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On the continuity property of L_p balls and an application ☆

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Abstract

In this paper continuity properties of the set-valued map $p \to B_p(\mu_0)$, $p \in (1, +\infty)$, are considered where $B_p(\mu_0)$ is the closed ball of the space $L_p([t_0, \theta]; R^m)$ centered at the origin with radius μ_0. It is proved that the set-valued map $p \to B_p(\mu_0)$, $p \in (1, +\infty)$, is continuous. Applying obtained results, the attainable set of the nonlinear control system with integral constraint on the control is studied. The admissible control functions are chosen from $B_p(\mu_0)$. It is shown that the attainable set of the system is continuous with respect to p.

Keywords: L_p space; Set-valued map; Control system; Attainable set

1. Introduction

Let $\|\cdot\|$ be Euclidean norm in R^m, $\|u(\cdot)\|_p$ $(1 \leqslant p < +\infty)$ be a norm in $L_p([t_0, \theta], R^m)$, where $L_p([t_0, \theta], R^m)$ denotes the space of measurable functions $u(\cdot) : [t_0, \theta] \to R^m$ with bounded $\|u(\cdot)\|_p$ norm and

$$\|u(\cdot)\|_p = \left(\int_{t_0}^{\theta} \|u(t)\|^p \, dt \right)^{\frac{1}{p}}.$$

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For $p \geqslant 1$ and $\mu_0 > 0$ we set

$$B_p(\mu_0) = \{u(\cdot) \in L_p([t_0, \theta], R^m) \colon \|u(\cdot)\|_p \leqslant \mu_0\}. \tag{1.1}$$

It is obvious that $B_p(\mu_0)$ is the closed ball centered at the origin with radius μ_0 in $L_p([t_0, \theta], R^m)$.

The Hausdorff distance between the sets $A \subset R^m$ and $E \subset R^m$ is denoted by $h(A, E)$ and is defined as

$$h(A, E) = \max\Big\{\sup_{x\in A} d(x, E), \sup_{y\in E} d(y, A)\Big\}$$

where $d(x, E) = \inf\{\|x - y\| \colon y \in E\}$.

The Hausdorff distance between the sets $U \subset L_{p_1}([t_0, \theta], R^m)$ and $V \subset L_{p_2}([t_0, \theta], R^m)$ is denoted by $h_1(U, V)$ and is defined as

$$h_1(U, V) = \max\Big\{\sup_{x(\cdot)\in V} d_1(x(\cdot), U), \sup_{y(\cdot)\in U} d_1(y(\cdot), V)\Big\}$$

where $d_1(x(\cdot), U) = \inf\{\|x(\cdot) - y(\cdot)\|_1 \colon y(\cdot) \in U\}$, $p_1 \in [1, \infty)$, $p_2 \in [1, \infty)$.

For $\Omega \subset R^n$ we denote by $\mu(\Omega)$ the Lebesgue measure of the set Ω.

The need to evaluate the distance between the sets arises in various problems of theory and applications (see, e.g., [1,2,4–6,8–10,13] and references therein).

In this paper, the Hausdorff distance between the sets $B_p(\mu_0)$ and $B_{p_*}(\mu_0)$ is studied where $p > 1$ and $p_* > 1$. In Section 2 we prove that $h_1(B_p(\mu_0), B_{p_*}(\mu_0)) \to 0$ as $p \to p_* - 0$ (Proposition 2.5). In Section 3 it is shown that $h_1(B_p(\mu_0), B_{p_*}(\mu_0)) \to 0$ as $p \to p_* + 0$ (Proposition 3.5). As a corollary of Propositions 2.5 and 3.5, Theorem 3.6 concludes that $h_1(B_p(\mu_0), B_{p_*}(\mu_0)) \to 0$ as $p \to p_*$. In Section 4 we consider attainable sets of the nonlinear control system with integral constraints on control. $B_p(\mu_0)$ is chosen as the set of admissible control functions. As an application of Theorem 3.6, it is proved that the attainable set of the control system is continuous with respect to p (Theorem 4.2).

Let $H \in (0, \infty)$. We set

$$B_p^H(\mu_0) = \{u(\cdot) \in B_p(\mu_0) \colon \|u(t)\| \leqslant H \text{ for every } t \in [t_0, \theta]\}.$$

The following proposition characterizes the Hausdorff distance between the sets $B_p(\mu_0)$ and $B_p^H(\mu_0)$.

Proposition 1.1. *Let $p > 1$, $H > 0$. Then the inequality*

$$h_1(B_p(\mu_0), B_p^H(\mu_0)) \leqslant \frac{2\mu_0{}^p}{H^{p-1}}$$

holds.

Proof. Let us choose an arbitrary $u(\cdot) \in B_p(\mu_0)$ and define a function $u_*(\cdot) \colon [t_0, \theta] \to R^m$, setting for $t \in [t_0, \theta]$

$$u_*(t) = \begin{cases} u(t), & \|u(t)\| \leqslant H, \\ \frac{u(t)}{\|u(t)\|} H, & \|u(t)\| > H. \end{cases} \tag{1.2}$$

It is not difficult to verify that $u_*(\cdot) \in B_p^H(\mu_0)$. Let

$$\Omega = \{\tau \in [t_0, \theta]\colon \ \|u(\tau)\| > H\}.$$

Then using Hölder and Minkowski inequalities, we have from (1.2) that

$$\|u(\cdot) - u_*(\cdot)\|_1 = \int_\Omega \|u(t) - u_*(t)\|\, dt \leqslant 2\mu_0 \mu(\Omega)^{\frac{p-1}{p}}. \tag{1.3}$$

Since $u(\cdot) \in B_p(\mu_0)$ and $\|u(\tau)\| > H$ for every $\tau \in \Omega$, we obtain

$$H^p \mu(\Omega) \leqslant \int_\Omega \|u(\tau)\|^p\, d\tau \leqslant \int_{t_0}^{\theta} \|u(\tau)\|^p\, d\tau \leqslant \mu_0^p$$

and consequently

$$\mu(\Omega) \leqslant \frac{\mu_0^p}{H^p}. \tag{1.4}$$

Then it follows from (1.3) and (1.4)

$$\|u(\cdot) - u_*(\cdot)\|_1 \leqslant 2\mu_0 \left(\frac{\mu_0^p}{H^p}\right)^{\frac{p-1}{p}} = \frac{2\mu_0^p}{H^{p-1}}.$$

Since $u(\cdot) \in B_p(\mu_0)$ is arbitrarily chosen, we get the inequality

$$\sup_{u(\cdot) \in B_p(\mu_0)} d_1\big(u(\cdot), B_p^H(\mu_0)\big) \leqslant \frac{2\mu_0^p}{H^{p-1}}. \tag{1.5}$$

Since $B_p^H(\mu_0) \subset B_p(\mu_0)$ then (1.5) completes the proof of the proposition. □

We obtain the following corollary from Proposition 1.1.

Corollary 1.2. *Let $p_* > 1$ and $\varepsilon > 0$. Then there exists $H_*(\varepsilon) > 2\mu_0$ such that for all $H > H_*(\varepsilon)$ the inequality*

$$h_1\big(B_p(\mu_0), B_p^H(\mu_0)\big) \leqslant \varepsilon$$

holds for any $p \in [\frac{p_*+1}{2}, 2p_*]$.

2. Left evaluation of $B_p(\mu_0)$

In this section, we will evaluate the Hausdorff distance between the sets $B_p(\mu_0)$ and $B_{p_*}(\mu_0)$ as $p \to p_* - 0$.

Let

$$\alpha_0 = \min\left\{\frac{\mu_0}{2}, 1\right\}. \tag{2.1}$$

Proposition 2.1. *Let $p_* > 1$, $\varepsilon \in (0, \alpha_0)$ and $H_2 > H_1 > 2\mu_0$. Then there exists $\delta_1 = \delta_1(\varepsilon, H_1, H_2) \in (0, \frac{p_*-1}{2}]$ such that the inclusion*

$$B_{p_*}^{H_1}(\mu_0) \subset B_p^{H_2}(\mu_0) + 2(\theta - t_0)\varepsilon B_1(1)$$

holds for all $p \in (p_* - \delta_1, p_*)$ where $\alpha_0 > 0$ is defined by (2.1) *and $B_1(1)$ is defined by* (1.1).

Proof. Let

$$\delta_1(\varepsilon, H_1, H_2) = \min\bigl\{\beta_1(\varepsilon, H_1), \beta_2(\varepsilon, H_1), p_* - p_1(H_1, H_2)\bigr\}$$

where

$$p_1(H_1, H_2) = \max\left\{\frac{p_* + 1}{2}, \frac{p_*}{1 + \log_{\frac{H_1}{\mu_0}} \frac{H_2}{H_1}}\right\},$$

$$\beta_1(\varepsilon, H_1) = p_*\left(1 - \frac{1}{1 + \log_{\frac{H_1}{\mu_0}} \frac{H_1+\varepsilon}{H_1}}\right),$$

$$\beta_2(\varepsilon, H_1) = p_*\left(1 - \frac{1}{1 + \log_{\frac{\varepsilon}{\mu_0}} \frac{H_1-\varepsilon}{H_1}}\right).$$

It is not difficult to verify that $\delta_1(\varepsilon, H_1, H_2) \in (0, \frac{p_*-1}{2}]$.

Let $u_*(\cdot) \in B_{p_*}^{H_1}(\mu_0)$ be arbitrarily chosen and $p \in (p_* - \delta_1(\varepsilon, H_1, H_2), p_*)$. We set

$$u_p(t) = u_*(t)\bigl\|u_*(t)\bigr\|^{\frac{p_*-p}{p}} \mu_0^{\frac{p-p_*}{p}}, \quad t \in [t_0, \theta]. \tag{2.2}$$

Since $u_*(\cdot) \in B_{p_*}^{H_1}(\mu_0)$ and $p \in (p_* - \delta_1(\varepsilon, H_1, H_2), p_*)$, it is possible to prove that $u_p(\cdot) \in B_p^{H_2}(\mu_0)$.

Denote

$$A(\varepsilon) = \bigl\{t \in [t_0, \theta]:\ 0 \leqslant \bigl\|u_*(t)\bigr\| \leqslant \varepsilon\bigr\}, \qquad B(\varepsilon) = \bigl\{t \in [t_0, \theta]:\ \varepsilon < \bigl\|u_*(t)\bigr\| \leqslant H_1\bigr\}.$$

Let $t \in A(\varepsilon)$. Then $0 \leqslant \|u_*(t)\| \leqslant \varepsilon$. Since $\varepsilon < \frac{\mu_0}{2}$ and $p < p_*$ then we obtain

$$\bigl\|u_*(t)\bigr\|\left|1 - \left(\frac{\|u_*(t)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right| \leqslant \varepsilon \tag{2.3}$$

for every $t \in A(\varepsilon)$.

Let $t \in B(\varepsilon)$. Then $\varepsilon < \|u_*(t)\| \leqslant H_1$ and this gives

$$1 - \left(\frac{H_1}{\mu_0}\right)^{\frac{p_*-p}{p}} \leqslant 1 - \left(\frac{\|u_*(t)\|}{\mu_0}\right)^{\frac{p_*-p}{p}} \leqslant 1 - \left(\frac{\varepsilon}{\mu_0}\right)^{\frac{p_*-p}{p}}. \tag{2.4}$$

Since $p \in (p_* - \delta_1(\varepsilon, H_1, H_2), p_*)$, then we get from (2.4) that the inequality

$$\left|1 - \left(\frac{\|u_*(t)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right| \leqslant \frac{\varepsilon}{H_1}$$

holds for every $t \in B(\varepsilon)$ and consequently

$$\left\|u_*(t)\right\|\left|1-\left(\frac{\|u_*(t)\|}{\mu_0}\right)^{\frac{p_*-p}{p}}\right|\leqslant\varepsilon. \tag{2.5}$$

Finally, it follows from (2.3) and (2.5) that

$$\left\|u_p(\cdot)-u_*(\cdot)\right\|_1\leqslant\varepsilon\Big[\mu\big(A(\varepsilon)\big)+\mu\big(B(\varepsilon)\big)\Big]\leqslant 2\varepsilon(\theta-t_0). \tag{2.6}$$

Since $p\in(p_*-\delta_1(\varepsilon,H_1,H_2),p_*)$ and $u_*(\cdot)\in B_{p_*}^{H_1}(\mu_0)$ are arbitrarily chosen, (2.6) implies the validity of the proposition. □

From Corollary 1.2 and Proposition 2.1 the validity of the following proposition follows.

Proposition 2.2. *Let* $p_*>1$, $\varepsilon\in(0,\alpha_0)$ *where* $\alpha_0>0$ *is defined by* (2.1). *Then there exists* $\gamma_1=\gamma_1(\varepsilon)\in(0,\frac{p_*-1}{2}]$ *such that the inclusion*

$$B_{p_*}(\mu_0)\subset B_p(\mu_0)+\varepsilon B_1(1)$$

holds for any $p\in(p_*-\gamma_1,p_*)$.

Proof. By Corollary 1.2 there exists $H_*(\varepsilon)>2\mu_0$ such that for every $H>H_*(\varepsilon)$ the inclusions

$$B_p(\mu_0)\subset B_p^H(\mu_0)+\frac{\varepsilon}{3}B_1(1),\qquad B_p^H(\mu_0)\subset B_p(\mu_0)+\frac{\varepsilon}{3}B_1(1) \tag{2.7}$$

hold for any $p\in[\frac{p_*+1}{2},2p_*]$.

Let $H_1(\varepsilon)=2H_*(\varepsilon)$, $H_2(\varepsilon)=3H_*(\varepsilon)$. Then by virtue of Proposition 2.1 there exists $\gamma_1(\varepsilon)=\delta_1(\varepsilon,H_1(\varepsilon),H_2(\varepsilon))\in(0,\frac{p_*-1}{2}]$ such that the inclusion

$$B_{p_*}^{H_1(\varepsilon)}(\mu_0)\subset B_p^{H_2(\varepsilon)}(\mu_0)+\frac{\varepsilon}{3}B_1(1) \tag{2.8}$$

holds for all $p\in(p_*-\gamma_1(\varepsilon),p_*)$.

The proof of the proposition follows from (2.7) and (2.8). □

Now, let us give an upper estimation of the set $B_p^H(\mu_0)$ as $p\to p_*-0$. Let

$$\mu_*=\max\left\{\mu_0^{\frac{p_*-p}{p_*}}:\ p\in\left[\frac{p_*+1}{2},p_*\right]\right\}, \tag{2.9}$$

$$L_*=(2+\mu_*)(\theta-t_0). \tag{2.10}$$

Proposition 2.3. *Let* $p_*>1$, $\varepsilon\in(0,\alpha_0)$, $H>2\mu_0$ *where* $\alpha_0>0$ *is defined by* (2.1). *Then there exists* $\delta_2=\delta_2(\varepsilon,H)\in(0,\frac{p_*-1}{2}]$ *such that the inclusion*

$$B_p^H(\mu_0)\subset B_{p_*}^H(\mu_0)+L_*\varepsilon^{\frac{1}{2}}B_1(1)$$

holds for any $p\in(p_*-\delta_2,p_*)$.

Proof. Let

$$\delta_2(\varepsilon,H)=\min\big\{\beta_3(\varepsilon,H),\beta_4(\varepsilon,H)\big\}$$

where

$$\beta_3(\varepsilon, H) = \min\left\{p_* \log_{\frac{\mu_0}{\varepsilon}} \frac{H+\varepsilon}{H}, \frac{p_*-1}{2}\right\},$$
$$\beta_4(\varepsilon, H) = \min\left\{p_* \log_{\frac{\mu_0}{H}} \frac{H-\varepsilon}{H}, \frac{p_*-1}{2}\right\}.$$

It is obvious that $\delta_2(\varepsilon, H) \in (0, \frac{p_*-1}{2}]$.

Let $p \in (p_* - \delta_2(\varepsilon, H), p_*)$. Now let us choose an arbitrary $u(\cdot) \in B_p^H(\mu_0)$ and define a function $u_*(\cdot) : [t_0, \theta] \to R^m$ by setting

$$u_*(t) = u(t)\left\|u(t)\right\|^{\frac{p-p_*}{p_*}} \mu_0^{\frac{p_*-p}{p_*}}, \quad t \in [t_0, \theta]. \tag{2.11}$$

Since $u(\cdot) \in B_p^H(\mu_0)$ one can show that $u_*(\cdot) \in B_{p_*}^H(\mu_0)$.

Now let us set

$$A(\varepsilon) = \left\{t \in [t_0, \theta]:\ 0 \leqslant \left\|u(t)\right\| \leqslant \varepsilon\right\}, \qquad B(\varepsilon) = \left\{t \in [t_0, \theta]:\ \varepsilon < \left\|u(t)\right\| \leqslant H_1\right\}.$$

Let $t \in A(\varepsilon)$. Then $0 \leqslant \|u(t)\| \leqslant \varepsilon$. Since $\varepsilon < \alpha_0 \leqslant 1$ and $p \in (p_* - \delta_2(\varepsilon, H), p_*) \subset [\frac{p_*+1}{2}, p_*)$ we get from (2.11)

$$\begin{aligned}\int\limits_{A(\varepsilon)} \left\|u(t) - u_*(t)\right\| dt &\leqslant \varepsilon(\theta - t_0) + \mu_0^{\frac{p_*-p}{p_*}} \varepsilon^{\frac{p}{p_*}}(\theta - t_0) \\ &\leqslant \varepsilon(\theta - t_0) + \varepsilon^{\frac{1}{2}} \mu_*(\theta - t_0) \\ &\leqslant \varepsilon^{\frac{1}{2}}(1 + \mu_*)(\theta - t_0)\end{aligned} \tag{2.12}$$

where μ_* is defined by (2.9).

Let $t \in B(\varepsilon)$. Then $\varepsilon < \|u(t)\| \leqslant H$ and

$$1 - \left(\frac{\mu_0}{\varepsilon}\right)^{\frac{p_*-p}{p_*}} \leqslant 1 - \left(\frac{\mu_0}{\|u(t)\|}\right)^{\frac{p_*-p}{p_*}} \leqslant 1 - \left(\frac{\mu_0}{H}\right)^{\frac{p_*-p}{p_*}}. \tag{2.13}$$

Since $p \in (p_* - \delta_2(\varepsilon, H), p_*)$ then (2.13) implies

$$\left|1 - \left(\frac{\mu_0}{\|u(t)\|}\right)^{\frac{p_*-p}{p_*}}\right| \leqslant \frac{\varepsilon}{H}$$

and consequently

$$\left\|u(t)\right\| \left|1 - \left(\frac{\mu_0}{\|u(t)\|}\right)^{\frac{p_*-p}{p_*}}\right| \leqslant \varepsilon \tag{2.14}$$

for every $t \in B(\varepsilon)$. Thus, it follows from (2.12) and (2.14)

$$\begin{aligned}\left\|u(\cdot) - u_*(\cdot)\right\|_1 &\leqslant \varepsilon^{\frac{1}{2}}(\theta - t_0)\left[1 + \mu_* + \varepsilon^{\frac{1}{2}}\right] \\ &\leqslant \varepsilon^{\frac{1}{2}}(\theta - t_0)[2 + \mu_*] = \varepsilon^{\frac{1}{2}} L_*\end{aligned}$$

where L_* is defined by (2.10).

Since $p \in (p_* - \delta_2(\varepsilon, H), p_*)$ and $u(\cdot) \in B_p^H(\mu_0)$ are arbitrarily chosen, we obtain the validity of the proposition. □

The following proposition gives an upper estimation of the set $B_p(\mu_0)$ as $p \to p_* - 0$.

Proposition 2.4. *Let $p_* > 1$, $\varepsilon \in (0, \alpha_0)$ where $\alpha_0 > 0$ is defined by* (2.1). *Then there exists $\gamma_2 = \gamma_2(\varepsilon) \in (0, \frac{p_*-1}{2}]$ such that the inclusion*

$$B_p(\mu_0) \subset B_{p_*}(\mu_0) + \varepsilon B_1(1)$$

holds for any $p \in (p_* - \gamma_2, p_*)$.

Proof. By Corollary 1.2 there exists $H_*(\varepsilon) > 2\mu_0$ such that for all $H > H_*(\varepsilon)$ the inclusions

$$B_p(\mu_0) \subset B_p^H(\mu_0) + \frac{\varepsilon}{3} B_1(1), \qquad B_p^H(\mu_0) \subset B_p(\mu_0) + \frac{\varepsilon}{3} B_1(1) \tag{2.15}$$

hold for any $p \in [\frac{p_*+1}{2}, 2p_*]$.

Let $H(\varepsilon) = 2H_*(\varepsilon)$. Then due to Proposition 2.3 there exists $\delta_2(\varepsilon) = \delta_2(\varepsilon, H(\varepsilon)) \in (0, \frac{p_*-1}{2}]$ such that the inclusion

$$B_p^{H(\varepsilon)}(\mu_0) \subset B_{p_*}^{H(\varepsilon)}(\mu_0) + \frac{\varepsilon}{3} B_1(1) \tag{2.16}$$

holds for any $p \in (p_* - \delta_2(\varepsilon), p_*)$.

Let $\gamma_2 = \gamma_2(\varepsilon) = \delta_2(\varepsilon)$. Then (2.15) and (2.16) complete the proof. $\square$

From Propositions 2.2 and 2.4 we get the following proposition.

Proposition 2.5. *Let $p_* > 1$, $\varepsilon \in (0, \alpha_0)$ where $\alpha_0 > 0$ is defined by* (2.1). *Then there exists $\delta_* = \delta_*(\varepsilon) \in (0, \frac{p_*-1}{2}]$ such that the inequality*

$$h_1\big(B_p(\mu_0), B_{p_*}(\mu_0)\big) \leqslant \varepsilon$$

holds for all $p \in (p_* - \delta_*, p_*)$.

3. Right evaluation of $B_p(\mu_0)$

In this section we will study right continuity of the set-valued map $p \to B_p(\mu_0)$, $p \in (1, +\infty)$. The following proposition gives an upper estimation of the set $B_p^H(\mu_0)$ as $p \to p_* + 0$.

Proposition 3.1. *Let $p_* > 1$, $\varepsilon \in (0, \alpha_0)$, $H_1 > H_2 > \mu_0$ where $\alpha_0 > 0$ is defined by* (2.1). *Then, there exists $\nu_1 = \nu_1(\varepsilon, H_1, H_2) \in (0, p_*]$ such that the inclusion*

$$B_p^{H_2}(\mu_0) \subset B_{p_*}^{H_1}(\mu_0) + 2(\theta - t_0)\varepsilon B_1(1)$$

holds for any $p \in (p_*, p_* + \nu_1)$.

Proof. Let

$$\nu_1(\varepsilon, H_1, H_2) = \min\big\{\beta_1^*(\varepsilon, H_1, H_2), \beta_2^*(\varepsilon, H_1, H_2), p_2(H_1, H_2) - p_*\big\}$$

where

$$p_2(H_1, H_2) = \min\left\{2p_*, p_*\left(1 + \log_{\frac{H_2}{\mu_0}} \frac{H_1}{H_2}\right)\right\},$$

$$\beta_1^*(\varepsilon, H_1, H_2) = \min\left\{p_* \log_{\frac{H_2}{\mu_0}} \frac{H_2 + \varepsilon}{H_2}, p_*\right\},$$

$$\beta_2^*(\varepsilon, H_1, H_2) = \min\left\{ p_* \log_{\frac{\varepsilon}{\mu_0}} \frac{H_2 - \varepsilon}{H_2}, p_* \right\}.$$

It is obvious that $\nu_1(\varepsilon, H_1, H_2) \in (0, p_*]$.

Let $p \in (p_*, p_* + \nu_1(\varepsilon, H_1, H_2))$ and choose an arbitrary $u(\cdot) \in B_p^{H_2}(\mu_0)$. Define a function $u_*(\cdot) : [t_0, \theta] \to R^m$ by setting

$$u_*(t) = u(t)\left\|u(t)\right\|^{\frac{p-p_*}{p_*}} \mu_0^{\frac{p_*-p}{p_*}}, \quad t \in [t_0, \theta]. \tag{3.1}$$

It is not difficult to show that $u_*(\cdot) \in B_{p_*}^{H_1}(\mu_0)$.

Let us denote

$$A(\varepsilon) = \left\{t \in [t_0, \theta]\colon\ 0 \leqslant \left\|u(t)\right\| \leqslant \varepsilon\right\}, \qquad B(\varepsilon) = \left\{t \in [t_0, \theta]\colon\ \varepsilon < \left\|u(t)\right\| \leqslant H_2\right\}.$$

Let $t \in A(\varepsilon)$. Since $\varepsilon < \mu_0$ and $p > p_*$ then we obtain

$$\left\|u(t)\right\| \left| 1 - \left(\frac{\|u(t)\|}{\mu_0}\right)^{\frac{p-p_*}{p_*}} \right| \leqslant \varepsilon. \tag{3.2}$$

Let $t \in B(\varepsilon)$. Then $\varepsilon < \|u(t)\| \leqslant H_2$ and consequently

$$1 - \left(\frac{H_2}{\mu_0}\right)^{\frac{p-p_*}{p_*}} \leqslant 1 - \left(\frac{\|u(t)\|}{\mu_0}\right)^{\frac{p-p_*}{p_*}} \leqslant 1 - \left(\frac{\varepsilon}{\mu_0}\right)^{\frac{p-p_*}{p_*}}. \tag{3.3}$$

Since $p \in (p_*, p_* + \nu_1(\varepsilon, H_1, H_2))$, from (3.3) we see that the inequality

$$\left| 1 - \left(\frac{\|u(t)\|}{\mu_0}\right)^{\frac{p-p_*}{p_*}} \right| \leqslant \frac{\varepsilon}{H_2}$$

is satisfied and hence

$$\left\|u(t)\right\| \left| 1 - \left(\frac{\|u(t)\|}{\mu_0}\right)^{\frac{p-p_*}{p_*}} \right| \leqslant \varepsilon. \tag{3.4}$$

Thus, from (3.2) and (3.4) we obtain the inequality

$$\left\|u(\cdot) - u_*(\cdot)\right\|_1 \leqslant \varepsilon\mu\big(A(\varepsilon)\big) + \varepsilon\mu\big(B(\varepsilon)\big) \leqslant 2\varepsilon(\theta - t_0).$$

Since $p \in (p_*, p_* + \nu_1(\varepsilon, H_1, H_2))$ and $u(\cdot) \in B_p^{H_2}(\mu_0)$ are arbitrarily chosen, the proof is completed. □

Proposition 3.2. *Let $p_* > 1$, $\varepsilon \in (0, \alpha_0)$ where $\alpha_0 > 0$ is defined by (2.1). Then there exists $\gamma_1^* = \gamma_1^*(\varepsilon) \in (0, p_*]$ such that the inclusion*

$$B_p(\mu_0) \subset B_{p_*}(\mu_0) + \varepsilon B_1(1)$$

holds for any $p \in (p_*, p_* + \gamma_1^*)$.

The proof of Proposition 3.2 follows from Corollary 1.2 and Proposition 3.1.

Let us define constants

$$\mu^* = \max\left\{\mu_0^{\frac{p-p_*}{p}} : p \in [p_*, 2p_*]\right\}, \tag{3.5}$$

$$L^* = (2 + \mu^*)(\theta - t_0) \tag{3.6}$$

which are required in Proposition 3.3 and will be used in the sequel.

Proposition 3.3. *Let $p_* > 1$, $\varepsilon \in (0, \alpha_0)$ and $H > 2\mu_0$ where $\alpha_0 > 0$ is defined by* (2.1). *Then, there exists $\nu_2 = \nu_2(\varepsilon, H) \in (0, p_*]$ such that the inclusion*

$$B_{p_*}^H(\mu_0) \subset B_p^H(\mu_0) + L_* \varepsilon^{\frac{1}{2}} B_1(1)$$

holds for any $p \in (p_*, p_* + \nu_2)$.

Proof. Let

$$\nu_2(\varepsilon, H) = \min\{\beta_3^*(\varepsilon, H), \beta_4^*(\varepsilon, H)\}$$

where

$$\beta_3^*(\varepsilon, H) = \min\left\{p_*\left(\frac{1}{1 - \log_{\frac{\mu_0}{\varepsilon}} \frac{H+\varepsilon}{H}} - 1\right), p_*\right\},$$

$$\beta_4^*(\varepsilon, H) = \min\left\{p_*\left(\frac{1}{1 - \log_{\frac{\mu_0}{H}} \frac{H-\varepsilon}{H}} - 1\right), p_*\right\}.$$

It is obvious that $\nu_2(\varepsilon, H) \in (0, p_*]$.

Let $p \in (p_*, p_* + \nu_2(\varepsilon, H))$ and $u_*(\cdot) \in B_{p_*}^H(\mu_0)$ be arbitrarily chosen. We set

$$u(t) = u_*(t)\|u_*(t)\|^{\frac{p_*-p}{p}} \mu_0^{\frac{p-p_*}{p}}, \quad t \in [t_0, \theta]. \tag{3.7}$$

It can be shown that $u(\cdot) \in B_p^H(\mu_0)$.

We denote

$$A(\varepsilon) = \{t \in [t_0, \theta]:\ 0 \leqslant \|u_*(t)\| \leqslant \varepsilon\}, \qquad B(\varepsilon) = \{t \in [t_0, \theta]:\ \varepsilon < \|u_*(t)\| \leqslant H\}.$$

Let $t \in A(\varepsilon)$. Then $0 \leqslant \|u_*(t)\| \leqslant \varepsilon$. Since $\varepsilon < \alpha_0 < 1$, $p \in [p_*, 2p_*]$ then we obtain that

$$\begin{aligned}\int\limits_{A(\varepsilon)} \left\|u_*(t) - u_*(t)\|u_*(t)\|^{\frac{p_*-p}{p}} \mu_0^{\frac{p-p_*}{p}}\right\| dt &\leqslant \varepsilon(\theta - t_0) + \mu_0^{\frac{p-p_*}{p}} \varepsilon^{\frac{p_*}{p}} (\theta - t_0) \\ &\leqslant \varepsilon^{\frac{1}{2}}(1 + \mu^*)(\theta - t_0)\end{aligned} \tag{3.8}$$

where μ^* is defined by (3.5).

Let $t \in B(\varepsilon)$. Then, $\varepsilon < \|u_*(t)\| \leqslant H$ and consequently

$$1 - \left(\frac{\mu_0}{\varepsilon}\right)^{\frac{p-p_*}{p}} \leqslant 1 - \left(\frac{\mu_0}{\|u_*(t)\|}\right)^{\frac{p-p_*}{p}} \leqslant 1 - \left(\frac{\mu_0}{H}\right)^{\frac{p-p_*}{p}}. \tag{3.9}$$

Since $p \in (p_*, p_* + \upsilon_2(\varepsilon, H))$ then from (3.9) we get

$$\left|1 - \left(\frac{\mu_0}{\|u_*(t)\|}\right)^{\frac{p-p_*}{p}}\right| \leqslant \frac{\varepsilon}{H_2}$$

and hence

$$\|u_*(t)\|\left|1 - \left(\frac{\mu_0}{\|u_*(t)\|}\right)^{\frac{p-p_*}{p}}\right| \leqslant \varepsilon \tag{3.10}$$

for every $t \in B(\varepsilon)$. From (3.8) and (3.10) we conclude that

$$\begin{aligned}\left\|u(\cdot)-u_*(\cdot)\right\|_1 &\leqslant \varepsilon^{\frac{1}{2}}(1+\mu^*)(\theta-t_0)+\varepsilon(\theta-t_0)\\ &\leqslant \varepsilon^{\frac{1}{2}}(\theta-t_0)[2+\mu^*]=\varepsilon^{\frac{1}{2}}L^*\end{aligned}$$

where L^* is defined by (3.6).

Since $p \in (p_*, p_* + \nu_2(\varepsilon, H))$ and $u_*(\cdot) \in B_{p_*}^H(\mu_0)$ are arbitrarily chosen, this completes the proof of the proposition. □

Proposition 3.4. *Let $p_* > 1$, $\varepsilon \in (0, \alpha_0)$ where $\alpha_0 > 0$ is defined by* (2.1). *Then there exists $\gamma_2^* = \gamma_2^*(\varepsilon) \in (0, p_*]$ such that the inclusion*

$$B_{p_*}(\mu_0) \subset B_p(\mu_0) + \varepsilon B_1(1)$$

holds for any $p \in (p_*, p_* + \gamma_2^*)$.

The proof of Proposition 3.4 follows from Corollary 1.2 and Proposition 3.3.

Propositions 3.2 and 3.4 imply the validity of the following proposition.

Proposition 3.5. *Let $p_* > 1$ and $\varepsilon \in (0, \alpha_0)$ where $\alpha_0 > 0$ is defined by* (2.1). *Then there exists $\delta^* = \delta^*(\varepsilon) \in (0, p_*]$ such that the inequality*

$$h_1\big(B_p(\mu_0), B_{p_*}(\mu_0)\big) \leqslant \varepsilon$$

holds for all $p \in (p_*, p_* + \delta^*)$.

Finally, from Propositions 2.5 and 3.5 we obtain the validity of following theorem, which characterizes continuity of the set-valued map $p \to B_p(\mu_0)$ with respect to p where $p \in (1+\infty)$.

Theorem 3.6. *Let $p_* > 1$ and $\varepsilon \in (0, \alpha_0)$ where $\alpha_0 > 0$ is defined by* (2.1). *Then there exists $\delta = \delta(\varepsilon) > 0$ such that the inequality*

$$h_1\big(B_p(\mu_0), B_{p_*}(\mu_0)\big) \leqslant \varepsilon$$

holds for all $p \in (p_* - \delta, p_* + \delta)$.

4. Attainable sets of control systems

Consider the control system the behavior of which is described by the differential equation

$$\dot{x}(t) = f\big(t, x(t), u(t)\big), \qquad x(t_0) \in X_0, \tag{4.1}$$

where $x \in R^n$ is the phase state vector of the system, $u \in R^m$ is the control vector, $t \in [t_0, \theta]$ is the time and $X_0 \subset R^n$ is a compact set.

It is assumed that the right-hand side of the system (4.1) satisfies the following conditions:

(4.A) the function $f(\cdot) : [t_0, \theta] \times R^n \times R^m \to R^n$ is continuous;

(4.B) for any bounded set $D \subset [t_0, \theta] \times R^n$ there exist constants $L_1 = L_1(D) > 0$, $L_2 = L_2(D) > 0$ and $L_3 = L_3(D) > 0$ such that

$$\big\|f(t, x_1, u_1) - f(t, x_2, u_2)\big\| \leqslant \big(L_1 + L_2\|u_2\|\big)\|x_1 - x_2\| + L_3\|u_1 - u_2\|$$

for any $(t, x_1) \in D$, $(t, x_2) \in D$, $u_1 \in R^m$ and $u_2 \in R^m$;

(4.C) There exists a constant $c > 0$ such that

$$\|f(t,x,u)\| \leqslant c(1+\|x\|)(1+\|u\|)$$

for every $(t,x,u) \in [t_0,\theta] \times R^n \times R^m$.

The set $B_p(\mu_0)$ is called the set of admissible control functions and a function $u(\cdot) \in B_p(\mu_0)$ is said to be an admissible control function where the set $B_p(\mu_0)$ is defined by (1.1).

Let $u_*(\cdot) \in B_p(\mu_0)$. The absolutely continuous function $x_*(\cdot) : [t_0,\theta] \to R^n$ which satisfies the equation $\dot{x}_*(t) = f(t, x_*(t), u_*(t))$ a.e. in $[t_0,\theta]$ and the initial condition $x_*(t_0) = x_0 \in X_0$ is said to be a solution of the system (4.1) with initial condition $x_*(t_0) = x_0$, generated by the admissible control function $u_*(\cdot)$. The symbol $x(\cdot; t_0, x_0, u(\cdot))$ denotes a solution of the system (4.1) with initial condition $x(t_0) = x_0$, generated by the admissible control function $u(\cdot)$.

Let us define the sets

$$X_p(t_0, X_0, \mu_0) = \{x(\cdot; t_0, x_0, u(\cdot)) : [t_0,\theta] \to R^n \colon x_0 \in X_0,\ u(\cdot) \in B_p(\mu_0)\}$$

and

$$X_p(t; t_0, X_0, \mu_0) = \{x(t) \in R^n \colon x(\cdot) \in X_p(t_0, X_0, \mu_0)\}$$

where $t \in [t_0,\theta]$.

The set $X_p(t; t_0, X_0, \mu_0)$ is called the attainable set of the system (4.1) at the instant of time t. It is clear that the set $X_p(t; t_0, X_0, \mu_0)$ consists of all $x \in R^n$ to which the system (4.1) can be steered at the instant of time $t \in [t_0,\theta]$. In general the set $X_p(t; t_0, X_0, \mu_0) \subset R^n$ is not closed (see, e.g., [3]) and it is not difficult to verify that it depends on t, t_0, X_0 and μ_0 continuously. Other properties of the attainable set $X_p(t; t_0, X_0, \mu_0) \subset R^n$ and approximation methods for its numerical construction have been considered in [7,8,10–13]. In this section we show that the attainable set $X_p(t; t_0, X_0, \mu_0) \subset R^n$ depends on p continuously.

The following proposition characterizes boundedness of the attainable sets of the system (4.1).

Proposition 4.1. *The inequality*

$$\|x(t)\| \leqslant (\rho_* + r_*)\exp(r_*)$$

holds for every $p \in (1,+\infty)$, $x(\cdot) \in X_p(t_0, X_0, \mu_0)$ *and* $t \in [t_0,\theta]$ *where*

$$\rho_* = \max\{\|x\| \colon x \in X_0\}, \qquad r_* = cr_0(1+\mu_0), \quad r_0 = \max\{\theta - t_0, 1\}, \tag{4.2}$$

$c > 0$ *is defined by condition* (4.C).

The proof of the proposition follows from conditions (4.A)–(4.C) and Gronwall inequality.

Denote

$$D = \{(t,x) \in [t_0,\theta] \times R^n \colon \|x\| \leqslant (\rho_* + r_*)\exp(r_*)\} \tag{4.3}$$

where ρ_* and r_* are defined by (4.2).

According to Proposition 4.1 we get that $(t, x(t)) \in D$ for every $p \in (1,+\infty)$, $x(\cdot) \in X_p(t_0, X_0, \mu_0)$ and $t \in [t_0,\theta]$. Therefore, here and henceforth we will have in mind the cylinder (4.3) as the set D in condition (4.B).

Note that the continuity property of the set-valued map $p \to B_p(\mu_0)$, $p \in (1,+\infty)$, implies the uniform continuity (with respect to t) of the set-valued map $p \to X_p(t; t_0, X_0, \mu_0)$, $p \in (1,+\infty)$.

Theorem 4.2. *Let $p_* > 1$, $\varepsilon \in (0, \alpha_0)$ where $\alpha_0 > 0$ is defined by (2.1). Then there exists $\xi = \xi(\varepsilon) > 0$ such that the inequality*

$$h_c\big(X_p(t_0, X_0, \mu_0), X_{p_*}(t_0, X_0, \mu_0)\big) \leqslant \varepsilon$$

holds for any $p \in (p_* - \xi, p_* + \xi)$ and consequently for all $t \in [t_0, \theta]$

$$h\big(X_p(t; t_0, X_0, \mu_0), X_{p_*}(t; t_0, X_0, \mu_0)\big) \leqslant \varepsilon.$$

Here $h_c(S, G)$ denotes the Hausdorff distance between the sets $S \subset C([t_0, \theta]; R^n)$ and $G \subset C([t_0, \theta]; R^n)$ and is defined as

$$h_c(S, G) = \max\Big\{ \sup_{x(\cdot)\in S} d_c\big(x(\cdot), G\big), \sup_{y(\cdot)\in G} d_c\big(y(\cdot), S\big)\Big\}$$

where $d_c(x(\cdot), G) = \inf\{\|x(\cdot) - y(\cdot)\|_c\colon y(\cdot) \in G\}$, $\|z(\cdot)\|_c = \max\{\|z(t)\|\colon t \in [t_0, \theta]\}$, $C([t_0, \theta]; R^n)$ is the space of continuous functions $x(\cdot) : [t_0, \theta] \to R^n$.

Proof. Let

$$a_0 = L_1 r_0 + L_2 r_0 \mu_0, \qquad b_0 = L_3 \exp(a_0) \tag{4.4}$$

where r_0 is defined by (4.2).

By virtue of Proposition 3.6 for $\frac{\varepsilon}{b_0}$ there exists $\xi = \xi(\varepsilon)$ such that the inequality

$$h_1\big(B_p(\mu_0), B_{p_*}(\mu_0)\big) \leqslant \frac{\varepsilon}{b_0} \tag{4.5}$$

holds for all $p \in (p_* - \xi(\varepsilon), p_* + \xi(\varepsilon))$.

Let us choose an arbitrary $p \in (p_* - \xi(\varepsilon), p_* + \xi(\varepsilon))$ and $x(\cdot) \in X_p(t_0, X_0, \mu_0)$. Then there exist $x_0 \in X_0$ and $u(\cdot) \in B_p(\mu_0)$ such that the equality

$$x(t) = x_0 + \int_{t_0}^{t} f\big(\tau, x(\tau), u(\tau)\big)\, d\tau \tag{4.6}$$

holds for every $t \in [t_0, \theta]$.

According to (4.5) there exists $u_*(\cdot) \in B_{p_*}(\mu_0)$ such that

$$\big\|u(\cdot) - u_*(\cdot)\big\|_1 \leqslant \frac{\varepsilon}{b_0}. \tag{4.7}$$

Let

$$x_*(t) = x_0 + \int_{t_0}^{t} f\big(\tau, x_*(\tau), u_*(\tau)\big)\, d\tau \tag{4.8}$$

where $t \in [t_0, \theta]$. Then $x_*(\cdot) \in X_{p_*}(t_0, X_0, \mu_0)$. It follows from (4.B), (4.6)–(4.8) that

$$\big\|x(t) - x_*(t)\big\| \leqslant L_3 \frac{\varepsilon}{b_0} + \int_{t_0}^{t} \big(L_1 + L_2\big\|u_*(\tau)\big\|\big)\big\|x(\tau) - x_*(\tau)\big\|\, d\tau \tag{4.9}$$

for all $t \in [t_0, \theta]$. From (4.4), (4.9) and Gronwall's inequality we obtain that the inequality

$$\|x(t) - x_*(t)\| \leqslant \frac{\varepsilon}{b_0} L_3 \exp\left[\int_{t_0}^{\theta} \left(L_1 + L_2 \|u_*(\tau)\|\right) d\tau\right]$$
$$= \frac{\varepsilon}{b_0} L_3 \exp(a_0) = \varepsilon \tag{4.10}$$

holds for every $t \in [t_0, \theta]$. Since $p \in (p_* - \xi(\varepsilon), p_* + \xi(\varepsilon))$ and $x(\cdot) \in X_p(t_0, X_0, \mu_0)$ are arbitrarily chosen, we get from (4.10) that

$$X_p(t_0, X_0, \mu_0) \subset X_{p_*}(t_0, X_0, \mu_0) + \varepsilon B_c \tag{4.11}$$

holds for every $p \in (p_* - \xi(\varepsilon), p_* + \xi(\varepsilon))$ where B_c is unique ball of the space $C([t_0, \theta]; R^n)$.

Analogously, it is possible to prove that

$$X_{p_*}(t_0, X_0, \mu_0) \subset X_p(t_0, X_0, \mu_0) + \varepsilon B_c \tag{4.12}$$

for every $p \in (p_* - \xi(\varepsilon), p_* + \xi(\varepsilon))$.

Thus, inclusions (4.11) and (4.12) imply the validity of the theorem. □

References

[1] J.P. Aubin, H. Frankowska, Set Valued Analysis, Birkhäuser, Boston, 1990.
[2] D. Burago, Yu. Burago, S. Ivanov, A Course in Metric Geometry, Grad. Stud. Math., vol. 33, American Mathematical Society, Providence, RI, 2001.
[3] R. Conti, Problemi di controllo e di controllo ottimale, UTET, Torino, 1974.
[4] K. Deimling, Multivalued Differential Equations, Walter de Gruyter, New York, 1992.
[5] A.F. Filippov, Differential Equations with Discontinuous Right-Hand Sides, Kluwer Acad. Publ., Dordrecht, 1988.
[6] Sh. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis, vol. 1, Theory, Kluwer Acad. Publ., Dordrecht, 1997.
[7] F. Gozzi, P. Loreti, Regularity of the minimum time function and minimum energy problems: The linear case, SIAM J. Control Optim. 37 (1999) 1195–1221.
[8] Kh.G. Guseinov, A.N. Moiseev, V.N. Ushakov, On the approximation of reachable domains of control systems, J. Appl. Math. Mech. 62 (2) (1998) 169–175.
[9] Kh.G. Guseinov, O. Ozer, S.A. Duzce, On differential inclusions with prescribed attainable sets, J. Math. Anal. Appl. 277 (2) (2003) 701–713.
[10] Kh.G. Guseinov, O. Ozer, E. Akyar, V.N. Ushakov, The approximation of reachable sets of control systems with integral constraint on controls, NoDEA Nonlinear Differential Equations Appl. (2007), in press.
[11] H.W. Lou, On the attainable sets of control systems with p-integrable controls, J. Optim. Theory Appl. 123 (1) (2004) 123–147.
[12] M. Motta, C. Sartori, Minimum time with bounded energy, minimum energy with bounded time, SIAM J. Control Optim. 42 (2003) 789–809.
[13] A.I. Panasyuk, Equations of attainable set dynamics, Part 1: Integral Funnel equations, J. Optim. Theory Appl. 64 (2) (1990) 349–366.