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On differential inclusions with prescribed attainable sets ☆

Kh.G. Guseinov,* O. Ozer, and S.A. Duzce

Anadolu University, Faculty of Science, Department of Mathematics, Eskisehir 26470, Turkey

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Abstract

In this article, the inverse problem of the differential inclusion theory is studied. For a given $\varepsilon > 0$ and a continuous set valued map $t \to W(t)$, $t \in [t_0, \theta]$, where $W(t) \subset R^n$ is compact and convex for every $t \in [t_0, \theta]$, it is required to define differential inclusion so that the Hausdorff distance between the attainable set of the differential inclusion at the time moment t with initial set $(t_0, W(t_0))$ and $W(t)$ would be less than ε for every $t \in [t_0, \theta]$.

Keywords: Differential inclusion; Set valued map; Attainable set

1. Introduction

In many papers (see, e.g., [1,2,6–10,15,17–19,21], and references therein) the various properties of integral funnels and attainable sets of the given differential inclusion (DI) were studied. In this article, the inverse problem of the DI theory is studied.

We denote by symbol $h(E, D)$ the Hausdorff distance between the sets $E, D \subset R^n$. It is defined as

$$h(E, D) = \max\left\{\sup_{x \in E} d(x, D),\ \sup_{y \in D} d(y, E)\right\}$$

where $d(x, D) = \inf_{y \in D} \|x - y\|$, $\|\cdot\|$ means the Euclidean norm.

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* Corresponding author.

E-mail addresses: kguseynov@anadolu.edu.tr (Kh.G. Guseinov), oozer@anadolu.edu.tr (O. Ozer), saduzce@anadolu.edu.tr (S.A. Duzce).

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Consider the DI

$$\dot{x} \in F(t, x) \tag{1.1}$$

where $x \in R^n$ is the phase state vector, $t \in [t_0, \theta]$ is the time.

The absolutely continuous function $x(\cdot) : [t_0, \theta] \to R^n$ satisfying the inclusion $\dot{x}(t) \in F(t, x(t))$ for almost all $t \in [t_0, \theta]$ is said to be a solution of the DI (1.1) (see, e.g., [4]). By symbol $X(t_*, X_*)$ we denote the set of all solutions of the DI (1.1) satisfying the condition $x(t_*) \in X_*$, where $X_* \subset R^n$, $t_* \in [t_0, \theta]$. We set

$$X(t; t_*, X_*) = \{x(t) \in R^n\colon\ x(\cdot) \in X(t_*, X_*)\},$$
$$H(t_*, X_*) = \{(t, x) \in [t_*, \theta] \times R^n\colon\ x(t) \in X(t; t_*, X_*)\}.$$

Let $\varepsilon > 0$ be a given number and $t \to W(t)$, $t \in [t_0, \theta]$, be a given set valued map. We will study the following problem. It is required to define DI (1.1) so that the inequality

$$h\big(X\big(t; t_0, W(t_0)\big), W(t)\big) \leqslant \varepsilon$$

would be fulfilled for every $t \in [t_0, \theta]$.

Such problems may appear in mathematics modelling where it is required to specify the dynamic of the system through measurement of the phase state of the system.

The inverse problem was investigated in the works [3,5,13]. In this article, the desired DI is defined so that the right hand side of the DI satisfies the conditions which guarantee existence and extendability of the solutions. Note that the notions strong and weak invariant sets with respect to DI play an important role in construction of such DI (see, e.g., [1,7,8, 11,16]).

To solve the problem, we take a small enough partition of the time interval $[t_0, \theta]$ and on each of the subintervals of the partition we build a piecewise affine interpolation of the set valued map $t \to W(t)$, $t \in [t_0, \theta]$. Using such approximated affine tube, a linear DI is defined which solves the problem.

Let $t \to W(t)$, $t \in [t_0, \theta]$, be a set valued map,

$$W = gr\ W(\cdot) = \{(t, x) \in [t_0, \theta] \times R^n\colon\ x \in W(t)\}$$

be a closed set. For $(t, x) \in [t_0, \theta] \times R^n$ we denote

$$D_*^+ W(t, x) = \left\{d \in R^n\colon\ \exists x(\tau) \in W(\tau),\ \tau > t,\ \lim_{\tau \to t+0} \frac{x(\tau) - x}{\tau - t} = d\right\},$$
$$D_*^- W(t, x) = \left\{d \in R^n\colon\ \exists x(\tau) \in W(\tau),\ \tau < t,\ \lim_{\tau \to t-0} \frac{x(\tau) - x}{\tau - t} = d\right\}.$$

The sets $D_*^+ W(t, x)$ and $D_*^- W(t, x)$ are respectively said to be lower right hand side and lower left hand side derivative sets of the set valued map $t \to W(t)$ calculated at the point (t, x). These sets are closed and near connected by the lower Bouligand contingent cone to the set W at (t, x) (see, e.g., [1,2,8,14]). Note that the sets $D_*^+ W(t, x)$ and $D_*^- W(t, x)$ can be empty for some $(t, x) \in W$. It is not difficult to show that, if $W \subset [t_0, \theta] \times R^n$ is convex and closed set, then $D_*^+ W(t, x) \neq \emptyset$ for every $(t, x) \in W$, $t \in [t_0, \theta)$, and $D_*^- W(t, x) \neq \emptyset$ for every $(t, x) \in W$, $t \in (t_0, \theta]$.

2. Special case

In this section, we consider the special case where the set valued map $t \to W(t)$, $t \in [t_0, \theta]$, is affine tube.

Let $t_*, t^* \in R$, $V_*, V^* \subset R^n$, $\alpha > 0$. From now on, it will be assumed that $V_*, V^* \subset R^n$ are convex and compact sets. Define the set valued map $t \to V_\alpha(t)$, $t \in [t_* - \alpha, t^* + \alpha]$, where

$$V_\alpha(t) = \left(1 - \frac{t - t_* + \alpha}{t^* - t_* + 2\alpha}\right) V_* + \frac{t - t_* + \alpha}{t^* - t_* + 2\alpha} V^*. \tag{2.1}$$

We set

$$V_\alpha = \left\{(t, x) \in [t_* + \alpha, t^* + \alpha] \times R^n \colon x \in V_\alpha(t)\right\}. \tag{2.2}$$

It is obvious that $V_\alpha \subset [t_* - \alpha, t^* + \alpha] \times R^n$ and $V_\alpha(t) \subset R^n$ are convex, compact sets for every $t \in [t_* - \alpha, t^* + \alpha]$ and the set valued map $t \to V_\alpha(t)$, $t \in [t_* - \alpha, t^* + \alpha]$, is continuous.

We will study the following problem. It is required to define a DI so that the equality $X(t; t_*, V_\alpha(t_*)) = V_\alpha(t)$ holds for every $t \in [t_*, t^*]$. Here $X(t; t_*, V_\alpha(t_*))$ is the attainable set of the desired DI at the time moment t with initial set $(t_*, V_\alpha(t_*))$.

For $(t, x) \in [t_*, t^*] \times R^n$, $\nu \in (0, \alpha)$ we set

$$F_\nu(t, x) = \frac{1}{\nu}\left[V_\alpha(t + \nu) - x\right], \tag{2.3}$$

$$\Phi_\nu(t, x) = -\frac{1}{\nu}\left[V_\alpha(t - \nu) - x\right]. \tag{2.4}$$

Consider some properties of the set valued maps $(t, x) \to F_\nu(t, x)$ and $(t, x) \to \Phi_\nu(t, x)$ which are defined on the space $[t_*, t^*] \times R^n$. Denote

$$a = \max\left\{\|x\| \colon (t, x) \in V_\alpha\right\}. \tag{2.5}$$

Proposition 2.1. *The sets $F_\nu(t, x) \subset R^n$ and $\Phi_\nu(t, x) \subset R^n$ are convex and compact for any $(t, x) \in [t_*, t^*] \times R^n$, the set valued maps $(t, x) \to F_\nu(t, x)$ and $(t, x) \to \Phi_\nu(t, x)$ are continuous with respect to (t, x) in $[t_*, t^*] \times R^n$ and are Lipschitz with respect to x with constant $\frac{1}{\nu}$. The inequalities*

$$\max\left\{\|f\| \colon f \in F_\nu(t, x)\right\} \leqslant \frac{1}{\nu}\left(a + \|x\|\right)$$

$$\max\left\{\|\varphi\| \colon \varphi \in \Phi_\nu(t, x)\right\} \leqslant \frac{1}{\nu}\left(a + \|x\|\right)$$

are true for any $(t, x) \in [t_*, t^*] \times R^n$, where $a \geqslant 0$ is defined by relation (2.5).

Proposition 2.2. *If $(t, x) \in V_\alpha$, $t \in [t_*, t^*]$ and $f \in F_\nu(t, x)$ then $x + \delta f \in V_\alpha(t + \delta)$ for any $\delta \in (0, \nu)$. If $(t, x) \in V_\alpha$, $t \in [t_*, t^*]$ and $\varphi \in \Phi_\nu(t, x)$ then $x - \delta\varphi \in V_\alpha(t - \delta)$ for any $\delta \in (0, \nu)$.*

Here the set valued map $t \to V_\alpha(t)$, $t \in [t_* - \alpha, t_* + \alpha]$ is defined by relation (2.1).

Proposition 2.3. *For every* $(t,x) \in V_\alpha,\ t \in [t_*, t^*]$

$$F_\nu(t,x) \cap \Phi_\nu(t,x) \neq \emptyset.$$

Proof. Let $(t_0, x_0) \in V_\alpha,\ t_0 \in [t_*, t^*]$. Then from (2.1) and (2.2) we have that there exist $v_* \in V_*$ and $v^* \in V^*$ such that

$$x_0 = \left(1 - \frac{t_0 - t_* + \alpha}{t^* - t_* + 2\alpha}\right) v_* + \frac{t_0 - t_* + \alpha}{t^* - t_* + 2\alpha} v^*.$$

Denote $d_0 = \frac{v^* - v_*}{t^* - t_*}$. It is not difficult to prove that $d_0 \in F_\nu(t,x)$, $d_0 \in \Phi_\nu(t,x)$ and consequently $d_0 \in F_\nu(t,x) \cap \Phi_\nu(t,x)$. □

Proposition 2.4. *The inclusions*

$$F_\nu(t,x) \subset D_*^+ V_\alpha(t,x) \quad \textit{for any } (t,x) \in V_\alpha,\ t \in [t_*, t^*],$$
$$\Phi_\nu(t,x) \subset D_*^- V_\alpha(t,x) \quad \textit{for any } (t,x) \in V_\alpha,\ t \in [t_*, t^*],$$

hold where the set $V_\alpha \subset [t_* - \alpha, t^* + \alpha] \times R^n$ *is defined by* (2.2).

It follows from Proposition 2.4 that $D_*^+ V_\alpha(t,x) \neq \emptyset$, $D_*^- V_\alpha(t,x) \neq \emptyset$ for any $(t,x) \in V_\alpha,\ t \in [t_*, t^*]$.

Consider DI

$$\dot{x} \in F_\nu(t,x) \tag{2.6}$$

where $(t,x) \in [t_*, t^*] \times R^n$ and $F_\nu(t,x)$ is defined by (2.3). We denote by symbol $X_\nu(t_0, X_0)$ the set of all solutions of the DI (2.6) satisfying the condition $x(t_0) \in X_0$ where $X_0 \subset R^n$, $t_0 \in [t_*, t^*]$. Further we set

$$X_\nu(t; t_0, X_0) = \big\{x(t) \in R^n\colon\ x(\cdot) \in X_\nu(t_0, X_0)\big\},$$
$$H_\nu(t_0, X_0) = \big\{(t,x) \in [t_*, t^*] \times R^n\colon\ x \in X_\nu(t; t_0, X_0)\big\}.$$

Let

$$V = \big\{(t,x) \in [t_*, t^*] \times R^n\colon\ x \in V_\alpha(t)\big\},$$
$$V(t) = \big\{x \in R^n\colon\ (t,x) \in V\big\}$$

where $t \in [t_*, t^*]$. It is obvious that $V(t) = V_\alpha(t)$ for every $t \in [t_*, t^*]$.

Formulate the theorem establishing that integral funnel of the DI (2.6) with initial set $(t_*, V(t_*))$ coincides with the set $V \subset [t_*, t^*] \times R^n$.

Theorem 2.1. *For every* $\nu \in (0, \alpha)$ *the equalities*

$$H_\nu\big(t_*, V(t_*)\big) = V, \qquad X_\nu\big(t; t_*, V(t_*)\big) = V(t), \qquad X_\nu\big(t; t_*, V_\alpha(t_*)\big) = V_\alpha(t)$$

are fulfilled for every $t \in [t_*, t^*]$.

Proof. According to the Proposition 2.4,

$$F_\nu(t,x) \subset D_*^+ V(t,x)$$

for any $(t,x) \in V$, $t \in [t_*, t^*)$ and according to the Proposition 2.1, set valued map $(t,x) \to F_\nu(t,x)$, $((t,x) \in [t_*, t^*] \times R^n)$, is continuous with respect to (t,x) and is Lipschitz with respect to x. Then, it follows from Theorem 2 of [1, p. 202] (see, also [8,11,20]), that the set $V \subset [t_*, t^*] \times R^n$ is positively strongly invariant with respect to the DI (2.6). It means that for any $(t_0, x_0) \in V$, $x(\cdot) \in X_\nu(t_0, x_0)$ the inclusion $x(t) \in V(t)$ is verified for any $t \in [t_0, t^*]$. It follows from here that

$$H_\nu\big(t_*, V(t_*)\big) \subset V, \qquad X_\nu\big(t; t_*, V(t_*)\big) \subset V(t) \tag{2.7}$$

for every $t \in [t_*, t^*]$.

According to the Proposition 2.4,

$$\Phi_\nu(t,x) \subset D_*^- V(t,x)$$

for any $(t,x) \in V$, $t \in [t_*, t^*]$, where $\Phi_\nu(t,x)$ is defined by (2.4). Then it follows from Proposition 2.3 that

$$F_\nu(t,x) \cap D_*^- V(t,x) \neq \emptyset.$$

Then, it follows from here and Theorem 1 of [1, p. 191] (see, also [12,16,20]), that the set V is negatively weakly invariant with respect to the DI (2.6). This means that for every fixed $(t_0, x_0) \in V$ there exists $x(\cdot) \in X_\nu(t_0, x_0)$ such that the inclusion $x(t) \in V(t)$ is verified for every $t \in [t_*, t_0]$. One can have from here that

$$V \subset H_\nu\big(t_*, V(t_*)\big), \qquad V(t) \subset X_\nu\big(t; t_*, V(t_*)\big) \tag{2.8}$$

for every $t \in [t_*, t^*]$. Since $V(t) = V_\alpha(t)$ for every $t \in [t_*, t^*]$, the validity of the theorem follows from (2.7) and (2.8). □

3. Main result

The construction which have built in Section 2 for the set valued map $t \to V_\alpha(t)$, $t \in [t_*, t^*]$, defined by relation (2.1) will be extended to continuous, convex and compact valued map $t \to W(t)$, $t \in [t_0, \theta]$.

It will be assumed that the set valued map $t \to W(t)$, $t \in [t_0, \theta]$, satisfies the following condition.

(A) The set valued map $t \to W(t)$ is continuous on $[t_0, \theta]$ and $W(t) \subset R^n$ is convex, compact for every $t \in [t_0, \theta]$.

Since the set valued map $t \to W(t)$ is continuous on $[t_0, \theta]$, it is uniformly continuous on $[t_0, \theta]$. Then, for every $\sigma > 0$, there exists $\delta = \delta(\sigma) > 0$ such that for any t, $\tau \in [t_0, \theta]$, where $|t - \tau| \leqslant \delta$, the inequality

$$h\big(W(t), W(\tau)\big) \leqslant \sigma \tag{3.1}$$

holds. Choose an arbitrary uniform partition $\Gamma = \{t_0 < t_1 < \cdots < t_m = \theta\}$ of the segment $[t_0, \theta]$ such that $t_{i+1} - t_i = \Delta$ for any $i = 0, 1, \ldots, m-1$, and

$$\Delta \leqslant \delta. \tag{3.2}$$

According to the (3.1) and (3.2)

$$h\big(W(t_i), W(t_{i+1})\big) \leqslant \sigma \tag{3.3}$$

for any $i = 0, 1, \ldots, m-1$. We set $W_i = W(t_i)$, $i = 0, 1, \ldots, m-1$, and define the set valued map $t \to W^0(t)$, $t \in [t_0, \theta]$, where

$$W^0(t) = \left(1 - \frac{t - t_i}{t_{i+1} - t_i}\right) W_i + \frac{t - t_i}{t_{i+1} - t_i} W_{i+1} \tag{3.4}$$

as $t \in [t_i, t_{i+1})$, $i = 0, 1, \ldots, m-1$. It is obvious that $W^0(t_i) = W(t_i) = W_i$ for every $i = 0, 1, \ldots, m$.

Proposition 3.1. *For any $t \in [t_0, \theta]$, the inequality*

$$h\big(W(t), W^0(t)\big) \leqslant \sigma \tag{3.5}$$

holds.

Now we set

$$\begin{aligned}
&W^0 = \big\{(t, x) \in [t_0, \theta] \times R^n \colon x \in W^0(t)\big\},\\
&W_i^0 = \big\{(t, x) \in [t_i, t_{i+1}] \times R^n \colon x \in W^0(t)\big\}, \quad i = 0, 1, \ldots, m-1,\\
&W_i^0(t) = \big\{x \in R^n \colon (t, x) \in W_i^0\big\}, \quad t \in [t_i, t_{i+1}].
\end{aligned} \tag{3.6}$$

Choose $\alpha \in (0, \frac{\Delta}{4})$ where $\Delta = t_{i+1} - t_i$, $i = 0, 1, \ldots, m-1$, and define set valued maps $t \to V_i^*(t)$, $[t_i - \alpha, t_{i+1} + \alpha]$, $i = 0, 1, \ldots, m-1$, setting

$$V_i^*(t) = \left(1 - \frac{t - t_i + \alpha}{t_{i+1} - t_i + 2\alpha}\right) W_i + \frac{t - t_i + \alpha}{t_{i+1} - t_i + 2\alpha} W_{i+1}. \tag{3.7}$$

Proposition 3.2. *Let the set valued maps $t \to W_i^0(t)$, $t \in [t_i, t_{i+1}]$ and $t \to V_i^*(t)$, $t \in [t_i - \alpha, t_{i+1} + \alpha]$, $i = 0, 1, \ldots, m-1$, be defined by the relations* (3.6) *and* (3.7)*, respectively. Then,*

$$h\big(W_i^0(t), V_i^*(t)\big) \leqslant \frac{\alpha}{\Delta} M$$

for every $t \in [t_i, t_{i+1}]$ where $i = 0, 1, \ldots, m-1$, $M = \max\{M_0, M_1, \ldots, M_{m-1}\}$, $M_i = \max\{\|x - y\| \colon x \in W_i,\ y \in W_{i+1}\}$.

Proof. Let $i = 0, 1, \ldots, m-1$ be fixed. Choose arbitrary $t_* \in [t_i, t_{i+1}]$ and $x_* \in W_i^0(t_*)$. Then, it follows from the relation (3.4) that there exist $x_i \in W_i$, $x_{i+1} \in W_{i+1}$ such that

$$x_* = \left(1 - \frac{t_* - t_i}{t_{i+1} - t_i}\right) x_i + \frac{t_* - t_i}{t_{i+1} - t_i} x_{i+1}. \tag{3.8}$$

Choose the point

$$x^* = \left(1 - \frac{t_* - t_i + \alpha}{t_{i+1} - t_i + 2\alpha}\right) x_i + \frac{t_* - t_i + \alpha}{t_{i+1} - t_i + 2a} x_{i+1}. \tag{3.9}$$

It is obvious that $x^* \in V_i^*(t_*)$. It can be obtained from (3.8) and (3.9) that

$$\|x_* - x^*\| \leqslant \alpha \frac{|t_{i+1} + t_i - 2t_*|}{(t_{i+1} - t_i)(t_{i+1} - t_i + 2\alpha)} \|x_{i+1} - x_i\|$$
$$\leqslant 2\alpha \frac{|(t_{i+1} + t_i)/2 - t_*|}{(t_{i+1} - t_i)(t_{i+1} - t_i + 2\alpha)} M_i \leqslant \alpha \frac{\Delta}{\Delta(\Delta + 2\alpha)} M \leqslant \frac{\alpha}{\Delta} M.$$

This implies

$$W_i^0(t_*) \subset V_i^*(t_*) + \frac{\alpha}{\Delta} MB. \tag{3.10}$$

where $B = \{x \in R^n\colon \|x\| \leqslant 1\}$. Analogously, it is possible to show that

$$V_i^*(t_*) \subset W_i^0(t_*) + \frac{\alpha}{\Delta} MB. \tag{3.11}$$

From (3.10) and (3.11) we have the validity of the proposition. □

Let $i = 0, 1, \ldots, m-1$ be fixed. We set

$$V_i^* = \bigl\{(t, x) \in [t_i - \alpha, t_{i+1} + \alpha] \times R^n\colon x \in V_i^*(t)\bigr\},$$
$$V_i^0 = \bigl\{(t, x) \in [t_i, t_{i+1}] \times R^n\colon x \in V_i^*(t)\bigr\},$$
$$V_i^0(t) = \bigl\{x \in R^n\colon (t, x) \in V_i^0\bigr\}, \quad t \in [t_i, t_{i+1}], \tag{3.12}$$

where $V_i^*(t)$ is defined by (3.7). It is obvious that the sets $V_i^0 \subset [t_i, t_{i+1}] \times R^n$ and $V_i^* \subset [t_i - \alpha, t_{i+1} + \alpha] \times R^n$ are convex, compact and $V_i^0(t) = V_i^*(t)$ for every $t \in [t_i, t_{i+1}]$.

Let $\nu \in (0, \alpha)$. Define set valued map $(t, x) \to P_\nu^i(t, x)$, $(t, x) \in [t_i, t_{i+1}] \times R^n$, $i = 0, 1, \ldots, m-1$, setting

$$P_\nu^i(t, x) = \frac{1}{\nu}\bigl[V_i^*(t + \nu) - x\bigr]. \tag{3.13}$$

Now, we consider DI

$$\dot{x} \in P_\nu^i(t, x) \tag{3.14}$$

where $(t, x) \in [t_i, t_{i+1}] \times R^n$, $i = 0, 1, \ldots, m-1$, is fixed. Analogously to Proposition 2.1, it is not difficult to prove that the set $P_\nu^i(t, x)$ is convex and compact for every $(t, x) \in [t_i, t_{i+1}] \times R^n$, the set valued maps $(t, x) \to P_\nu^i(t, x)$ $(i = 0, 1, \ldots, m-1)$ are continuous on the space $[t_i, t_{i+1}] \times R^n$ and Lipschitz with respect to x with constant $\frac{1}{\nu}$, the inequality

$$\max\bigl\{\|f\|\colon f \in P_\nu^i(t, x)\bigr\} \leqslant \frac{1}{\nu}\bigl(b_i + \|x\|\bigr)$$

is verified for every $(t, x) \in [t_i, t_{i+1}] \times R^n$, where $b_i = \max\{\|x\|\colon (t, x) \in V_i^*\}$.

By the symbol $Y_\nu^i(t_*, X_*)$, we denote the set of all solutions of the DI (3.14) satisfying the condition $x(t_*) \in X_*$ where $X_* \subset R^n$, $t_* \in [t_i, t_{i+1}]$. We set

$$Y_\nu^i(t; t_*, X_*) = \bigl\{x(t) \in R^n\colon x(\cdot) \in Y_\nu^i(t_*, X_*)\bigr\},$$
$$R_\nu^i(t_*, X_*) = \bigl\{(t, x) \in [t_*, t_{i+1}] \times R^n : x \in Y_\nu^i(t; t_*, X_*)\bigr\}.$$

Proposition 3.3. *For any fixed* $\nu \in (0, \alpha)$

$$R_\nu^i\big(t_i, V_i^0(t_i)\big) = V_i^0, \qquad Y_\nu^i\big(t; t_i, V_i^0(t_i)\big) = V_i^0(t)$$

for every $t \in [t_i, t_{i+1}]$.

The proof of the proposition follows from Theorem 2.1.

We formulate the proposition which characterises Hausdorff distance between the attainable sets of the DI (3.14).

Proposition 3.4. *Let* $X_1, X_2 \subset R^n$ *be compact sets. Then*

$$h\big(Y_\nu^i(t; t_i, X_1), Y_\nu^i(t; t_i, X_2)\big) \leqslant \alpha(X_1, X_2) \cdot \exp\left[-\frac{1}{\nu}(t - t_i)\right]$$

for every $t \in [t_i, t_{i+1}]$.

Consider DI

$$\dot{x} \in P_\nu(t, x), \quad (t, x) \in [t_0, \theta] \times R^n, \tag{3.15}$$

where $P_\nu(t, x) = P_\nu^i(t, x)$ as $(t, x) \in [t_i, t_{i+1}) \times R^n$, $i = 0, 1, \ldots, m-1$, the set $P_\nu^i(t, x)$ is defined by (3.13).

It is obvious that the set $P_\nu(t, x)$ is convex and compact for every $(t, x) \in [t_0, \theta] \times R^n$, the set valued map $(t, x) \to P_\nu(t, x)$ is continuous at the points $(t, x) \in [t_0, \theta] \times R^n$ where $t \neq t_i$, $i = 0, 1, \ldots, m$. Further, the set valued map $(t, x) \to P_\nu(t, x)$ is right continuous as $t = t_i$, $i = 0, 1, \ldots, m-1$, left continuous as $t = t_m = \theta$, is Lipschitz with respect to x with constant $\frac{1}{\nu}$ and

$$\max\big\{\|f\|\colon f \in P_\nu(t, x)\big\} \leqslant \frac{1}{\nu}\big(b + \|x\|\big),$$

where $b = \max\{b_0, b_1, \ldots, b_{m-1}\}$, $b_i = \max\{\|x\|\colon (t, x) \in V_i^*\}$.

By symbol $Y_\nu(t_*, X_*)$, we denote the set of all solutions of the DI (3.15) satisfying the condition $x(t_*) \in X_*$ where $X_* \subset R^n$, $t_* \in [t_0, \theta]$. We set

$$Y_\nu(t; t_*, X_*) = \big\{x(t) \in R^n\colon x(\cdot) \in Y_\nu(t_*, X_*)\big\}.$$

Theorem 3.1. *For every fixed* $\nu \in (0, \alpha)$ *the inequality*

$$h\big(Y_\nu\big(t; t_0, W(t_0)\big), W^0(t)\big) \leqslant 3\frac{\alpha}{\Delta}M$$

holds for any $t \in [t_0, \theta]$, *where* $M \geqslant 0$ *is defined in Proposition* 3.2.

Proof. Let $t_* \in [t_0, \theta]$. If $t_* = t_0$, then the validity of the theorem is obvious. If $t_* \in (t_0, \theta]$, then there exists $k = 0, 1, \ldots, m-1$ such that $t_* \in (t_k, t_{k+1}]$. Consider the sets $W(t_0)$ and $V_0^0(t_0)$. Since $V_0^0(t) = V_0^*(t)$ as $t \in [t_0, t_1]$, $W(t_0) = W_0 = W_0^0(t_0)$, then we obtain from Proposition 3.2 that

$$h\big(V_0^0(t_0), W_0\big) \leqslant \frac{\alpha}{\Delta}M. \tag{3.16}$$

Consequently, it follows from relation (3.16) and Proposition 3.4 that

$$h\big(Y_\nu(t;t_0,W_0),\,Y_\nu\big(t;t_0,V_0^0(t_0)\big)\big)\leqslant\frac{\alpha}{\Delta}M\exp\left[-\frac{1}{\nu}(t-t_0)\right]\tag{3.17}$$

for any $t\in[t_0,t_1]$.

According to Proposition 3.3, we have

$$Y_\nu\big(t;t_0,V_0^0(t_0)\big)=V_0^0(t)\tag{3.18}$$

for any $t\in[t_0,t_1]$.

Since $V_0^0(t)=V_0^*(t)$ as $t\in[t_0,t_1]$, where the set valued map $t\to V_0^*(t),\ t\in[t_0,t_1]$, is defined by (3.7), the set valued map $t\to V_0^0(t),\ t\in[t_0,t_1]$, is defined by (3.12), it follows from (3.17), (3.18), and Proposition 3.2 that

$$\begin{aligned}&h\big(Y_\nu(t;t_0,W_0),\,W_0^0(t)\big)\\&\quad\leqslant h\big(Y_\nu(t;t_0,W_0),\,V_0^0(t)\big)+h\big(V_0^0(t),\,W_0^0(t)\big)\\&\quad\leqslant\frac{\alpha}{\Delta}M+\frac{\alpha}{\Delta}M\exp\left[-\frac{1}{\nu}(t-t_0)\right]\leqslant\frac{\alpha}{\Delta}M\left(1+\exp\left[-\frac{1}{\nu}(t-t_0)\right]\right)\end{aligned}$$

for any $t\in[t_0,t_1]$.

Since $W_0^0(t)=W^0(t)$ for every $t\in[t_0,t_1]$, where the set valued map $t\to W_0^0(t)$, $t\in[t_0,t_1]$, is defined by (3.6), we obtain

$$h\big(Y_\nu(t;t_0,W_0),\,W^0(t)\big)\leqslant\frac{\alpha}{\Delta}M\left(1+\exp\left[-\frac{1}{\nu}\Delta\right]\right)\tag{3.19}$$

for any $t\in[t_0,t_1]$.

Now, we consider the segment $[t_1,t_2]$. Denote $Y_\nu^1=Y_\nu(t_1;t_0,W_0)$. Since $W_1^0(t_1)=W(t_1)=W_1$, $\Delta=t_1-t_0$, then from (3.19) we have

$$h\big(Y_\nu^1,\,W_1\big)\leqslant\frac{\alpha}{\Delta}M\left(1+\exp\left[-\frac{\Delta}{\nu}\right]\right).\tag{3.20}$$

Since $V_1^0(t_1)=V_1^*(t_1),\ W_1^0(t_1)=W(t_1)=W_1$, then from Proposition 3.2 we obtain that

$$h\big(V_1^0(t_1),\,W_1\big)\leqslant\frac{\alpha}{\Delta}M.\tag{3.21}$$

It follows from (3.20) and (3.21)

$$h\big(Y_\nu^1,\,V_1^0(t_1)\big)\leqslant\frac{\alpha}{\Delta}M\left(2+\exp\left[-\frac{\Delta}{\nu}\right]\right).\tag{3.22}$$

Then, according to the (3.22), Propositions 3.3 and 3.4, we have

$$\begin{aligned}h\big(Y_\nu\big(t;t_1,Y_\nu^1\big),\,V_1^0(t)\big)&=h\big(Y_\nu\big(t;t_1,Y_\nu^1\big),\,Y_\nu\big(t;t_1,V_1^0(t_1)\big)\big)\\&\leqslant\frac{\alpha}{\Delta}M\left(2+\exp\left[-\frac{\Delta}{\nu}\right]\right)\exp\left[-\frac{1}{\nu}(t-t_1)\right]\end{aligned}\tag{3.23}$$

for every $t\in[t_1,t_2]$. Since $\Delta=t_2-t_1,\ V_1^0(t)=V_1^*(t),\ Y_\nu(t;t_1,Y_\nu^1)=Y_\nu(t;t_0,W_0)$ for any $t\in[t_1,t_2]$, then it follows from (3.23), Propositions 3.2 and 3.3 that

$$\begin{aligned} h\big(Y_\nu(t;t_0,W_0),W_1^0(t)\big) &\leqslant h\big(Y_\nu\big(t;t_1,Y_\nu^1\big),V_1^0(t)\big)+h\big(V_1^0(t),W_1^0(t)\big)\\ &\leqslant \frac{\alpha}{\Delta}M+\frac{\alpha}{\Delta}M\left(2+\exp\left[-\frac{\Delta}{\nu}\right]\right)\exp\left[-\frac{1}{\nu}(t-t_1)\right]\\ &\leqslant \frac{\alpha}{\Delta}M\left(1+2\exp\left[-\frac{\Delta}{\nu}\right]+\exp\left[-\frac{1}{\nu}(t-t_0)\right]\right) \end{aligned}$$

for any $t\in[t_1,t_2]$.

Since $W_1^0(t)=W^0(t)$ for any $t\in[t_1,t_2]$, then we obtain that

$$h\big(Y_\nu(t;t_0,W_0),W^0(t)\big)\leqslant \frac{\alpha}{\Delta}M\left(1+2\exp\left[-\frac{\Delta}{\nu}\right]+\exp\left[-\frac{2}{\nu}\Delta\right]\right)$$

for every $t\in[t_1,t_2]$.

It is possible analogously to show that

$$h\big(Y_\nu(t;t_0,W_0),W^0(t)\big)\leqslant \frac{\alpha}{\Delta}M\left(1+2\exp\left[-\frac{\Delta}{\nu}\right]+2\exp\left[-\frac{2\Delta}{\nu}\right]+\exp\left[-\frac{3}{\nu}\Delta\right]\right)$$

for any $t\in[t_2,t_3]$.

Further assuming that the inequality

$$h\big(Y_\nu(t;t_0,W_0),W^0(t)\big)\leqslant \frac{\alpha}{\Delta}M\left(1+2\sum_{j=1}^{i-1}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{i}{\nu}\Delta\right]\right) \tag{3.24}$$

holds for any $t\in[t_{i-1},t_i]$ $(i=2,3,\dots,k)$, let us prove that the inequality

$$h\big(Y_\nu(t;t_0,W_0),W^0(t)\big)\leqslant \frac{\alpha}{\Delta}M\left(1+2\sum_{j=1}^{i}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{i+1}{\nu}\Delta\right]\right) \tag{3.25}$$

is fulfilled for every $t\in[t_i,t_{i+1}]$.

Let $Y_\nu^i=Y_\nu(t_i;t_0,W_0)$. Since $W_i^0(t_i)=W(t_i)=W_i$ then we obtain from (3.24) that

$$h\big(Y_\nu^i,W_i\big)\leqslant \frac{\alpha}{\Delta}M\left(1+2\sum_{j=1}^{i-1}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{i}{\nu}\Delta\right]\right). \tag{3.26}$$

Since $V_i^0(t_i)=V_i^*(t_i)$, $W_i^0(t_i)=W(t_i)=W_i$, then from Proposition 3.2 we have that

$$h\big(V_i^0(t_i),W_i\big)\leqslant \frac{\alpha}{\Delta}M. \tag{3.27}$$

It follows from (3.26) and (3.27)

$$h\big(Y_\nu^i,V_i^0(t_i)\big)\leqslant \frac{\alpha}{\Delta}M\left(2+2\sum_{j=1}^{i-1}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{i}{\nu}\Delta\right]\right).$$

Since $\Delta=t_{i+1}-t_i$, then according to Propositions 3.3 and 3.4 we have from here that

$$
\begin{aligned}
h\big(Y_\nu\big(t;t_i,Y_\nu^i\big),V_i^0(t)\big) &= h\big(Y_\nu\big(t;t_i,Y_\nu^i\big),Y_\nu\big(t;t_i,V_i^0(t_i)\big)\big)\\
&\leqslant \frac{\alpha}{\Delta}M\left(2+2\sum_{j=1}^{i-1}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{i}{\nu}\Delta\right]\right)\exp\left[-\frac{1}{\nu}(t-t_i)\right]\\
&\leqslant \frac{\alpha}{\Delta}M\left(2\exp\left[-\frac{\Delta}{\nu}\right]+2\sum_{j=1}^{i-1}\exp\left[-\frac{j+1}{\nu}\Delta\right]+\exp\left[-\frac{i+1}{\nu}\Delta\right]\right)\\
&= \frac{\alpha}{\Delta}M\left(2\sum_{j=1}^{i}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{i+1}{\nu}\Delta\right]\right) \qquad (3.28)
\end{aligned}
$$

for every $t \in [t_i, t_{i+1}]$. Since $V_i^0(t) = V_i^*(t)$, $Y_\nu(t;t_i,Y_\nu^i) = Y_\nu(t;t_0,W_0)$, $W_i^0(t) = W^0(t)$ for any $t \in [t_i, t_{i+1}]$, then it follows from (3.28), Proposition 3.2 and Proposition 3.3 that

$$
\begin{aligned}
h\big(Y_\nu(t;t_0,W_0),W^0(t)\big) &= h\big(Y_\nu(t;t_0,W_0),W_i^0(t)\big)\\
&\leqslant h\big(Y_\nu\big(t;t_i,Y_\nu^i\big),V_i^0(t)\big)+h\big(V_i^0(t),W_i^0(t)\big)\\
&\leqslant \frac{\alpha}{\Delta}M+\frac{\alpha}{\Delta}M\left(2\sum_{j=1}^{i}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{i+1}{\nu}\Delta\right]\right)\\
&= \frac{\alpha}{\Delta}M\left(1+2\sum_{j=1}^{i}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{i+1}{\nu}\Delta\right]\right)
\end{aligned}
$$

for any $t \in [t_i, t_{i+1}]$.

So validity of the inequality (3.25) is proved. Thus we obtain that for $t_* \in (t_k, t_{k+1}]$ the inequality

$$
\begin{aligned}
&h\big(Y_\nu(t_*;t_0,W_0),W^0(t_*)\big)\\
&\quad\leqslant \frac{\alpha}{\Delta}M\left(1+2\sum_{j=1}^{k}\exp\left[-\frac{j}{\nu}\Delta\right]+\exp\left[-\frac{k+1}{\nu}\Delta\right]\right) \qquad (3.29)
\end{aligned}
$$

holds. Since $t_* \in [t_0, \theta]$ is arbitrary chosen moment of time, it follows from (3.19) and (3.29) that

$$
h\big(Y_\nu(t;t_0,W_0),W^0(t)\big) \leqslant \frac{\alpha}{\Delta}M\left(1+2\sum_{j=1}^{m}\exp\left[-\frac{j}{\nu}\Delta\right]\right) \qquad (3.30)
$$

for any $t \in [t_0, \theta]$. It is not difficult to verify that

$$
\sum_{j=1}^{m}\exp\left[-\frac{j}{\nu}\Delta\right] \leqslant \int_0^\infty \exp\left[-\frac{\Delta}{\nu}\tau\right]d\tau \leqslant 1.
$$

Then, from here and from (3.30) it follows validity of the theorem. □

Theorem 3.2. *Let $\varepsilon > 0$ be a given number. Then there exist $\Delta > 0$, $\alpha > 0$ such that for any $\nu \in (0, \alpha)$ the inequality*

$$h\big(Y_\nu\big(t; t_0, W(t_0)\big), W(t)\big) \leqslant \varepsilon$$

holds for every $t \in [t_0, \theta]$.

Proof. Let $\varepsilon > 0$ be a given number. Then, for $\sigma = \frac{\varepsilon}{2}$, it can be defined $\delta = \delta(\varepsilon/2) > 0$ such that for the partition $\Gamma = \{t_0 < t_1 < \cdots < t_m = \theta\}$ of the segment $[t_0, \theta]$, where $\Delta \leqslant \delta$ and $\Delta = t_{i+1} - t_i$ for any $i = 0, 1, \ldots, m-1$, the relation (3.5) is verified.

We choose $\alpha \in (0, \frac{\Delta}{4})$ so that the inequality

$$\alpha < \frac{\varepsilon}{6M}\Delta$$

holds where $M > 0$ is defined in Theorem 3.1. Now, let $\nu \in (0, \alpha)$ be a fixed number. Then, it follows from (3.5) and Theorem 3.1 that

$$\begin{aligned} h\big(Y_\nu(t; t_0, W_0), W(t)\big) &\leqslant h\big(Y_\nu(t; t_0, W_0), W^0(t)\big) + h\big(W^0(t), W(t)\big) \\ &\leqslant \sigma + 3\frac{\alpha}{\Delta}M \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for any $t \in [t_0, \theta]$. □

From Theorem 3.1 and 3.2, it follows that if $\alpha > 0$ has the smallest order relative to Δ, which is greater than one, then the Hausdorff distance between the reachable sets $Y_\nu(t; t_0, W_0)$ and $W(t)$, $t \in [t_0, \theta]$ tends to zero as $\Delta \to 0$ for every $t \in [t_0, \theta]$.

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