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J. Math. Anal. Appl. 314 (2006) 672–688

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Convex extensions of the convex set valued maps

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Received 17 February 2005

Available online 24 May 2005

Submitted by T. Akahori

Abstract

In this article the existence of the convex extension of convex set valued map is considered. Conditions are obtained, based on the notion of the derivative of set valued maps, which guarantee the existence of convex extension. The conditions are given, when the convex set valued map has no convex extension. The convex set valued map is specified, which is the maximal convex extension of the given convex set valued map and includes all other convex extensions. The connection between Lipschitz continuity and existence of convex extension of the given convex set valued map is studied.

Keywords: Set valued map; Convex extension; Derivative set; Lipschitz continuity

1. Introduction

The family of all nonempty, compact subsets of R^n is denoted by $\text{comp}(R^n)$.

We denote by symbol $h(E,D)$ the Hausdorff distance between the sets $E, D \in \text{comp}(R^n)$. It is defined as

$$h(E,D) = \max\left\{\sup_{x\in E} d(x,D), \sup_{y\in D} d(y,E)\right\},$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.04.019

where $d(x, D) = \inf_{y \in D} \parallel x - y \parallel$ and $\parallel \cdot \parallel$ means the Euclidean norm. It is known that, $(\mathrm{comp}(R^n), h(\cdot,\cdot))$ is a metric space (see, e.g., [2,4]).

Let $V(\cdot) : [t_0, \theta] \to \mathrm{comp}(R^n)$ be a set valued map the graph of which is denoted by $\mathrm{gr}\, V(\cdot)$ and defined as

$$\mathrm{gr}\, V(\cdot) = \big\{(t, x) \in [t_0, \theta] \times R^n \colon\ x \in V(t)\big\}.$$

Now, we give the definitions of the locally Lipschitz and Lipschitz continuity of the set valued map.

Definition 1.1. If for every $t_* \in (t_0, \theta)$ there exist $\sigma_* = \sigma(t_*) > 0$, $L_* = L(t_*, \sigma_*) > 0$ such that for every $t_1 \in (t_* - \sigma_*, t_* + \sigma_*)$, $t_2 \in (t_* - \sigma_*, t_* + \sigma_*)$ the inequality $h(V(t_1), V(t_2)) \leqslant L_*|t_1 - t_2|$ holds, then the set valued map $V(\cdot) : (t_0, \theta) \to \mathrm{comp}(R^n)$ is said to be locally Lipschitz continuous on the open interval (t_0, θ).

If there exists $K > 0$ such that $h(V(t_1), V(t_2)) \leqslant K|t_1 - t_2|$ for every $t_1 \in [t_0, \theta]$, $t_2 \in [t_0, \theta]$ then, the set valued map $V(\cdot) : [t_0, \theta] \to \mathrm{comp}(R^n)$ is said to be Lipschitz continuous on the closed interval $[t_0, \theta]$ with Lipschitz constant K.

For $A \subset R^n$, we denote the convex hull of A as $\mathrm{co}(A)$, the interior of A as $\mathrm{int}\, A$. $\langle \cdot, \cdot \rangle$ denotes the inner product.

Now let us give the definitions of the derivative sets of the given set valued map. For $(t, x) \in [t_0, \theta] \times R^n$ and the set valued map $V(\cdot) : [t_0, \theta] \to \mathrm{comp}(R^n)$, we define

$$D^+V(t, x) = \left\{ v \in R^n \colon\ \liminf_{\delta \to 0^+} \frac{1}{\delta} d\big(x + \delta v, V(t + \delta)\big) = 0 \right\},$$

$$D^-V(t, x) = \left\{ v \in R^n \colon\ \liminf_{\delta \to 0^+} \frac{1}{\delta} d\big(x - \delta v, V(t - \delta)\big) = 0 \right\}.$$

The set $D^+V(t, x)$ ($D^-V(t, x)$) is said to be an upper right-hand side (left-hand side) derivative set of the set valued map $t \to V(t)$ calculated at the point (t, x). Note that the upper right-hand side (left-hand side) derivative set is closed and it has a close connection with upper Bouligand contingent cone, used in the study of many problems of the set valued and nonsmooth analysis (see, e.g., [2,4,5,9]).

A set valued map $V(\cdot) : [t_0, \theta] \to \mathrm{comp}(R^n)$ is said to be convex (compact) if $\mathrm{gr}\, V(\cdot)$ is convex (compact) set. It is obvious that the set valued map $V(\cdot)$ is convex iff

$$\lambda V(t_1) + (1 - \lambda)V(t_2) \subset V\big(\lambda t_1 + (1 - \lambda)t_2\big)$$

for all $t_1, t_2 \in [t_0, \theta]$ and $\lambda \in [0, 1]$.

Let us give the definition of the convex extension of the convex set valued map.

Definition 1.2. Let $V(\cdot) : [t_0, \theta] \to \mathrm{comp}(R^n)$ be a convex set valued map and $\alpha > 0$ be a fixed number. If there exists a convex set valued map

$$W(\cdot) : [t_0 - \alpha, \theta] \to \mathrm{comp}(R^n) \quad \big(W(\cdot) : [t_0, \theta + \alpha] \to \mathrm{comp}(R^n)\big)$$

such that $W(t) = V(t)$ for all $t \in [t_0, \theta]$, then the set valued map $W(\cdot)$ is said to be a left-hand (right-hand) convex α-extension of the set valued map $V(\cdot)$.

If there exists a convex set valued map $W(\cdot):[t_0-\alpha,\theta+\alpha]\to \mathrm{comp}(R^n)$ such that $W(t)=V(t)$ for all $t\in[t_0,\theta]$, then the set valued map $W(\cdot)$ is said to be a convex α-extension of the set valued map $V(\cdot)$.

The existence of the convex extension of the convex set valued map arises in the solutions of some inverse problems of the differential inclusion theory, where it is required to construct a differential inclusion with prescribed attainable sets or integral funnel (see [3,6–9]). For a special case, the existence of the convex extension of convex set valued map was studied in [3,8]. Some maximality properties of the convex extension is considered in [7].

In this article, in Section 2, conditions, based on the notion of the derivative of set valued maps, are obtained, which guarantee the existence of left-hand and right-hand convex extensions of the convex set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ (Theorems 2.3–2.5). In Section 3 we give conditions, which guarantee nonexistence of the left-hand and right-hand convex extensions of the convex set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ (Theorems 3.3 and 3.5) and formulate a necessary and sufficient condition for the existence of left-hand and right-hand convex extensions (Theorem 3.6). In Section 4 the maximality property of the convex extension is studied. The convex set valued maps are defined, which are convex left-hand and right-hand extensions of the convex set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ and includes all their other convex left-hand and right-hand extensions respectively (Theorems 4.4 and 4.5). At the end, in Section 5, the connection between the existence of the convex extension and Lipschitz continuity of the convex set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ is studied. It is shown that if the convex set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ has a convex extension, then it is Lipschitz continuous (Theorem 5.2) and vice versa, if the convex set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ is Lipschitz continuous and $\mathrm{int}\,V(t_0)\neq\emptyset$ and $\mathrm{int}\,V(\theta)\neq\emptyset$, then it has a convex extension (Theorem 5.3).

2. The existence of convex extension

For given $\alpha>0$, $x_*\in R^n$ and the set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$, we define the set valued maps

$$V_\alpha^L(x_*)\mid(\cdot):[t_0-\alpha,\theta+\alpha]\to \mathrm{comp}(R^n),$$
$$V_\alpha^R(x_*)\mid(\cdot):[t_0-\alpha,\theta+\alpha]\to \mathrm{comp}(R^n)$$

setting

$$V_\alpha^L(x_*)\mid(t)=\left(1-\frac{t-t_0+\alpha}{\alpha}\right)x_*+\frac{t-t_0+\alpha}{\alpha}V(t_0), \tag{2.1}$$

$$V_\alpha^R(x_*)\mid(t)=\left(1-\frac{\theta+\alpha-t}{\alpha}\right)x_*+\frac{\theta+\alpha-t}{\alpha}V(\theta). \tag{2.2}$$

It is obvious that $V_\alpha^L(x_*)\mid(t_0)=V(t_0)$ and $V_\alpha^R(x_*)\mid(\theta)=V(\theta)$. If the set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ is convex, then $V_\alpha^L(x_*)\mid(\cdot)$ and $V_\alpha^R(x_*)\mid(\cdot)$ are convex compact and Lipschitz continuous set valued maps.

The following proposition characterizes the existence of the left-hand convex extension of the given convex set valued map.

Proposition 2.1. *Let* $\alpha > 0$, $V_\alpha(\cdot) : [t_0-\alpha, \theta] \to \mathrm{comp}(R^n)$ *and* $V(\cdot) : [t_0, \theta] \to \mathrm{comp}(R^n)$ *be convex set valued maps. Suppose that* $V_\alpha(t_0) = V(t_0)$ *and* $V(t) \subset V_\alpha(t)$ *for all* $t \in (t_0, \theta]$. *Then the set valued map* $W(\cdot) : [t_0 - \alpha, \theta] \to \mathrm{comp}(R^n)$ *defined as*

$$W(t) = \begin{cases} V_\alpha(t), & t \in [t_0 - \alpha, t_0), \\ V(t), & t \in [t_0, \theta], \end{cases}$$

is a left-hand convex α*-extension of the set valued map* $V(\cdot)$.

Proposition 2.2. *Let* $\alpha > 0$, $x_0 \in R^n$ *and* $V(\cdot) : [t_0, \theta] \to \mathrm{comp}(R^n)$ *be a convex set valued map. Suppose that*

$$D^+V(t_0, v) \subset D^+V_\alpha^L(x_0) \mid (t_0, v) \quad \text{for all } v \in V(t_0).$$

Then $V(t) \subset V_\alpha^L(x_0) \mid (t)$ *for every* $t \in (t_0, \theta]$, *where the set* $V_\alpha^L(x_0) \mid (t)$, $t \in [t_0, \theta]$, *is defined by* (2.1).

Proof. Assume the contrary. Let there exist $t_* \in (t_0, \theta]$ and $x_* \in V(t_*)$ such that $x_* \notin V_\alpha^L(x_0) \mid (t_*)$. Since $V_\alpha^L(x_0) \mid (t_*)$ is a convex compact set, then there exists $w_* \in V_\alpha^L(x_0) \mid (t_*)$ such that

$$\|x_* - w_*\| = d\big(x_*, V_\alpha^L(x_0) \bigm| (t_*)\big) > 0.$$

According to the Theorem 2.3 from [1], we have

$$\langle x_* - w_*, w_* - w\rangle \geqslant 0 \quad \text{for all } w \in V_\alpha^L(x_0) \mid (t_*). \tag{2.3}$$

Since $w_* \in V_\alpha^L(x_0) \mid (t_*)$, then we get that there exists a $v_0 \in V(t_0)$ such that

$$w_* = \left(1 - \frac{t_* - t_0 + \alpha}{\alpha}\right)x_0 + \frac{t_* - t_0 + \alpha}{\alpha}v_0. \tag{2.4}$$

Let $v \in V(t_0)$ be an arbitrary chosen element. Then

$$w = \left(1 - \frac{t_* - t_0 + \alpha}{\alpha}\right)x_0 + \frac{t_* - t_0 + \alpha}{\alpha}v \in V_\alpha^L(x_0) \mid (t_*). \tag{2.5}$$

From (2.3)–(2.5) it can be obtained that

$$\langle x_* - w_*, v_0 - v\rangle \geqslant 0 \quad \text{for every } v \in V(t_0). \tag{2.6}$$

Now let us set

$$x(t) = \left(1 - \frac{t - t_0}{t_* - t_0}\right)v_0 + \frac{t - t_0}{t_* - t_0}x_*, \quad t \in [t_0, t_*], \tag{2.7}$$

$$w(t) = \left(1 - \frac{t - t_0}{t_* - t_0}\right)v_0 + \frac{t - t_0}{t_* - t_0}w_*, \quad t \in [t_0, t_*]. \tag{2.8}$$

It follows from (2.1), (2.4) and (2.8) that

$$w(t) = \left(1 - \frac{t - t_0 + \alpha}{\alpha}\right)x_0 + \frac{t - t_0 + \alpha}{\alpha} v_0, \quad t \in [t_0, t_*], \tag{2.9}$$

and

$$w(t) \in V_\alpha^L(x_0) \mid (t) \quad \text{for every } t \in [t_0, t_*]. \tag{2.10}$$

Let $t \in (t_0, t_*]$ be fixed. Now let us take an arbitrary $w \in V_\alpha^L(x_0) \mid (t)$. Then there exists $v \in V(t_0)$ such that

$$w = \left(1 - \frac{t - t_0 + \alpha}{\alpha}\right)x_0 + \frac{t - t_0 + \alpha}{\alpha} v.$$

By virtue of (2.6)–(2.9), we get

$$\langle x(t) - w(t), w(t) - w\rangle \geqslant 0 \quad \text{for all } w \in V_\alpha^L(x_0) \mid (t). \tag{2.11}$$

Since $\|x(t) - w(t)\| = \frac{t-t_0}{t_*-t_0}\|x_* - w_*\| > 0$, according to [1, Theorem 2.3], it follows from (2.10) and (2.11) that

$$d\big(x(t), V_\alpha^L(x_0) \mid (t)\big) = \big\|x(t) - w(t)\big\| = (t - t_0)r_*, \tag{2.12}$$

where $r_* = \frac{\|x_*-w_*\|}{t_*-t_0} > 0,\ t \in (t_0, t_*]$.

Let $u_* = \frac{x_*-v_0}{t_*-t_0}$. Then (2.7) implies that

$$x(t) = v_0 + (t - t_0)u_*, \quad t \in [t_0, t_*]. \tag{2.13}$$

So, we get from (2.12) and (2.13) that

$$\lim_{t\to t_0^+} \frac{d(v_0 + (t - t_0)u_*, V_\alpha^L(x_0) \mid (t))}{t - t_0} = r_* > 0,$$

which means that

$$u_* \notin D^+V_\alpha^L(x_0) \mid (t_0, v_0). \tag{2.14}$$

Since $x_* \in V(t_*)$, $v_0 \in V(t_0)$ and $V(\cdot)$ is a convex set valued map, it follows from (2.7) that $x(t) \in V(t)$ for every $t \in [t_0, t_*]$ and therefore $d(x(t), V(t)) = 0$ for every $t \in [t_0, t_*]$. Consequently we get from (2.13) that

$$\liminf_{t\to t_0^+} \frac{d(v_0 + (t - t_0)u_*, V(t))}{t - t_0} = 0,$$

and therefore

$$u_* \in D^+V(t_0, v_0). \tag{2.15}$$

It follows from (2.14) and (2.15) that $D^+V(t_0, v_0) \nsubseteq D^+V_\alpha^L(x_0) \mid (t_0, v_0)$, which is a contradiction of the condition of the theorem. So, this concludes the proof. □

From Propositions 2.1 and 2.2 we obtain following theorems which specify sufficient conditions for the existence of the left-hand and right-hand convex extensions of the given convex set valued map in terms of derivative sets.

Theorem 2.3. *Let $\alpha > 0$, $x_0 \in R^n$ and $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ be a convex set valued map. Assume that*

$$D^+V(t_0, v) \subset D^+V_\alpha^L(x_0) \mid (t_0, v) \quad \textit{for every } v \in V(t_0).$$

Then the set valued map $W(\cdot) : [t_0 - \alpha, \theta] \to \text{comp}(R^n)$ defined as

$$W(t) = \begin{cases} V_\alpha^L(x_0) \mid (t), & t \in [t_0 - \alpha, t_0), \\ V(t), & t \in [t_0, \theta], \end{cases}$$

is a left-hand convex α-extension of the set valued map $V(\cdot)$, where the set $V_\alpha^L(x_0) \mid (t)$ is defined by relation (2.1)*.*

Analogously, the following right-hand convex extension existence theorem is valid.

Theorem 2.4. *Let $\alpha > 0$, $x_1 \in R^n$ and $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ be a convex set valued map. Assume that*

$$D^-V(\theta, v) \subset D^-V_\alpha^R(x_1) \mid (\theta, v) \quad \textit{for every } v \in V(\theta).$$

Then the set valued map $W(\cdot) : [t_0, \theta + \alpha] \to \text{comp}(R^n)$ defined as

$$W(t) = \begin{cases} V(t), & t \in [t_0, \theta], \\ V_\alpha^R(x_1) \mid (t), & t \in (\theta, \theta + \alpha], \end{cases}$$

is a right-hand convex α-extension of the set valued map $V(\cdot)$, where the set $V_\alpha^R(x_1) \mid (t)$ is defined by relation (2.2)*.*

Now, let us formulate the convex extension existence theorem.

Theorem 2.5. *Let $\alpha > 0$, $x_0 \in R^n$, $x_1 \in R^n$ and $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ be a convex set valued map. Assume that*

$$D^+V(t_0, v) \subset D^+V_\alpha^L(x_0) \mid (t_0, v) \quad \textit{for every } v \in V(t_0), \tag{2.16}$$

$$D^-V(\theta, v) \subset D^-V_\alpha^R(x_1) \mid (\theta, v) \quad \textit{for every } v \in V(\theta). \tag{2.17}$$

Then the set valued map $W(\cdot) : [t_0 - \alpha, \theta + \alpha] \to \text{comp}(R^n)$ defined as

$$W(t) = \begin{cases} V_\alpha^L(x_0) \mid (t), & t \in [t_0 - \alpha, t_0), \\ V(t), & t \in [t_0, \theta], \\ V_\alpha^R(x_1) \mid (t), & t \in (\theta, \theta + \alpha], \end{cases} \tag{2.18}$$

is a convex α-extension of the set valued map $V(\cdot)$, where the sets $V_\alpha^L(x_0) \mid (t)$ and $V_\alpha^R(x_1) \mid (t)$ are defined by relations (2.1) *and* (2.2)*, respectively.*

Proof. Since for $\alpha > 0$ and $x_1 \in R^n$ the inclusion (2.17) is satisfied, then it follows from Theorem 2.4 that the set valued map $U(\cdot) : [t_0, \theta + \alpha] \to \text{comp}(R^n)$ defined as

$$U(t) = \begin{cases} V(t), & t \in [t_0, \theta], \\ V_\alpha^R(x_1) \mid (t), & t \in (\theta, \theta + \alpha], \end{cases} \tag{2.19}$$

is a right-hand convex α-extension of the set valued map $V(\cdot)$. From the definition of the right-hand convex α-extension, $U(\cdot):[t_0,\theta+\alpha]\to \mathrm{comp}(R^n)$ is convex set valued map, $U(t)=V(t)$ for all $t\in[t_0,\theta]$, and consequently

$$D^+U(t_0,v)=D^+V(t_0,v) \quad \text{for all } v\in V(t_0). \tag{2.20}$$

Then from (2.16) and (2.20) we obtain that

$$D^+U(t_0,v)\subset D^+V_\alpha^L(x_0)\mid(t_0,v) \quad \text{for all } v\in V(t_0).$$

According to the Theorem 2.3, the set valued map $U_L(\cdot):[t_0-\alpha,\theta+\alpha]\to \mathrm{comp}(R^n)$ defined as

$$U_L(t)=\begin{cases} V_\alpha^L(x_0)\mid(t), & t\in[t_0-\alpha,t_0),\\ U(t), & t\in[t_0,\theta+\alpha], \end{cases} \tag{2.21}$$

is a left-hand convex extension of $U(\cdot):[t_0,\theta+\alpha]\to \mathrm{comp}(R^n)$. From (2.18), (2.19) and (2.21) the proof of the theorem follows. □

3. Impossibility of convex extension

In this section, we will study the case where the convex set valued map has no convex extension and will formulate a necessary and sufficient condition for the existence of the convex extension of the given convex set valued map. At first let us give some auxiliary propositions.

Proposition 3.1. *Let $W(\cdot):[t_0-\alpha,\theta]\to \mathrm{comp}(R^n)$ be a left-hand convex α-extension of the convex set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$. Then for every fixed $\alpha_*\in(0,\alpha]$ and $x_*\in W(t_0-\alpha_*)$ the inclusion*

$$V(t)\subset V_{\alpha_*}^L(x_*)\mid(t)$$

holds for any $t\in[t_0,\theta]$, where the set $V_{\alpha_*}^L(x_*)\mid(t)$, $t\in[t_0-\alpha_*,\theta]$, is defined by relation (2.1).

Proof. Choose an arbitrary $t^*\in(t_0,\theta]$ and define the set valued map $P(\cdot):[t_0-\alpha_*,\theta]\to \mathrm{comp}(R^n)$ where

$$P(t)=\left(1-\frac{t-t_0+\alpha_*}{t^*-t_0+\alpha_*}\right)x_*+\frac{t-t_0+\alpha_*}{t^*-t_0+\alpha_*}V(t^*).$$

It is not difficult to show that

$$P(t_0)\subset V(t_0). \tag{3.1}$$

Now, let us prove that

$$P(t)\subset V(t) \quad \text{for all } t\in[t_0,t^*].$$

We define the set valued map $Q(\cdot):[t_0-\alpha_*,t^*]\to \mathrm{comp}(R^n)$ setting

$$Q(t) = \left(1 - \frac{t - t_0 + \alpha_*}{\alpha_*}\right)x_* + \frac{t - t_0 + \alpha_*}{\alpha_*} P(t_0). \tag{3.2}$$

It follows from (3.1) and the definition of the set valued map $V^L_{\alpha_*}(x_*) \mid (\cdot)$ that

$$Q(t) \subset V^L_{\alpha_*}(x_*) \mid (t) \quad \text{for all } t \in [t_0 - \alpha_*, \theta]. \tag{3.3}$$

Now, let us take $t \in [t_0 - \alpha_*, t^*]$. From (3.2), it can be obtained that $Q(t) = P(t)$ and consequently, it follows from (3.3) that

$$P(t) \subset V^L_{\alpha_*}(x_*) \mid (t) \tag{3.4}$$

for all $t \in [t_0 - \alpha_*, t^*]$. Since $P(t^*) = V(t^*)$, then it follows from (3.4) that

$$V(t^*) \subset V^L_{\alpha_*}(x_*) \mid (t^*).$$

Since $t^* \in [t_0, \theta]$ is arbitrary chosen, we have $V(t) \subset V^L_{\alpha_*}(x_*) \mid (t)$ for all $t \in [t_0, \theta]$. □

Proposition 3.2. *Let the set valued map* $W(\cdot) : [t_0, \theta + \alpha] \to \operatorname{comp}(R^n)$ *be a right-hand convex* α*-extension of the convex set valued map* $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$*. Then for every fixed* $\alpha_* \in (0, \alpha]$ *and* $x_* \in W(\theta + \alpha_*)$ *the inclusion*

$$V(t) \subset V^R_{\alpha_*}(x_*) \mid (t)$$

holds for any $t \in [t_0, \theta]$*, where the set* $V^R_{\alpha_*}(x_*) \mid (t)$, $t \in [t_0, \theta + \alpha_*]$, *is defined by relation* (2.2).

The proof of Proposition 3.2 is similar to proof of the Proposition 3.1.

In the next theorem we give a sufficient condition, when the convex set valued map has no convex extension.

Theorem 3.3. *Let* $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ *be a convex set valued map. Suppose that for every fixed* $\alpha > 0$ *and* $x \in R^n$ *there exists* $v \in V(t_0)$ *such that*

$$D^+V(t_0, v) \not\subseteq D^+V^L_{\alpha}(x) \mid (t_0, v).$$

Then the set valued map $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ *has no left-hand convex extension, where the set valued map* $t \to V^L_{\alpha}(x) \mid (t)$, $t \in [t_0 - \alpha, \theta]$, *is defined by relation* (2.1).

Proof. Assume the contrary. Let $\alpha_* > 0$ and the set valued map $W(\cdot) : [t_0 - \alpha_*, \theta] \to \operatorname{comp}(R^n)$ be a left-hand convex α_*-extension of the set valued map $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$. Choose $x_* \in W(t_0 - \alpha_*)$. According to the Proposition 3.1, we have

$$V(t) \subset V^L_{\alpha_*}(x_*) \mid (t) \quad \text{for all } t \in [t_0, \theta], \tag{3.5}$$

where the set $V^L_{\alpha_*}(x_*) \mid (t)$, $t \in [t_0 - \alpha_*, \theta]$, is defined by relation (2.1). Since $V(t_0) = V^L_{\alpha_*}(x_*) \mid (t_0)$, then from (3.5) it follows that $D^+V(t_0, v) \subset D^+V^L_{\alpha_*}(x_*) \mid (t_0, v)$ for all $v \in V(t_0)$ which is a contradiction to assumption of the theorem. Therefore the proof is complete. □

Example 3.4. Let the set valued map $V(\cdot):[0,1]\to \mathrm{comp}(R^1)$ is defined as

$$V(t)=\{x\in R^1\colon\ |x|\leqslant 1+\sqrt{t}\},\quad t\in[0,1].$$

It is not difficult to show that the convex set valued map $V(\cdot):[0,1]\to \mathrm{comp}(R^1)$ has no convex left-hand extension.

Analogously, the following theorem is true.

Theorem 3.5. *Let $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ be a convex set valued map. Suppose that for every fixed $\alpha>0$ and $x\in R^n$ there exists $v\in V(\theta)$ such that*

$$D^-V(\theta,v)\not\subseteq D^-V_\alpha^R(x)\mid(\theta,v).$$

Then the set valued map $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ has no right-hand convex extension, where the set valued map $t\to V_\alpha^R(x)\mid(t)$, $t\in[t_0,\theta+\alpha]$, is defined by relation (2.2).

So we obtain from Theorems 2.3, 2.4, 3.3 and 3.5 the validity of the following theorem which gives us a necessary and sufficient condition for the existence of left-hand and right-hand convex extensions of the given convex set valued map.

Theorem 3.6. *Let $\alpha>0$, $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ be a convex set valued map. The set valued map $V(\cdot)$ has a left-hand* (*right-hand*) *convex α-extension if and only if there exists $x_0\in R^n$ such that*

$$D^+V(t_0,v)\subset D^+V_\alpha^L(x_0)\mid(t_0,v)\quad \textit{for every } v\in V(t_0)$$
$$\big(D^-V(\theta,v)\subset D^-V_\alpha^R(x_0)\mid(\theta,v)\quad \textit{for every } v\in V(\theta)\big).$$

4. Maximal convex extension

Let us give the definition of maximal convex α-extension of the given convex set valued map.

Definition 4.1. Let $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$ be a convex set valued map, $\alpha_*>0$ and $W(\cdot):[t_0-\alpha_*,\theta]\to \mathrm{comp}(R^n)$ $(W(\cdot):[t_0,\theta+\alpha_*]\to \mathrm{comp}(R^n))$ be a left-hand (right-hand) convex α_*-extension of the set valued map $V(\cdot)$. If

$$\mathrm{gr}\,V_\alpha(\cdot)\subset \mathrm{gr}\,W(\cdot)$$

for every $\alpha\in(0,\alpha_*]$ and $V_\alpha(\cdot):[t_0-\alpha,\theta]\to \mathrm{comp}(R^n)$, $(V_\alpha(\cdot):[t_0,\theta+\alpha]\to \mathrm{comp}(R^n))$, where $V_\alpha(\cdot)$ is left-hand (right-hand) convex α-extension of $V(\cdot)$, then $W(\cdot)$ is called maximal left-hand (right-hand) convex α_*-extension of the set valued map $V(\cdot)$.

Now let us give one auxiliary proposition.

Proposition 4.2. *Let $V(\cdot):[t_0,\theta]\to \operatorname{comp}(R^n)$ be a convex set valued map, $V_1(\cdot):[t_0-\alpha_1,\theta]\to \operatorname{comp}(R^n)$, $V_2(\cdot):[t_0-\alpha_2,\theta]\to \operatorname{comp}(R^n)$ be left-hand convex α_1 and α_2-extensions of $V(\cdot)$, respectively, $\alpha=\max\{\alpha_1,\alpha_2\}$ and*

$$E=\operatorname{co}\big(\operatorname{gr} V_1(\cdot)\cup \operatorname{gr} V_2(\cdot)\big),$$
$$E(t)=\big\{x\in R^n\colon (t,x)\in E\big\},\quad t\in[t_0-\alpha,\theta].$$

Then the set valued map $E(\cdot):[t_0-\alpha,\theta]\to\operatorname{comp}(R^n)$ is a left-hand convex α-extension of the set valued map $V(\cdot)$.

Proof. It is obvious that the set valued map $E(\cdot):[t_0-\alpha,\theta]\to\operatorname{comp}(R^n)$ is convex. Now, let us prove that

$$E(t)=V(t)\quad \text{for all } t\in[t_0,\theta]. \tag{4.1}$$

Since $\operatorname{gr} V(\cdot)\subset \operatorname{gr} V_i(\cdot)\subset E=\operatorname{gr} E(\cdot)$ $(i=1,2)$, then $\operatorname{gr} V(\cdot)\subset\operatorname{gr} E(\cdot)$ and consequently

$$V(t)\subset E(t)\quad \text{for all } t\in[t_0,\theta]. \tag{4.2}$$

Now let us show that

$$E(t)\subset V(t)\quad \text{for all } t\in[t_0,\theta]. \tag{4.3}$$

Let $t\in[t_0,\theta]$ and choose an arbitrary $x\in E(t)$. Then $(t,x)\in E=\operatorname{gr} E(\cdot)$. From the Caratheodory theorem (see, e.g., [10, Theorem 17.1]), there exist $k\in N=\{1,2,\ldots\}$, $(t_i,x_i)\in \operatorname{gr} V_1(\cdot)\cup\operatorname{gr} V_2(\cdot)$ and $\lambda_i>0$ $(i=1,\ldots,k)$, such that

$$(t,x)=\sum_{i=1}^{k}\lambda_i(t_i,x_i),\qquad \sum_{i=1}^{k}\lambda_i=1. \tag{4.4}$$

If $(t_i,x_i)\in\operatorname{gr} V_1(\cdot)$ for all $i=1,\ldots,k$ or $(t_i,x_i)\in\operatorname{gr} V_2(\cdot)$ for all $i=1,\ldots,k$, then it is not difficult to verify that $x\in V(t)$.

Now, we assume that neither $(t_i,x_i)\in\operatorname{gr} V_1(\cdot)$ nor $(t_i,x_i)\in\operatorname{gr} V_2(\cdot)$ for all $i=1,\ldots,k$. Denote

$$I_1=\big\{i=1,2,\ldots,k\colon (t_i,x_i)\in\operatorname{gr} V_1(\cdot)\big\},\qquad I_2=\{1,2,\ldots,k\}\backslash I_1,$$
$$\mu_1=\sum_{i\in I_1}\lambda_i,\qquad \mu_2=\sum_{i\in I_2}\lambda_i. \tag{4.5}$$

It is obvious that $I_1\neq\emptyset$, $I_2\neq\emptyset$, $\mu_1>0$, $\mu_2>0$ and $\mu_1+\mu_2=1$.

Then from (4.4) and (4.5) we obtain that

$$(t,x)=\mu_1(t_{I_1},x_{I_1})+\mu_2(t_{I_2},x_{I_2}), \tag{4.6}$$

where

$$(t_{I_j},x_{I_j})=\sum_{i\in I_j}\left(\frac{\lambda_i}{\mu_j}(t_i,x_i)\right)\quad (j=1,2).$$

Since $(t_i, x_i) \in \operatorname{gr} V_1(\cdot)$ for every $i \in I_1$, $\sum_{i \in I_1} \frac{\lambda_i}{\mu_1} = 1$ and $V_1(\cdot)$ is a convex set valued map, then we get that $(t_{I_1}, x_{I_1}) \in \operatorname{gr} V_1(\cdot)$. Similarly, one can be shown that $(t_{I_2}, x_{I_2}) \in \operatorname{gr} V_2(\cdot)$.

Since $t \in [t_0, \theta]$ and $t = \mu_1 t_{I_1} + \mu_2 t_{I_2}$, then

$$t_{I_1} \geqslant t_0 \quad \text{or} \quad t_{I_2} \geqslant t_0.$$

Assume that $t_{I_1} \geqslant t_0$. Since $(t_{I_1}, x_{I_1}) \in \operatorname{gr} V_1(\cdot)$, then $x_{I_1} \in V_1(t_{I_1})$. Since the set valued maps $V_1(\cdot) : [t_0 - \alpha_1, \theta] \to \operatorname{comp}(R^n)$ and $V_2(\cdot) : [t_0 - \alpha_2, \theta] \to \operatorname{comp}(R^n)$ are convex extensions of the set valued map $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ and $t_{I_1} \geqslant t_0$, we obtain that

$$x_{I_1} \in V_1(t_{I_1}) = V(t_{I_1}) = V_2(t_{I_1}).$$

Thus, $(t_{I_1}, x_{I_1}) \in \operatorname{gr} V_2(\cdot)$. Since $(t_{I_2}, x_{I_2}) \in \operatorname{gr} V_2(\cdot)$, $V_2(\cdot) : [t_0 - \alpha_2, \theta] \to \operatorname{comp}(R^n)$ is a convex set valued map, then we get from (4.6) that $(t, x) \in \operatorname{gr} V_2(\cdot)$, that is $x \in V_2(t)$. Since $t \geqslant t_0$ then $V_2(t) = V(t)$ and consequently $x \in V(t)$.

If $t_{I_2} \geqslant t_0$, then the inclusion $x \in V(t)$ is proved analogously.

Since $x \in V(t)$ is chosen arbitrarily, we obtain the validity of the inclusion (4.3).

From (4.2) and (4.3) it follows the validity of the equality (4.1), which completes the proof of the proposition. □

Let $\alpha > 0$ and $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ be a convex set valued map. Define

$$M^L(\alpha) = \bigl\{x \in R^n:\ D^+ V(t_0, v) \subset D^+ V_\alpha^L(x) \mid (t_0, v),\ \forall v \in V(t_0)\bigr\}, \tag{4.7}$$

$$M^R(\alpha) = \bigl\{x \in R^n:\ D^- V(\theta, v) \subset D^- V_\alpha^R(x) \mid (\theta, v),\ \forall v \in V(\theta)\bigr\}. \tag{4.8}$$

Here the set valued maps $t \to V_\alpha^L(x) \mid (t)$, $t \in [t_0 - \alpha, \theta]$, and $t \to V_\alpha^R(x) \mid (t)$, $t \in [t_0, \theta + \alpha]$, are defined by (2.1) and (2.2), respectively.

Proposition 4.3. *If the convex set valued map* $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ *has a left-hand* (*right-hand*) *convex* α_**-extension, then* $M^L(\alpha) \subset R^n$ $(M^R(\alpha) \subset R^n)$ *is a nonempty convex compact set for every* $\alpha \in (0, \alpha_*]$.

The following theorem asserts that if the convex set valued map $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ has a left-hand convex α_*-extension, then the set valued map $V(\cdot)$ has the maximal left-hand convex α_*-extension.

Theorem 4.4. *Suppose that the convex set valued map* $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ *has a left-hand convex* α_**-extension.*

Then the set valued map $V_{\alpha_*}^{(ML)}(\cdot) : [t_0 - \alpha_*, \theta] \to \operatorname{comp}(R^n)$ *defined as*

$$V_{\alpha_*}^{(ML)}(t) = \begin{cases} M^L(t_0 - t), & t \in [t_0 - \alpha_*, t_0), \\ V(t), & t \in [t_0, \theta], \end{cases}$$

is the maximal left-hand convex α_**-extension of the set valued map* $V(\cdot)$, *where the set* $M^L(t_0 - t)$, $t \in [t_0 - \alpha_*, t_0)$, *is defined by* (4.7).

Proof. Since the set valued map $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ has a left-hand convex α_*-extension then according to Proposition 4.3 the set $M^L(t_0 - t)$ is nonempty convex and compact for every $t \in [t_0 - \alpha_*, t_0]$.

Firstly, let us prove that the set valued map $M^L(\cdot):(0,\alpha_*]\to \mathrm{comp}(R^n)$ is convex. Let $x_1\in M^L(\alpha_1),\ x_2\in M^L(\alpha_2)$ where $\alpha_1,\alpha_2\in(0,\alpha_*]$ and $\lambda\in[0,1]$. It is sufficient to show that

$$\lambda x_1+(1-\lambda)x_2\in M^L\big(\lambda\alpha_1+(1-\lambda)\alpha_2\big). \tag{4.9}$$

Since $x_1\in M^L(\alpha_1)$, $x_2\in M^L(\alpha_2)$, then from (4.7) it follows that

$$D^+V(t_0,v)\subset D^+V^L_{\alpha_i}(x_i)\mid(t_0,v)\quad (i=1,2)$$

for all $v\in V(t_0)$. Theorem 2.3 implies that the set valued map $V_i(\cdot):[t_0-\alpha_i,\theta]\to \mathrm{comp}(R^n)$ $(i=1,2)$ defined as

$$V_i(t)=\begin{cases} V^L_{\alpha_i}(x_i)\mid(t), & t\in[t_0-\alpha_i,t_0),\\ V(t), & t\in[t_0,\theta],\end{cases}$$

is a left-hand convex α_i-extension of the set valued map $V(\cdot)$. Let us denote

$$E=\mathrm{co}\big(\mathrm{gr}\,V_1(\cdot)\cup \mathrm{gr}\,V_2(\cdot)\big),$$
$$E(t)=\big\{x\in R^n:\ (t,x)\in E\big\},\quad t\in[t_0-\alpha,\theta],$$

where $\alpha=\max\{\alpha_1,\alpha_2\}$. Then from Proposition 4.2 we have that the set valued map $E(\cdot):[t_0-\alpha,\theta]\to \mathrm{comp}(R^n)$ is also a left-hand convex α-extension of $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$.

Since $(t_0-\alpha_i,x_i)\in \mathrm{gr}\,V_i(\cdot)$ $(i=1,2)$ then it follows from the definition of the set valued map $E(\cdot)$ that $(t_0-\alpha_i,x_i)\in \mathrm{gr}\,E(\cdot)$ $(i=1,2)$. Since the set valued map $E(\cdot):[t_0-\alpha,\theta]\to \mathrm{comp}(R^n)$ is convex, then we get

$$\lambda x_1+(1-\lambda)x_2\in E\big(t_0-\big(\lambda\alpha_1+(1-\lambda)\alpha_2\big)\big). \tag{4.10}$$

Denote $x_0=\lambda x_1+(1-\lambda)x_2,\ \alpha_0=\lambda\alpha_1+(1-\lambda)\alpha_2$. Then (4.10) implies $x_0\in E(t_0-\alpha_0)$. Since the set valued map $E(\cdot):[t_0-\alpha,\theta]\to \mathrm{comp}(R^n)$ is left-hand convex α-extension of $V(\cdot):[t_0,\theta]\to \mathrm{comp}(R^n)$, we obtain from Proposition 3.1 that

$$D^+V(t_0,v)\subset D^+V^L_{\alpha_0}(x_0)\mid(t_0,v)\quad \text{for all } v\in V(t_0).$$

Hence $x_0\in M^L(\alpha_0)$. So, (4.9) is satisfied.

Now, we should prove the set valued map $V^{(ML)}_{\alpha_*}(\cdot):[t_0-\alpha_*,\theta]\to \mathrm{comp}(R^n)$ is convex. Let $y_1\in V^{(ML)}_{\alpha_*}(t_1),\ y_2\in V^{(ML)}_{\alpha_*}(t_2),\ \lambda\in[0,1]$ and denote $y_\lambda=\lambda y_1+(1-\lambda)y_2$, $t_\lambda=\lambda t_1+(1-\lambda)t_2$. We have to prove that

$$y_\lambda\in V^{(ML)}_{\alpha_*}(t_\lambda). \tag{4.11}$$

If $t_1,t_2\in[t_0,\theta]$ or $t_1,t_2\in[t_0-\alpha_*,t_0)$, then validity of the inclusion (4.11) follows from definition of the set valued map $V^{(ML)}_{\alpha_*}(\cdot):[t_0-\alpha_*,\theta]\to \mathrm{comp}(R^n)$ and from convexity of the set valued map $M^L(\cdot):(0,\alpha_*]\to \mathrm{comp}(R^n)$.

Let $t_0-\alpha_*\leqslant t_1<t_0\leqslant t_2$. Then, from the definition of the set valued map $V^{(ML)}_{\alpha_*}(\cdot)$, we have $y_2\in V(t_2)$ and $y_1\in M^L(\beta_1)$, where $\beta_1=t_0-t_1$. Since $y_1\in M^L(\beta_1)$, then we obtain

$$D^+V(t_0,v)\subset D^+V^L_{\beta_1}(y_1)\mid(t_0,v)\quad \text{for all } v\in V(t_0).$$

According to Theorem 2.3, set valued map $V_*(\cdot):[t_0-\beta_1,\theta]\to \text{comp}(R^n)$ defined as

$$V_*(t)=\begin{cases} V^L_{\beta_1}(y_1)\mid(t), & t\in[t_0-\beta_1,t_0),\\ V(t), & t\in[t_0,\theta],\end{cases}$$

is a left-hand convex β_1-extension of the set valued map $V(\cdot):[t_0,\theta]\to \text{comp}(R^n)$. Then $y_\lambda\in V_*(t_\lambda)$.

If $t_\lambda\geqslant t_0$, then $y_\lambda\in V_*(t_\lambda)=V(t_\lambda)=V^{(ML)}_{\alpha_*}(t_\lambda)$. So the inclusion (4.11) holds.

Let $t_\lambda<t_0$. Denote $\beta_\lambda=t_0-t_\lambda$. Since $\beta_1\geqslant\beta_\lambda$, $y_\lambda\in V_*(t_\lambda)=V_*(t_0-\beta_\lambda)$, set valued map $V_*(\cdot):[t_0-\beta_1,\theta]\to \text{comp}(R^n)$ is a left-hand convex β_1-extension of the set valued map $V(\cdot):[t_0,\theta]\to \text{comp}(R^n)$, then it follows from Proposition 3.1 that

$$V(t)\subset V_{\beta_\lambda}(y_\lambda)\mid(t)\quad \text{for every } t\in[t_0,\theta]$$

and consequently

$$D^+V(t_0,v)\subset D^+V^L_{\beta_\lambda}(y_\lambda)\mid(t_0,v)\quad \text{for all } v\in V(t_0).$$

It follows from the last inclusion that $y_\lambda\in M^L(\beta_\lambda)=M^L(t_0-t_\lambda)=V^{(ML)}_{\alpha_*}(t_\lambda)$, that is the inclusion (4.11) is satisfied.

Finally, we prove the maximality of the set valued map $V^{(ML)}_{\alpha_*}(\cdot)$. Let $\alpha\in(0,\alpha_*]$, $V_\alpha(\cdot):[t_0-\alpha,\theta]\to \text{comp}(R^n)$ be a convex α-extension of the set valued map $V(\cdot)$ and $x\in V_\alpha(t)$ where $t\in[t_0-\alpha,t_0)$. Let $t_0-t=\beta$. Then $0<\beta\leqslant\alpha\leqslant\alpha_*$. From Proposition 3.1 we get

$$D^+V(t_0,v)\subset D^+V^L_{\beta}(x)\mid(t_0,v)\quad \text{for all } v\in V(t_0).$$

So, $x\in M^L(\beta)=M(t_0-t)$ and consequently $x\in V^{(ML)}_{\alpha_*}(t)$. □

Analogously the following theorem is true, which characterizes the existence of the maximal right-hand convex α_*-extension of the given set valued map.

Theorem 4.5. *Suppose that the convex set valued map $V(\cdot):[t_0,\theta]\to \text{comp}(R^n)$ has a right-hand convex α_*-extension.*

Then the set valued map $V^{(MR)}_{\alpha_*}(\cdot):[t_0,\theta+\alpha_*]\to \text{comp}(R^n)$ defined as

$$V^{(MR)}_{\alpha_*}(t)=\begin{cases} V(t), & t\in[t_0,\theta],\\ M^R(t-\theta), & t\in(\theta,\theta+\alpha_*],\end{cases}$$

is the maximal right-hand convex α_*-extension of the set valued map $V(\cdot)$, where the set $M^R(t-\theta)$, $t\in(\theta,\theta+\alpha_*]$, is defined by (4.8).

5. Lipschitz continuity and existence of convex extension

In this section the connection between the existence of the convex extension and Lipschitz continuity of the given set valued map is studied.

It follows from [11, Proposition 2.3], that if the set valued map $V(\cdot):[t_0,\theta]\to \text{comp}(R^n)$ is convex, then it is locally Lipschitz continuous on the open interval (t_0,θ). It is not difficult to give an example that the set valued map $V(\cdot):[t_0,\theta]\to \text{comp}(R^n)$ is a convex one, but it is not continuous on the closed interval $[t_0, \theta]$. It is possible to prove that the convex set valued map $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ is continuous on the closed interval $[t_0, \theta]$ if and only if it is a compact set valued map. But Example 3.4 illustrates that the continuity of the convex set valued map $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ is not sufficient for the existence of the convex α-extension.

Denote $B = \{x \in R^n\colon \| x \| \leqslant 1\}$.

Proposition 5.1. *Let* $W(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ *be a Lipschitz continuous set valued map on the closed interval* $[t_0, \theta]$ *with Lipschitz constant* $L > 0$. *Suppose that* $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ *is convex set valued map satisfying the conditions* $V(t_0) = W(t_0)$, $V(\theta) = W(\theta)$ *and* $\text{gr}\, V(\cdot) \subset \text{gr}\, W(\cdot)$. *Then the set valued map* $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ *is Lipschitz continuous on the closed interval* $[t_0, \theta]$ *with Lipschitz constant* $L > 0$.

Proof. Since $\text{gr}\, V(\cdot) \subset \text{gr}\, W(\cdot)$, then

$$V(t) \subset W(t) \quad \text{for every } t \in [t_0, \theta]. \tag{5.1}$$

Let us choose arbitrary t_1, $t_2 \in [t_0, \theta]$ and let $t_1 < t_2$. Choose $v_2 \in V(t_2)$. Then we have from (5.1) that $v_2 \in W(t_2)$. Since $W(\cdot)$ is Lipschitz continuous with Lipschitz constant $L > 0$, $W(t_0) = V(t_0)$, then there exist $v_0 \in V(t_0)$ and $b \in B$ such that

$$v_2 = v_0 + L(t_2 - t_0)b. \tag{5.2}$$

Now we define the function $v(\cdot) : [t_0, t_2] \to R^n$, by

$$v(t) = \left(1 - \frac{t - t_0}{t_2 - t_0}\right) v_0 + \frac{t - t_0}{t_2 - t_0} v_2, \quad t \in [t_0, t_2]. \tag{5.3}$$

Since $V(\cdot)$ is a convex set valued map and $v_0 \in V(t_0)$, $v_2 \in V(t_2)$, then $v(t) \in V(t)$ for every $t \in [t_0, t_2]$ and consequently $v(t_1) \in V(t_1)$. (5.2) and (5.3) imply that $\|v_2 - v(t_1)\| \leqslant L(t_2 - t_1)$. So, $v_2 \in V(t_1) + L(t_2 - t_1)B$ and consequently

$$V(t_2) \subset V(t_1) + L(t_2 - t_1)B. \tag{5.4}$$

Analogously, it can be shown that

$$V(t_1) \subset V(t_2) + L(t_2 - t_1)B. \tag{5.5}$$

(5.4) and (5.5) complete the proof. □

Theorem 5.2. *If the convex set valued map* $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ *has a convex* α*-extension, then it is Lipschitz continuous on the closed interval* $[t_0, \theta]$.

Proof. Since $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ has convex α-extension, then from Propositions 3.1 and 3.2, there exist $x_0 \in R^n$, $x_1 \in R^n$, the set valued maps $V_\alpha^L(x_0) \mid (\cdot) : [t_0 - \alpha, \theta] \to \text{comp}(R^n)$, $V_\alpha^R(x_1) \mid (\cdot) : [t_0, \theta + \alpha] \to \text{comp}(R^n)$ such that

$$V(t) \subset V_\alpha^L(x_0) \mid (t), \qquad V(t) \subset V_\alpha^R(x_1) \mid (t) \tag{5.6}$$

for every $t \in [t_0, \theta]$ where the sets $V_\alpha^L(x_0) \mid (t)$, $t \in [t_0 - \alpha, \theta]$, and $V_\alpha^R(x_1) \mid (t)$, $t \in [\theta, \theta + \alpha]$, are defined by (2.1) and (2.2), respectively. Denote

$$W = \operatorname{gr} V_\alpha^L(x_0) \mid (\cdot) \cap \operatorname{gr} V_\alpha^R(x_1) \mid (\cdot), \tag{5.7}$$

$$W(t) = \{x \in R^n\colon\ (t, x) \in W\}, \quad t \in [t_0, \theta]. \tag{5.8}$$

Since $\operatorname{gr} V_\alpha^L(x_0) \mid (\cdot)$ and $\operatorname{gr} V_\alpha^R(x_1) \mid (\cdot)$ are convex compact sets, then $W \subset [t_0 - \alpha, \theta + \alpha] \times R^n$ is also convex and compact. So, the set valued map $W(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ defined by (5.8), is a convex compact one.

From (2.1), (2.2), (5.6)–(5.8) it follows that $W(t_0) = V(t_0)$, $W(\theta) = V(\theta)$, $V(t) \subset W(t)$ for every $t \in (t_0, \theta)$. It is not difficult to verify that the set valued map $W(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ is Lipschitz continuous on the closed interval $[t_0, \theta]$. Then according to Proposition 5.1, the set valued map $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ is Lipschitz continuous. □

Theorem 5.3. *Let $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ be a convex and Lipschitz continuous set valued map. If $\operatorname{int} V(t_0) \neq \emptyset$ and $\operatorname{int} V(\theta) \neq \emptyset$, then the set valued map $V(\cdot)$ has a convex α-extension.*

Proof. Let the set valued map $V(\cdot) : [t_0, \theta] \to \operatorname{comp}(R^n)$ be Lipschitz continuous with Lipschitz constant $L_0 > 0$. Then

$$V(t) \subset V_1(t) \quad \text{for every } t \in [t_0, \theta], \tag{5.9}$$

where

$$V_1(t) = V(t_0) + L_0(t - t_0)B, \quad t \in [t_0, \theta]. \tag{5.10}$$

Now, let us show that there exist $\alpha_0 > 0$ and $x_0^* \in R^n$ such that $V_1(t) \subset V_{\alpha_0}^L(x_0^*) \mid (t)$ for every $t \in [t_0, \theta]$, where the set $V_{\alpha_0}^L(x_0^*) \mid (t)$ is defined by (2.1).

Let $x_0^* \in \operatorname{int} V(t_0)$. Then there exists $\varepsilon_0 > 0$ such that $B(x_0^*, \varepsilon_0) \subset V(t_0)$, where $B(x_0^*, \varepsilon_0) = \{x \in R^n\colon \| x - x_0^* \| \leqslant \varepsilon_0\}$.

Choose $\alpha_0 > 0$ such that

$$0 < \alpha_0 \leqslant \frac{\varepsilon_0}{L_0}. \tag{5.11}$$

Since $B(x_0^*, \varepsilon_0) \subset V(t_0)$, then from (2.1), (5.10) and (5.11) we obtain

$$\begin{aligned} V_1(t) &= V(t_0) + L_0(t - t_0)B = V(t_0) + \frac{L_0}{\varepsilon_0}(t - t_0)\big(B\big(x_0^*, \varepsilon_0\big) - x_0^*\big) \\ &\subset V(t_0) + \frac{1}{\alpha_0}(t - t_0)\big(V(t_0) - x_0^*\big) = V_{\alpha_0}^L\big(x_0^*\big) \mid (t) \end{aligned}$$

for every $t \in [t_0, \theta]$. So we have from (5.9) that

$$V(t) \subset V_{\alpha_0}^L\big(x_0^*\big) \mid (t) \quad \text{for every } t \in [t_0, \theta],$$

where $\alpha_0 \in (0, \frac{\varepsilon_0}{L_0}]$. Since $V_{\alpha_0}^L(t_0, x_0^*) \mid (t_0) = V(t_0)$, then

$$D^+V(t_0, v) \subset D^+V_{\alpha_0}^L\big(t_0, x_0^*\big) \mid (t_0, v) \quad \text{for every } v \in V(t_0). \tag{5.12}$$

Similarly, it can be proved that there exist $x_1^* \in R^n$ and $\alpha_1 > 0$ such that

$$D^- V(\theta, v_1) \subset D^- V_{\alpha_1}^R\big(x_1^*\big) \mid (\theta, v_1) \quad \text{for every } v_1 \in V(\theta). \tag{5.13}$$

It follows from (5.12) and Theorem 2.3 that the set valued map $W_1(\cdot) : [t_0 - \alpha_0, \theta] \to \text{comp}(R^n)$ defined as

$$W_1(t) = \begin{cases} V_{\alpha_0}^L(x_0^*) \mid (t), & t \in [t_0 - \alpha_0, t_0), \\ V(t), & t \in [t_0, \theta], \end{cases}$$

is a left-hand convex α_0-extension of the set valued map $V(\cdot)$.

Analogously we get from (5.13) and Theorem 2.4 that the set valued map $W_2(\cdot) : [t_0, \theta + \alpha_1] \to \text{comp}(R^n)$ defined as

$$W_2(t) = \begin{cases} V(t), & t \in [t_0, \theta], \\ V_{\alpha_1}^R(x_1^*) \mid (t), & t \in (\theta, \theta + \alpha_1], \end{cases}$$

is a right-hand convex α_1-extension of the set valued map $V(\cdot)$.

Let $\alpha = \min\{\alpha_0, \alpha_1\}$, $x_0 \in W_1(t_0 - \alpha)$, $x_1 \in W_2(\theta + \alpha)$. Then it follows from Propositions 3.1 and 3.2 that the inclusions

$$V(t) \subset V_\alpha^L(x_0) \mid (t), \qquad V(t) \subset V_\alpha^R(x_1) \mid (t)$$

hold for any $t \in [t_0, \theta]$ and consequently

$$D^+ V(t_0, v_0) \subset D^+ V_\alpha^L(x_0) \mid (t_0, v_0) \quad \text{for every } v_0 \in V(t_0), \tag{5.14}$$

$$D^- V(\theta, v_1) \subset D^- V_\alpha^R(x_1) \mid (\theta, v_1) \quad \text{for every } v_1 \in V(\theta). \tag{5.15}$$

Hence we get from (5.14), (5.15) and Theorem 2.5 that the set valued map $W(\cdot) : [t_0 - \alpha, \theta + \alpha] \to \text{comp}(R^n)$ defined as

$$W(t) = \begin{cases} V_\alpha^L(x_0) \mid (t), & t \in [t_0 - \alpha, t_0), \\ V(t), & t \in [t_0, \theta], \\ V_\alpha^R(x_1) \mid (t), & t \in (\theta, \theta + \alpha], \end{cases}$$

is a convex α-extension of the set valued map $V(\cdot)$. □

Example 5.4. Let $V_* \subset R^n$, $V^* \subset R^n$ be compact, convex sets and $\text{int}\, V_* \neq \emptyset$, $\text{int}\, V^* \neq \emptyset$. Then the set valued map $V(\cdot) : [t_0, \theta] \to \text{comp}(R^n)$ defined as

$$V(t) = \left(1 - \frac{t - t_0}{\theta - t_0}\right) V_* + \frac{t - t_0}{\theta - t_0} V^*$$

has a convex α-extension.

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