Cl(8,6) H(64)
Cl(0,7) C(8)	Cl(1,7) H(8)	Cl(2,7) H(8) $\oplus$ H(8)	Cl(3,7) H(16)	Cl(4,7) C(32)	Cl(5,7) R(64)	Cl(6,7) R(64) $\oplus$ R(64)	Cl(7,7) R(128)	Cl(8,7) C(128)
Cl(0,8) R(16)	Cl(1,8) C(16)	Cl(2,8) H(16)	Cl(3,8) H(16) $\oplus$ H(16)	Cl(4,8) H(32)	Cl(5,8) C(64)	Cl(6,8) R(128)	Cl(7,8) R(128) $\oplus$ R(128)	Cl(8,8) R(256)

In the construction of canonical representations, if r and s are both non-zero, one can move diagonally backwards on Table 1, constructing representations of $Cl(r, s)$ in terms of the representations of $Cl(r-1, s-1)$, while on the vertical edge of Table 1, the representations of $Cl(0, s)$ with $s \geqslant 2$ can be obtained from the representations of $Cl(s-2, 0)$. These constructions are trivial, as one just replaces the standard generators with their 2-dimensional representations. Hence the non-trivial part is to give representations of $Cl(r, 0)$ in terms of a representation in half dimension.

3.2. Representations of $Cl(r, s)$ with $s \geqslant 1$

We now give the representations of $Cl(r, s)$ with $s \geqslant 1$ on R^{2N} in terms of the representations of $Cl(r-1, s-1)$ or $Cl(s-2, r)$ on R^N.

Proposition 3.4. *Let ρ be a real representation of the Clifford algebra $Cl(r, s)$ on R^N and let a_i, b_i be the canonical generators of the representation, let A_i, B_i be the complementary generators satisfying*

$$a_i^2 + I = 0, \quad A_i^2 + I = 0, \quad b_i^2 - I = 0, \quad B_i^2 - I = 0$$

and let J_i be the generators of the maximal commuting subalgebra. Then

(i) *the canonical generators, the complementary generators and the generators of the maximal commuting subalgebra of the representation of $Cl(r+1, s+1)$ on R^{2N} are given by*

$$\begin{aligned} &\widetilde{a}_i = \sigma \otimes a_i, \quad \widetilde{b}_i = \sigma \otimes b_i, \quad \widetilde{b}_{s+1} = \tau \otimes 1, \quad \widetilde{a}_{r+1} = \epsilon \otimes 1, \\ &\widetilde{A}_i = \sigma \otimes A_i, \quad \widetilde{B}_i = \sigma \otimes B_i, \\ &\widetilde{J}_i = 1 \otimes J_i, \end{aligned} \tag{3.1}$$

(ii) *the canonical generators, the complementary generators and the generators of the maximal commuting subalgebra of the representation of $Cl(s, r+2)$ on R^{2N} are given by*

$$\begin{aligned} &\widetilde{a}_i = \epsilon \otimes b_i, \quad \widetilde{b}_i = \epsilon \otimes a_i, \quad \widetilde{b}_{r+1} = \sigma \otimes 1, \quad \widetilde{b}_{r+2} = \tau \otimes 1, \\ &\widetilde{A}_i = \epsilon \otimes B_i, \quad \widetilde{B}_i = \epsilon \otimes A_i, \\ &\widetilde{J}_i = 1 \otimes J_i. \end{aligned} \tag{3.2}$$

3.3. Representations of $Cl(r, 0)$

We have summarized the structure of the maximal representations of $Cl(8d + c, 0)$ for $c = 0, 1, 3, 7$ in Table 2 where it can be seen that when in passing from a representation of real type on N dimensions to a maximal representation in $2N$ dimensions, there is always a single generator to be added, which can be chosen in the form $\epsilon \otimes I_N$. On the other hand when the representation in the half dimension is complex with complex a structure J, we see that one can add $\epsilon \otimes I_N$ and $\tau \otimes J$, while on quaternionic backgrounds with quaternionic structures J_i, $i = 1, 2, 3$ one can add $\epsilon \otimes I_N$ and $\tau \otimes J_i$. On the other hand the existence of the complex or quaternionic structures is related to the complementary generators with positive squares that can be added

Table 2
Construction of the generators, commutant and complementary generators of the representation of $Cl(r, 0)$ in terms of the data in half dimension

Clifford algebra Representation space	Cl(8d,0) $R^{2^{4d}}$	Cl(8d+1,0) $R^{2^{4d+1}}$	Cl(8d+3,0) $R^{2^{4d+2}}$	Cl(8d+7,0) $R^{2^{4d+3}}$
Generators	$a_i^{(n)}$:	$b_i^{(n)}$:	$c_i^{(n)}$:	$d_i^{(n)}$:
	$\sigma \otimes d^{(n-1)}$	$\sigma \otimes a_i^{(n)}$	$\sigma \otimes b_i^{(n)}$	$\sigma \otimes c_i^{(n)}$
	$\epsilon \otimes I^{(n-1)}$	$\epsilon \otimes I^{(n)}$	$\epsilon \otimes I^{(n)}$	$\epsilon \otimes I^{(n)}$
			$\tau \otimes J^{(n)}$	$\tau \otimes J_1^{(n)}$
				$\tau \otimes J_2^{(n)}$
				$\tau \otimes J_3^{(n)}$
Commutant	–	$J^{(n)} = \epsilon \otimes \alpha^{(n)}$	$J_1^{(n)} = 1 \otimes J^{(n)}$	
	–	–	$J_2^{(n)} = \epsilon \otimes \beta_1^{(n)}$	
	–	–	$J_3^{(n)} = \epsilon \otimes \beta_2^{(n)}$	
Complementary	$\alpha^{(n)} = \tau \otimes I^{(n-1)}$	$\beta_1^{(n)} = \tau \otimes I^{(n)}$		
Generators		$\beta_2^{(n)} = \sigma \otimes \alpha^{(n)}$		

to the representation without increasing the dimension. We denote the representation in the half dimension as the "background".

Representation of $Cl(8n, 0)$ *on* $R^{2^{4n}}$: We start with a representation of $Cl(8n, 0)$ on $R^{2^{4n}}$. The representation is real, hence the maximal commuting subalgebra is R, generated by the identity only. $Cl(8n, 0)$ can be extended to $Cl(8n, 1)$ hence there is a single complementary generator. The data of the representation is below.

$$\text{Canonical generators:} \quad a_i^{(n)}, \quad i = 1, \ldots, 8n, \tag{3.3a}$$

$$\text{Complementary generator:} \quad \alpha^{(n)}. \tag{3.3b}$$

Representation of $Cl(8n + 1, 0)$ *on* $R^{2^{4n+1}}$: From the data above we can obtain the representations of $Cl(8n + 1, 0)$ on double dimension. The representation is complex and extendible to a representation of $Cl(8n + 1, 2)$.

$$\text{Canonical generators:} \quad b_i^{(n)} = \sigma \otimes a_i^{(n)}, \quad i = 1, \ldots, 8n,$$

$$b_{8n+1}^{(n)} = \epsilon \otimes I. \tag{3.4a}$$

$$\text{Generator of the commutant:} \quad J^{(n)} = \epsilon \otimes \alpha^{(n)}. \tag{3.4b}$$

$$\text{Complementary generators:} \quad \beta_1^{(n)} = \tau \otimes I,$$

$$\beta_2^{(n)} = \sigma \otimes \alpha^{(n)}. \tag{3.4c}$$

Note that increasing the number of generators by 1 is trivial, it is sufficient to tensor the old generators by say σ (tensoring with τ would work as well) and add the generator $\epsilon \otimes 1$. The change of the type from real to complex is tied to the existence of a complementary generator as follows. A matrix which commutes with $\epsilon \otimes 1$ has to be either $\epsilon \otimes b$ or $1 \otimes c$, where b is symmetric and c is skew-symmetric. But if $1 \otimes c$ commutes with $\sigma \otimes a_i$, then $a_i c - c a_i = 0$, which is not possible because the background is of real type. On the other hand if $\epsilon \otimes b$ commutes with $\sigma \otimes a_i$, then $ba_i + a_i b = 0$, and as the background admits a complementary generator, it is possible to choose $J = \epsilon \otimes A_i$. The existence of one complementary generator, namely $\tau \otimes 1$ is trivial. The existence of a second one is again tied to the existence of the complementary generator in the background.

Representation of $Cl(8n+3, 0)$ *on* $R^{2^{4n+2}}$: In passing from representations of $Cl(8n+1, 0)$ to the representations of $Cl(8n+3, 0)$, the background is of complex type with the commutant constructed as above.

$$
\begin{aligned}
\text{Canonical generators:}\quad & c_i^{(n)} = \sigma \otimes b_i^{(n)}, \quad i = 1, \ldots, 8n+1,\\
& c_{8n+2}^{(n)} = \epsilon \otimes I,\\
& c_{8n+3}^{(n)} = \tau \otimes J^{(n)}, \qquad (3.5a)\\
\text{Generators of the commutant:}\quad & J_1^{(n)} = 1 \otimes J^{(n)},\\
& J_2^{(n)} = \epsilon \otimes \beta_1^{(n)},\\
& J_3^{(n)} = \epsilon \otimes \beta_2^{(n)}. \qquad (3.5b)
\end{aligned}
$$

As J commutes with the b_i's we can add the canonical generator $\tau \otimes J$ which anti-commutes with $\sigma \otimes b_i$ and $\epsilon \otimes 1$. Since the image is quaternionic, we should construct the commutant. As above, candidates for the commutant are $1 \otimes c$ and $\epsilon \otimes b$. It can be seen that as the background is complex, $c = J$ is possible, hence the matrices of the quaternionic structure, commuting every canonical generator, are $\epsilon \otimes \beta_{8n+1}$. As the representation is maximal, there are no complementary generators.

Representation of $Cl(8n+7, 0)$ *on* $R^{2^{4n+3}}$: In this case the background is quaternionic and the dimension of the linear subspace should increase by 4. The c_i's form an anti-commuting set while the J_i's anti-commute among each other but commute with all the c_i's. Thus in the double dimension we have the representations $\epsilon \otimes 1$, $\sigma \otimes c_i$ and $\tau \otimes J_i$. The canonical generators are thus

$$
\begin{aligned}
\text{Canonical generators:}\quad & d_i^{(n)} = \sigma \otimes c_i^{(n)}, \quad i = 1, \ldots, 8n+3,\\
& d_{8n+4}^{(n)} = \epsilon \otimes I,\\
& d_{8n+5}^{(n)} = \tau \otimes J_1^{(n)},\\
& d_{8n+6}^{(n)} = \tau \otimes J_2^{(n)},\\
& d_{8d+7}^{(n)} = \tau \otimes J_3^{(n)}. \qquad (3.6a)
\end{aligned}
$$

The representation is real and maximal. Hence the maximal commuting subalgebra is generated by the identity only and there are no complementary generators.

Representation of $Cl(8n+8, 0)$ *on* $R^{2^{4n+4}}$: Here there is a single generator to be added. As the representation is real, the commutant is generated by the identity and there is a single complementary generator.

$$\text{Canonical generators:} \quad a_i^{(n+1)} = \sigma \otimes d_i^{(n)}, \quad i = 1, \ldots, 8n+7,$$
$$a_{8n+8}^{(n+1)} = \epsilon \otimes I. \tag{3.7a}$$
$$\text{Complementary generator:} \quad \alpha^{(n+1)} = \tau \otimes I. \tag{3.7b}$$

These results are summarized in below and in Table 2.

Proposition 3.5. *Let $d_i, i = 1, \ldots, 8(n-1)+7$ be a set of generators for a representation of $Cl(8(d-1)+7, 0)$ on R^N. Then the set of canonical generators, the commutant and complementary generators of representations of $Cl(8d, 0)$, $Cl(8d+1, 0)$, $Cl(8d+3, 0)$ and $Cl(8d+7, 0)$ are determined in terms of these as given in Table 2.*

The product of the generators of a Clifford algebra is called the "volume element" and denoted by ω. The structure of the volume element in $Cl(r, s)$ is useful in working with iterative constructions. We first quote below the following result.

Proposition 3.6 (Proposition 3.3 in [1]). *Let $\{e_1, \ldots, e_{r+s}\}$ be an orthonormal set of generators for $Cl(r, s)$ and let $\omega = e_1, \ldots, e_{r+s}$. Then*

$$\omega^2 = (-1)^{\frac{n(n+1)}{2}+s}, \tag{4.4}$$

where $n = r + s$. Furthermore, for n odd, ω is a central element in $Cl(r, s)$ while for n even,

$$\varphi\omega = \omega\alpha(\varphi), \quad \text{for all } \varphi \in Cl(r, s), \tag{4.5}$$

where $\alpha(\varphi) = \pm\varphi$ respectively for even and odd elements.

It follows that when ω is central $\rho(\omega)$ belongs to the commutant and it can be checked that for $\omega^2 = 1$, $\rho(\omega) = I$ while for $\omega^2 = -1$, $\rho(\omega)$ is pure imaginary. Thus for $Cl(8d+3, 0)$ and $Cl(8d+7, 0)$, as $\omega^2 = 1$ and ω is a central element, hence $\rho(\omega)$ has to be proportional to identity and the choice of the sign leads to inequivalent representations. In our canonical representation, the product of the generators of $Cl(3, 0)$ is -1, and we stick with this convention. The general form of the volume elements can be obtained as follows.

Proposition 3.7. *Let $a_i^{(d)}, b_i^{(d)}, c_i^{(d)}$ and $d_i^{(d)}$ be canonical generators as given in Table 2, and assume that $c_1^{(0)} c_2^{(0)} c_3^{(0)} = -I$. Then for all d,*

$$\prod_i^{8d} a_i = \tau \otimes 1, \quad \prod_i^{8d+1} b_i = \epsilon \otimes \tau \quad \prod_i^{8d+3} c_i = -I, \quad \prod_i^{8d+7} d_i = I.$$

Proof. From Table 2, it is easy to see that $\prod_i^{8d} a_i = (\sigma \otimes 1)(\epsilon \otimes 1) = \tau \otimes 1$. Then $\prod_i^{8d+1} b_i = (1 \otimes \tau)(\epsilon \otimes 1) = \epsilon \otimes \tau$. The proofs of the remaining products are similar. □

4. Transformation to canonical forms

In Section 3 we have given the construction of canonical forms recursively, starting from lower dimensions. Here we consider the converse problem: Given an orthonormal basis for the image of $V \subset Cl(r, s)$ in R^N, find an orthonormal basis for R^N with respect to which the matrices of the basis elements have desired canonical forms. From the Wedderburn theorem, we know that such a basis is unique for Clifford algebras of real type, while it is determined up to the commutants for Clifford algebras of complex and quaternionic types.

We would like to start by noting that although the existence of the change of basis matrix is guaranteed by the Wedderburn theorem, its direct determination is not practically feasible. If an anti-commuting set of matrices with squares $-I$ is given, it is in principle possible to put them to canonical forms iteratively by restricting the change of basis matrix P at each step, but this procedure gives non-linear equations for the components of P and is not useful beyond a set of just two matrices.

It is clear that if $Cl(r, s)$ has an irreducible representation on R^N, then any reducible representation on R^{kN} is block diagonal: One chooses N linearly independent vectors with respect to which the given matrices have desired forms and then apply Gram–Schmidt orthogonalization procedure to obtain a direct sum splitting and reiterate. Henceforth we consider irreducible representations only.

We start with the representations of $Cl(r, s)$ with $s \geqslant 1$ in Section 4.1. Section 4.2 is devoted to a detailed the study of the representations of $Cl(7, 0)$ and the general results are given in Section 4.3.

4.1. Representations of $Cl(r, s)$ with $s \geqslant 1$

We have seen that a representation of $Cl(r, s)$ with $s \geqslant 1$ can be constructed by tensoring with the representations in half dimension with σ and adding two new generators $\epsilon \otimes 1$ and $\sigma \otimes 1$.

In the transformation of a given representation to a canonical one we follow the reverse path. Assume that we know how to transform a given representation of $Cl(r, s)$ on R^N to a canonical form. For $Cl(r + 1, s + 1)$ we want to identify one generator with negative square as $\epsilon \otimes 1$ and another with positive square as $\tau \otimes 1$. Once we find a basis with respect to which two generators have matrices $\epsilon \otimes 1$ and $\tau \otimes 1$, the remaining ones will be automatically of the form $A_i = \sigma \otimes a_i$, as they should anti-commute with both $\epsilon \otimes 1$ and $\tau \otimes 1$. The situation is similar for representations of $Cl(r, s + 2)$ where we identify two generators with positive squares as $\sigma \otimes 1$ and $\tau \otimes 1$. We thus start with proving the following Lemma.

Lemma 4.1. *Let A, B, C be a set of trace zero anti-commuting linear transformations with*

$$A^2 + I = 0, \quad B^2 - I = 0, \quad C^2 - I = 0. \tag{4.1}$$

Then, there are orthonormal bases of R^{2N} with respect to which

(i) $B = \sigma \otimes I, \quad C = \tau \otimes I,$
(ii) $A = \epsilon \otimes I, \quad B = \sigma \otimes I,$
(iii) $A = \epsilon \otimes I, \quad B = \tau \otimes I.$

Proof. As A, B and C are trace zero, the ± 1 and $\pm$i eigenspaces are N-dimensional. Thus one can take $A = \epsilon \otimes I$ or $B = \sigma \otimes I$. We give the proof of (i) as an example. Let $X_1, \ldots, X_N$ be an orthonormal basis for the $+1$ eigenspace of B and define $Y_i = CX_i$. Computing BY_i and CY_i it can be seen that B and C have desired canonical forms. The proofs of (ii) and (iii) are similar. □

It follows that given a representation of $Cl(r, s)$ with $r \geqslant 1$ and $s \geqslant 1$ or for $s \geqslant 2$, one can move diagonally backwards in constructing the representations.

Proposition 4.2. *Let $A_1, \ldots, A_r, B_1, \ldots, B_s$ belong to a an irreducible representation of $Cl(r, s)$ on R^{2N} with $A_i^2 + I = 0$, $B_j^2 - I = 0$ and assume that the transformation of a given representation to canonical forms is known on R^N. Then, there is an orthonormal basis of R^{2N} with respect to which*

(i) *If* $r \geqslant 1, s \geqslant 1$,

$$A_r = \epsilon \otimes I, \quad B_s = \tau \otimes I, \quad A_i = \sigma \otimes a_i, \quad i = 1, \ldots, r-1,$$
$$B_j = \sigma \otimes b_j, \quad j = 1, \ldots, s-1. \tag{4.2a}$$

(ii) *If* $s \geqslant 2$,

$$B_s = \sigma \otimes I, \quad B_{s-1} = \tau \otimes I, \quad A_i = \epsilon \otimes b_i, \quad i = 1, \ldots, s-2,$$
$$B_j = \epsilon \otimes a_j, \quad j = 1, \ldots, r, \tag{4.2b}$$

where

$$a_i^2 + I = 0, \quad b_j^2 - I = 0.$$

Proof. For (i), using Lemma 4.1(iii), one can choose an orthonormal basis $\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}$ with respect to which A_r and B_s have matrices $\epsilon \otimes I$ and $\tau \otimes I$. Then any matrix in the representation which anti-commutes with these has to be of the form $\sigma \otimes a$, with $a^2 = -I$ or $\sigma \otimes b$, with $b^2 = I$. By assumption, the transformations in N dimensions are known, hence we can put the remaining in canonical forms by orthogonal transformations on $\{X_1, \ldots, X_N\}$. The proof of (ii) is similar, but there is a twisting as in the proof of the construction of canonical forms. □

This complete the discussion for the case $s \geqslant 1$. Although the case $s = 0$ will be discussed in the following section we complete this section with two remarks on canonical forms for $Cl(r, 0)$.

Remark 4.3. Lemma 4.1 can be used to transform representations of $Cl(r + 1, 0)$ to canonical forms once the transformation for $Cl(r, 0)$ in half dimension is known and a complementary generator can be identified. This will be the case for $r = 8d$ as Proposition 3.6 implies that the volume element has square I and is not central.

Remark 4.4. $Cl(8d + 1, 0)$ is of complex type and from Proposition 3.6 it can be seen that the product of the generators is a central element with square $-I$, hence it is just J.

4.2. Representations of Cl(r, 0) for r = 3, 7

We start by giving the construction of canonical bases for $Cl(3, 0)$ and $Cl(7, 0)$. For $Cl(3, 0)$ the generators form a copy of the quaternionic subalgebra and we obtain the standard generators as in Proposition 4.5 below.

Representations of $Cl(3,0)$:

Proposition 4.5. *Let* A_1, A_2, A_3 *be an anti-commuting set of skew-symmetric endomorphisms in* $R(4)$ *with squares* $-I$ *and assume that* $A_1A_2A_3 = -I$. *Let* X *be a unit vector in* R^4 *and define*

$$X_1 = X, \quad X_2 = -A_1X, \quad X_3 = -A_2X, \quad X_4 = -A_1A_2X. \tag{4.6}$$

Then $\{X_1, X_2, X_3, X_4\}$ *is an orthonormal set with respect to which the matrices of* A_i, $i = 1, 2, 3$ *are*

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.7}$$

Proof. Using skew-symmetry and anti-commutativity, it is easy to check that the set $\{X, A_1X, A_2X, A_1A_2X\}$ is orthonormal. Then by relabelling these vectors, the matrices of A_1 and A_2 are of the form $A_1 = \sigma \otimes \epsilon$ and $A_2 = \epsilon \otimes 1$, as above. The form of A_3 follows from the fact that $A_3 = A_1A_2$. □

Representations of $Cl(7,0)$: The key fact in the construction above is that the triple product $A_1A_2A_3$ is proportional to identity. A similar approach does not work for the representations of $Cl(7,0)$, because the set consisting of the canonical generators and their products is linearly independent, consequently, none of the triple products is proportional to identity. Nevertheless, we show that these triple products contain an 8-dimensional abelian subalgebra consisting of simultaneously diagonalizable matrices with a common eigenvector corresponding to the eigenvalue 1. Hence, all triple products are proportional to identity on this one dimensional subspace and lead to the desired orthonormal basis. For this we need to determine conditions under which Clifford algebra elements commute.

Lemma 4.6. *Let* $e_1, \ldots, e_{r+s}$ *be an orthonormal set of generators of* $Cl(r,s)$ *and let* ω *and* η *be Clifford algebra elements of orders* $a+b$ *and* $a+c$ *of the form*

$$\omega = e_{i_1}e_{i_2}\ldots e_{i_{a+b}}, \quad \eta = e_{j_1}e_{j_2}\ldots e_{j_{a+c}}.$$

If ω *and* η *have a common factor of order* a, *then*

$$\omega\eta = (-1)^{ab+ac+bc}\eta\omega. \tag{4.8}$$

Proof. Without loss of generality we can write $\omega = \alpha\beta$ and $\eta = \alpha\gamma$, where α, β and γ are disjoint. Then the usual rules of exterior algebra applies and we obtain the result. □

As an immediate application we can see that a collection of odd Clifford algebra elements with odd order common factor form an abelian subalgebra. Hence we have the following.

Proposition 4.7. *Let A_i, $i = 1, \dots, 7$ be an anti-commuting set of skew-symmetric endomorphisms in $R(8)$ with squares $-I$ and assume that $A_1A_2 \dots A_7 = I$. Then the subgroup generated by*

$$\mu_1 = A_1A_2A_3, \quad \mu_2 = A_1A_4A_5, \quad \mu_3 = A_2A_4A_6 \tag{4.9}$$

is abelian and have exactly one common eigenvector X with eigenvalue $+1$. Hence the set

$$\{A_{123}, A_{145}, A_{167}, A_{246}, A_{257}, A_{347}, A_{356}, I\}, \tag{4.10}$$

where $A_{ijk} = A_iA_jA_k$ is a maximal abelian subalgebra.

Proof. From Lemma 4.6 it follows that distinct triple products commute if and only if they have exactly one common element. Thus the collection given in Eq. (4.10) is an abelian subalgebra which is clearly maximal in 8-dimensions, since they generate all diagonal matrices. As this is a commuting set of diagonalizable matrices, they can be simultaneously diagonalized. We have to be careful with the ordering of the factors to make sure that they have a common eigenvector with eigenvalue 1. For this we define a commuting set of generators μ_i as in Eq. (4.9). Then, it can be seen that

$$\mu_1\mu_2 = A_1A_2A_3A_1A_4A_5 = -A_2A_3A_4A_5 = -A_1A_6A_7,$$

where the first equality is obtained by anti-commutativity and using that $A_1^2 = -I$, for the second equality uses the fact that the product is equal to I. Similarly we can obtain

$$\mu_1\mu_3 = A_2A_5A_7, \quad \mu_2\mu_3 = A_3A_4A_7, \quad \mu_1\mu_2\mu_3 = -A_3A_5A_6.$$

Note that the image of $(\mu_i + I)$ is the $+1$ eigenspace of μ_i. Thus if μ_1, μ_2, μ_3 had no common eigenvector with eigenvalue 1, the product

$$\begin{aligned}(\mu_1 + I)(\mu_2 + I)(\mu_3 + I) = I &+ \mu_1 + \mu_2 + \mu_3 + \mu_1\mu_2 + \mu_1\mu_3 \\ &+ \mu_2\mu_3 + \mu_1\mu_2\mu_3\end{aligned} \tag{4.11}$$

would be identically zero. From Corollary 3.3, each triple product is trace zero, but the product in Eq. (4.11) cannot have trace zero, hence it is non-zero. Thus they have a common eigenvector X corresponding to the eigenvalue $+1$.

Let $\mathscr{A}$ be the subalgebra generated by $\{A_1, A_2, A_4\}$. It can be seen that $\mathscr{A}$ acting on X gives a linearly independent set of vectors in R^8 and as $\mathscr{A}$ is eight dimensional, X belongs to a one dimensional subspace. □

Once we find this preferred direction X on which the seven triple products act as identity, it is easy to construct the required orthonormal basis.

Remark 4.8. From a computational point of view, given any basis for a representation, one can use Gram–Schmidt orthogonalization to obtain an orthonormal anti-commuting set with squares $-I$ and form the symmetric matrices μ_1, μ_2 and μ_3 as above. Then in 8-dimensions, the matrix $(\mu_1 + I)(\mu_2 + I)(\mu_3 + I)$ has rank 1, and any of its columns yield the preferred direction without any need for eigenvalue computation. This is achieved in OCTAVE with the command $X = \text{orth}(Q)$ which gives an orthonormal basis for the range space of any matrix Q.

We will give in detail the construction of the orthonormal basis.

Proposition 4.9. *Let $A_i, i = 1, \ldots, 7$ be an anti-commuting set of skew-symmetric endomorphisms in $R(8)$ with squares $-I$, and let X be a common eigenvector of $\mu_1 = A_1A_2A_3$, $\mu_2 = A_1A_4A_5$, $\mu_3 = A_2A_4A_6$ with eigenvalue 1. Then, with respect to the basis*

$$\{X, A_1X, A_2X, \ldots, A_7X\} \tag{4.12}$$

the linear transformations A_i have matrices

$$\begin{aligned} &A_1 = -\sigma \otimes \sigma \otimes \epsilon, \quad A_2 = -\sigma \otimes \epsilon \otimes 1, \quad A_3 = -\sigma \otimes \tau \otimes \epsilon, \quad A_4 = -\epsilon \otimes 1 \otimes 1, \\ &A_5 = -\tau \otimes 1 \otimes \epsilon, \quad A_6 = -\tau \otimes \epsilon \otimes \sigma, \quad A_7 = -\tau \otimes \epsilon \otimes \tau. \end{aligned} \tag{4.13}$$

Proof. As the triple products in the subalgebra generated by the μ_i's act as identity on X we can compute the action of all double products as

$$\begin{array}{lll} A_1A_2X = -A_3X, & A_2A_4X = -A_6X, & A_3A_7X = A_4X, \\ A_1A_3X = A_2X, & A_2A_5X = -A_7X, & A_4A_5X = -A_1X, \\ A_1A_4X = -A_5X, & A_2A_6X = A_4X, & A_4A_6X = -A_2X, \\ A_1A_5X = A_4X, & A_2A_7X = A_5X, & A_4A_7X = -A_3X, \\ A_1A_6X = A_7X, & A_3A_4X = -A_7X, & A_5A_6X = A_3X, \\ A_1A_7X = -A_6X, & A_3A_5X = A_6X, & A_5A_7X = -A_2X, \\ A_2A_3X = -A_1X, & A_3A_6X = -A_5X, & A_6A_7X = A_1X. \end{array}$$

Using the above relations we can compute the action of the A_i's on the basis vectors as given in Table 3 from which it can be seen that the linear transformation have their matrices in the desired form with respect to this basis. □

4.3. Representations of $Cl(8d + c, 0)$, for $d \geqslant 1$

Now we give constructions for $Cl(8d + c, 0)$ for $d \geqslant 1$. The first step is to get an abelian subalgebra of $Cl(8d + c, 0)$. We start with determining a maximal abelian subalgebra of $Cl(8d, 0)$.

Lemma 4.10. *Let*

$$A_1^{(1)}, \ldots, A_8^{(1)}, A_1^{(2)}, \ldots, A_8^{(d)} \tag{4.14}$$

be generators of $Cl(8d, 0) \cong R(2^{8d})$. $Cl(8d, 0)$ have an abelian subalgebra $\mathscr{D}$ with $4d$ generators where $\mu_j^{(i)}, i = 1, \ldots, d, j = 1, \ldots, 4$ given by

$$\begin{aligned} &\mu_1^{(1)} = A_1^{(1)}A_2^{(1)}A_3^{(1)}, \quad \mu_2^{(1)} = A_1^{(1)}A_4^{(1)}A_5^{(1)}, \quad \mu_3^{(1)} = A_2^{(1)}A_4^{(1)}A_6^{(1)}, \\ &\mu_4^{(1)} = A_1^{(1)}A_2^{(1)}A_3^{(1)}A_4^{(1)}A_5^{(1)}A_6^{(1)}A_7^{(1)}, \\ &\ldots \quad \ldots \\ &\mu_1^{(d)} = \omega^{(d-1)}A_1^{(d)}A_2^{(d)}A_3^{(d)}, \quad \mu_2^{(d)} = \omega^{(d-1)}A_1^{(d)}A_4^{(d)}A_5^{(d)}, \\ &\mu_3^{(d)} = \omega^{(d-1)}A_2^{(d)}A_4^{(d)}A_6^{(d)}, \\ &\mu_4^{(d)} = \omega^{(d-1)}A_1^{(d)}A_2^{(d)}A_3^{(d)}A_4^{(d)}A_5^{(d)}A_6^{(d)}A_7^{(d)} \end{aligned} \tag{4.15}$$

and

$$\omega^{(k)} = A_1^{(1)} \ldots A_8^{(1)}A_1^{(2)} \ldots A_1^{(k-1)} \ldots A_8^{(k-1)}. \tag{4.16}$$

$\mathscr{D}$ is maximal and has a one dimensional invariant subspace corresponding to the eigenvalue $+1$.

Table 3
The action of the generators of $Cl(7,0)$ on the basis given by Eq. (4.12)

	X	A_1X	A_2X	A_3X	A_4X	A_5X	A_6X	A_7X
A_1:	A_1X	$-X$	$-A_3X$	A_2X	$-A_5X$	A_4X	A_7X	$-A_6X$
A_2:	A_2X	A_3X	$-X$	$-A_1X$	$-A_6X$	$-A_7X$	A_4X	A_5X
A_3:	A_3X	$-A_2X$	A_1X	$-X$	$-A_7X$	A_6X	$-A_5X$	A_4X
A_4:	A_4X	A_5X	A_6X	A_7X	$-X$	$-A_1X$	$-A_2X$	$-A_3X$
A_5:	A_5X	$-A_4X$	A_7X	$-A_6X$	A_1X	$-X$	A_3X	$-A_2X$
A_6:	A_6X	$-A_7X$	$-A_4X$	A_5X	A_2X	$-A_3X$	$-X$	A_1X
A_7:	A_7X	A_6X	$-A_5X$	$-A_4X$	A_3X	A_2X	$-A_1X$	$-X$

Proof. The generators given in Eq. (4.15) form an abelian subalgebra as each of the $\mu_j^{(i)}$'s is and odd algebra element they have odd common factors. As $\mathscr{D}$ is isomorphic to the subalgebra generated by diagonal matrices in $R(2^{4d})$ it is maximal. The proof of the existence of a common eigenvector and its uniqueness is similar to the proof of Proposition 4.9. and it is omitted. □

The abelian subalgebras of $Cl(8d+c,0)$ is constructed similarly but they are not maximal unless $Cl(8d+c,0)$ is of real type and their invariant subspaces is 2 or 4-dimensional according as they are of complex or quaternionic type.

Lemma 4.11. *Let* $A_1^{(1)},\dots,A_8^{(1)},A_1^{(2)},\dots,A_8^{(d)},A_1,\dots,A_c$ *be generators of* $Cl(8d+c,0)$ *and let* $\mu_j^{(i)}$ *be as in Lemma* 4.10 *and let* $\mathscr{D}$ *be an abelian subalgebra of* $Cl(8d+c,0)$. *Then for* $c=1,\dots,3$, $c=4$, $c=5$ *and* $c=6,7$, $\mathscr{D}$ *have respectively* $4d$, $4d+1$, $4d+2$ *and* $4d+3$ *generators given by*

$$
\begin{aligned}
c=0,1,2,3:&\quad \mu_1^{(1)},\dots,\mu_4^{(d)},\\
c=4:&\quad \mu_1^{(1)},\dots,\mu_4^{(d)},\quad \mu_{4d+1}=\omega^{(d)}A_1A_2A_3,\\
c=5:&\quad \mu_1^{(1)},\dots,\mu_4^{(d)},\quad \mu_{4d+1}=\omega^{(d)}A_1A_2A_3,\quad \mu_{4d+2}=\omega^{(d)}A_1A_4A_5,\\
c=6,7:&\quad \mu_1^{(1)},\dots,\mu_4^{(d)},\quad \mu_{4d+1}=\omega^{(d)}A_1A_2A_3,\quad \mu_{4d+2}=\omega^{(d)}A_1A_4A_5,\\
&\quad \mu_{4d+3}=\omega^{(d)}A_2A_4A_6.
\end{aligned}
\tag{4.17}
$$

The invariant subspace of $\mathscr{D}$ *corresponding to the eigenvalue* $+1$ *is one, two or four dimensional respectively for* $c=0,6,7$, $c=1,5$ *and* $c=2,3,4$.

Proof. The generators in Eq. (4.17) are commutative as they are odd algebra elements with odd common factors. For $c=6,7$, $\mathscr{D}$ has $4d+3$ generators in $R(2^{4d+3})$, hence it is isomorphic to the diagonal subalgebra and the invariant subspace of $\mathscr{D}$ is one dimensional. The dimension of other invariant subspaces can be determined by similar counting arguments. □

Let X be a unit vector belonging to the invariant subspace of $\mathscr{D}$ corresponding to the eigenvalue $+1$. The action of $\mathscr{D}$ on X is freely generated by a subset of $4d+j$, $j=0,1,2,3$ generators. We call these as "free generators" and denote the subalgebra they generate by $\mathscr{A}$. We show that the action of $\mathscr{A}$ on X is the required basis.

Remark 4.12. As the Clifford algebras $Cl(8d+3,0)$ and $Cl(8d+7,0)$ are isomorphic to a direct sum, their representations respectively on $R^{2^{4d+2}}$ and $R^{2^{4d+3}}$ are not faithful and the product of all generators is identity. Hence we can omit the last generator and work with the representations of $Cl(8d+2,0)$ and $Cl(8d+6,0)$. In Proposition 4.13 below we do not discuss the cases $c=3$ and $c=7$.

We now determine the set of free generators corresponding to $\mathscr{D}$.

Proposition 4.13. *Let $\mathscr{D}$ be the maximal abelian subalgebra of the Clifford algebra $Cl(8d+c,0)$ and let X be a common eigenvector of $\mathscr{D}$ corresponding to the eigenvalue $+1$. Then there are* $4d, 4d+1, 4d+2$ *and* $4d+3$ *free generators, respectively for* $c=0, c=1, c=2,$ *and* $c=4,5,6,$ *given by*

$$\begin{aligned}
c=0: &\quad A_1^{(1)}, A_2^{(1)}, A_4^{(1)}, A_8^{(1)}, \ldots, A_1^{(d)}, A_2^{(d)}, A_4^{(d)}, A_8^{(d)},\\
c=1: &\quad A_1^{(1)}, A_2^{(1)}, A_4^{(1)}, A_8^{(1)}, \ldots, A_1^{(d)}, A_2^{(d)}, A_4^{(d)}, A_8^{(d)}, A_1,\\
c=2: &\quad A_1^{(1)}, A_2^{(1)}, A_4^{(1)}, A_8^{(1)}, \ldots, A_1^{(d)}, A_2^{(d)}, A_4^{(d)}, A_8^{(d)}, A_1, A_2,\\
c=4,5,6: &\quad A_1^{(1)}, A_2^{(1)}, A_4^{(1)}, A_8^{(1)}, \ldots, A_1^{(d)}, A_2^{(d)}, A_4^{(d)}, A_8^{(d)}, A_1, A_2, A_4.
\end{aligned} \tag{4.18}$$

Proof. As the $\mu_k^{(i)}$ acting on X is identity, one of the $A_j^{(i)}$'s in each of them can be considered as generated by the other two. Eliminating these we arrive at the set given by Eq. (4.18) as free generators. □

We have thus an algorithm for constructing an orthonormal basis with respect to which a given representation will have canonical forms. The tools developed here can be used for the problem of transformation a given set of matrices between different reference frames which may have applications in robotics and computer graphics. In the appendix we give OCTAVE (a Linux shareware package similar to MATLAB) programs for the transformations.

Appendix A

OCTAVE programs for the transformation of arbitrary representations to canonical forms.

Canonical forms for a single generator: Let B be an endomorphism of R^{2N} with $B^2-I=0$. Then B has exactly N eigenvectors corresponding to the eigenvalues ± 1. Let $X^+=\mathrm{orth}(B+I)$ and $X^-=\mathrm{orth}(B-I)$. These are $2N\times N$ matrices and if $P=[X^+\ X^-]$ then $BP=P\sigma$ and if $P=[X^+\ BX^+]$ then $BP=P\tau$.

Similarly if A is a real endomorphism of R^{2N} with $A^2+I=0$, then A has exactly N eigenvectors corresponding to the eigenvalues $\pm\mathrm{i}$. Let $X^+=\mathrm{orth}(A+\mathrm{i}I)$ and $X^-=\mathrm{orth}(A-\mathrm{i}I)$. If $P=[X^+\ X^-]$ then $AP=\mathrm{i}P\sigma$ and if $P=\frac{1}{\sqrt{2}}[X^+ + \mathrm{i}X^-\ \ -\mathrm{i}X^+ - X^-]$ then $AP=\epsilon$.

Canonical forms for two generators: Let A, B be anti-commuting endomorphism of R^{2N} with $A^2+I=0$ and $B^2-I=0$ and let $X=\mathrm{orth}(B+I)$. If $P=[X\ \ -AX]$ then $BP=P\sigma$ and $AP=P\epsilon$.

Similarly if $P=\frac{1}{\sqrt{2}}[X+AX\ X-AX]$ then $BP=P\tau$ and $AP=P\epsilon$. In this case as the only matrix anti-commuting with ϵ and τ is σ, the remaining endomorphisms A_i in the representation are automatically of the form $\sigma\otimes a_i$.

If B and C are anti-commuting endomorphisms with squares $+I$ and $X = \mathrm{orth}(B + I)$ then $P = [X \;\; CX]$ results in $BP = P\sigma$, $CP = P\tau$. It follows that the remaining endomorphisms A_i in the representation are automatically of the form $\epsilon \otimes a_i$.

Representation of $Cl(7, 0)$: Let the representation of the basis elements be $A_1, A_2, \ldots, A_7$. Then the generators of abelian subalgebra are given by

$$\mu_1 = A_1A_2A_3, \quad \mu_2 = A_1A_4A_5, \quad \mu_3 = A_2A_4A_6 \tag{A.1a}$$

and $\mathrm{rank}((\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)) = 1$. The common eigenvector with eigenvalue $+1$ is

$$X = \mathrm{orth}((\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)). \tag{A.1b}$$

The action of $Cl(7, 0)$ on X is generated by A_1, A_2, A_4 and the transformation matrix is

$$P^{(7)} = \{X \; A_1X \; A_2X \; A_3X \; A_4X \; A_5X \; A_6X \; A_7X\}. \tag{A.1c}$$

Here the product of the generators is identity and all triples products appear in the abelian subalgebra.

Representation of $Cl(8, 0)$: Let the generators of the representation be $A_1, A_2, \ldots, A_7, A_8$. Then the abelian subalgebra is generated by $\mu_i,\; i = 1, 2, 3$ and

$$\mu_4 = A_1A_2A_3A_4A_5A_6A_7 \tag{A.2a}$$

with $\mathrm{rank}((\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)(\mu_4 + 1)) = 1$. The common eigenvector with eigenvalue $+1$ is

$$X = \mathrm{orth}((\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)(\mu_4 + 1)). \tag{A.2b}$$

The action of $Cl(8, 0)$ on X is generated by A_1, A_2, A_4, A_8 and the transformation matrix is

$$P^{(8)} = \{P^{(7)} \; A_8P^{(7)}\}. \tag{A.2c}$$

Representation of $Cl(9, 0)$: Let the generators of the representation be $A_1, A_2, \ldots, A_7, A_8, A_9$. The abelian subalgebra is generated by $\mu_i, i = 1, \ldots, 4$ as before but now $\mathrm{rank}((\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)(\mu_4 + 1)) = 2$. The common eigenvectors with eigenvalue $+1$ will belong to the 2-dimensional subspace

$$[X_aX_b] = \mathrm{orth}((\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)(\mu_4 + 1)). \tag{A.3a}$$

If X is any unit vector in the span of X_a, X_b, then the action of $Cl(9, 0)$ on X is generated by A_1, A_2, A_4, A_8, A_9 and the transformation matrix is

$$P^{(9)} = \{P^{(8)} \; A_9P^{(8)}\}. \tag{A.3b}$$

Representation of $Cl(11, 0)$: Let the generators of the representation be $A_1, A_2, \ldots, A_7, A_8, A_9, A_{10}, A_{11}$. The abelian subalgebra are again generated by $\mu_i,\; i = 1, \ldots, 4$ and rank $((\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)(\mu_4 + 1)) = 4$. The common eigenvectors with eigenvalue $+1$ will belong to the 4-dimensional space

$$[X_aX_bX_cX_d] = \mathrm{orth}((\mu_1 + 1)(\mu_2 + 1)(\mu_3 + 1)(\mu_4 + 1)). \tag{A.4a}$$

If X is any unit vector in the span of X_a, X_b, X_c, X_d, then the action of $Cl(11, 0)$ on X is generated by $A_1, A_2, A_4, A_8, A_9, A_{10}$ and the transformation matrix is

$$P^{(11)} = \{P^{(9)} \; A_{10}P^{(9)}\}. \tag{A.4b}$$

Representation of $Cl(15,0)$: Let the generators of the representation be $A_1, A_2, \ldots, A_{15}$. Note that the product of all generators is identity, hence once we put the first 14 generators in the required form, the last one will automatically be in the desired format. The maximal abelian subalgebra is generated by μ_i, $i = 1, \ldots, 4$ as above and

$$\mu_5 = A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}A_{11},$$
$$\mu_6 = A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{12}A_{13},$$
$$\mu_7 = A_1A_2A_3A_4A_5A_6A_7A_8A_{10}A_{12}A_{14} \tag{A.5a}$$

with $\operatorname{rank}((\mu_1+1)(\mu_2+1)(\mu_3+1)(\mu_4+1)(\mu_5+1)(\mu_6+1)(\mu_7+1)) = 1$. The common eigenvectors with eigenvalue $+1$ is

$$X = \operatorname{orth}((\mu_1+1)(\mu_2+1)(\mu_3+1)(\mu_4+1)(\mu_5+1)(\mu_6+1)(\mu_7+1)). \tag{A.5b}$$

The action of $Cl(15,0)$ on X is generated by $A_1, A_2, A_4, A_8, A_9, A_{10}, A_{12}$ and the transformation matrix is

$$P^{(15)} = \{P^{(11)}\ A_{12}P^{(11)}\}. \tag{A.5c}$$

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