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The circle as an attractor of an iterated function system on the plane

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Abstract

We give an iterated function system (IFS) on the plane with the circle as attractor. In doing this, we also give a sufficient condition for radially contracting functions on the plane (or on $\mathbb{R}^n$) to be a contraction. A counterexample shows that radial contractiveness is not enough to be a contraction.

Keywords: Iterated function system; Self-similarity

As is well known, there is not a generally agreed-upon definition of fractals. Self-similarity and fractional dimension are considered as characteristic features of fractals. Self-similarity is often understood as being the attractor of an iterated function system (IFS). But simple spaces such as an interval or a square are trivially attractors of IFS's, so that some dimension condition is considered to be desirable. We do not want to delve into these matters and refer to [1–3]. As an enrichment of the example-repertory, we want to show explicitly that the unit closed disc (the circle) can be realized as the attractor of an IFS on the plane.

We use the polar coordinates (ρ, θ) on the plane. We give the following IFS on the plane:

Let $k : [0, 2\pi] \to \mathbb{R}$ be the periodic function with period $\pi/2$ given by

$$k(\theta) = -\frac{3}{8}(\sin\theta + \cos\theta) + \sqrt{1 - \frac{9}{64}(\sin\theta - \cos\theta)^2} \quad \text{for } 0 \leqslant \theta \leqslant \pi/2.$$

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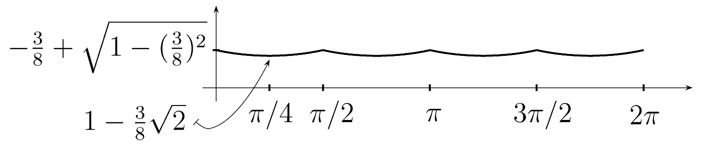

Fig. 1. Graph of $k(\theta)$.

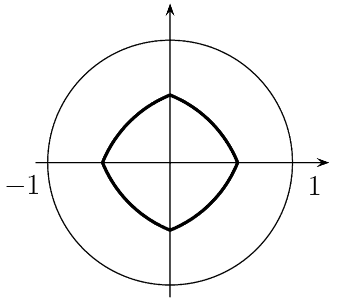

Fig. 2. Polar graph of $\rho = k(\theta)$.

(For the graph of k see Fig. 1 and for the polar graph of $\rho = k(\theta)$ see Fig. 2.)

Now we define an IFS on $\mathbb{R}^2$ consisting of 9 functions:

$$f_0(\rho e^{i\theta}) = \rho k(\theta) e^{i\theta},$$

$$f_n(\rho e^{i\theta}) = \rho k(\theta) e^{i\theta} + \frac{3\sqrt{2}}{8} e^{\frac{\pi}{4} ni}, \quad n = 1, 3, 5, 7,$$

$$f_n(\rho e^{i\theta}) = \rho k(\theta) e^{i\theta} e^{\frac{\pi}{4} i} + \frac{3\sqrt{2}}{8} e^{\frac{\pi}{4} ni}, \quad n = 2, 4, 6, 8.$$

Theorem 1. *The functions* $\{f_n\}_{n=0,1,\ldots,8}$ *are contractions on the plane and their attractor is the circle* $\mathbb{D}^2 = \{\rho e^{i\theta} \mid 0 \leqslant \rho \leqslant 1\}$.

Proof. That $\mathbb{D}^2 = \bigcup_{n=0}^{8} f_n(\mathbb{D}^2)$ is shown in Fig. 3 and results from the definition of f_0 which is obtained by translating the center of the circle to the point $\frac{3\sqrt{2}}{8} e^{\frac{5\pi}{4} i}$ as shown in Fig. 4 (we omit the straightforward, plane-geometric details). To show that the functions $\{f_n\}_{n=0,1,\ldots,8}$ are contractions on the plane, it is enough to see that f_0 is contraction, because the others are obtained by translations and rotations of it.

Now, f_0 is a function on the plane, whose restrictions to rays through the origin are obviously linear contractions. But interestingly, this property is not enough for a function on the plane to be a contraction of the plane! (For a counterexample see below.) Some additional constraints are necessary and we now interrupt this proof and give first the following Theorem 2 and come back to the proof of Theorem 1 as an application of Theorem 2. □

In the following, it will be more convenient to consider a 2π-periodic function on $\mathbb{R}$ as a function on the circle $\kappa : S^1 \to \mathbb{R}$ by identification $\theta \leftrightarrow e^{i\theta}$.

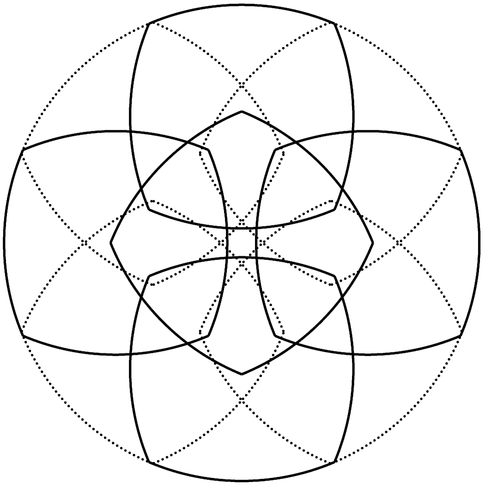

Fig. 3. Attractor of the IFS $\{f_0, f_1, \ldots, f_8\}$.

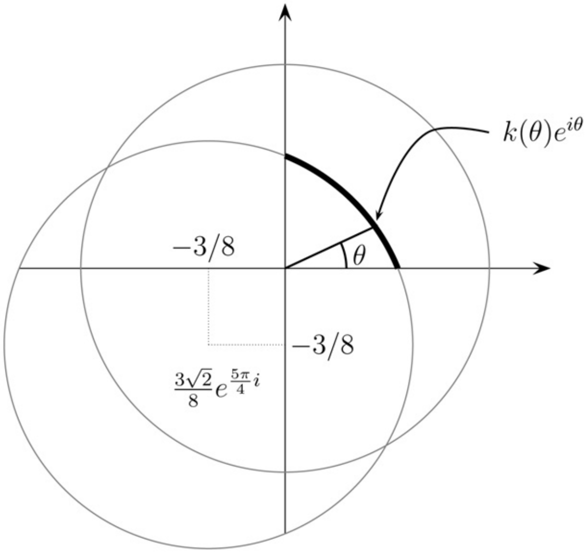

Fig. 4. Construction of f_0.

Theorem 2. *Let $\kappa : S^1 \to \mathbb{R}^+$ be a continuous function with*

$$r = \max_{e^{i\theta} \in S^1} \kappa\left(e^{i\theta}\right).$$

Assume, there exists s with $r < s < 1$ such that for $e^{i\theta_1}, e^{i\theta_2} \in S^1$ and $d(\theta_1, \theta_2) < \frac{\pi}{2}$ (where $d(\theta_1, \theta_2)$ denotes the shorter arc-length on S^1) the inequality

$$\left|\kappa\left(e^{i\theta_1}\right) - \kappa\left(e^{i\theta_2}\right)\right| < \frac{(s^2 - r^2)}{s}\, d(\theta_1, \theta_2)$$

holds. Then, the function $f:\mathbb{C}\to\mathbb{C}$ defined by $f(\rho e^{i\theta})=\rho\kappa(e^{i\theta})e^{i\theta}$ is a contraction on the plane with s being a contractivity coefficient.

Proof. Let $\rho_1 e^{i\theta_1}$ and $\rho_2 e^{i\theta_2}$ be two points on the plane. We have to show that the distance contracts with factor s under the mapping f:

$$\left\| f\left(\rho_1 e^{i\theta_1}\right) - f\left(\rho_2 e^{i\theta_2}\right)\right\| \leqslant s\left\|\rho_1 e^{i\theta_1} - \rho_2 e^{i\theta_2}\right\|.$$

Consider first the case where $d(\theta_1,\theta_2) < \frac{\pi}{2}$:

After squaring and applying the cosine theorem, it will be enough to show the following:

$$\rho_1^2\kappa^2\left(e^{i\theta_1}\right) + \rho_2^2\kappa^2\left(e^{i\theta_2}\right) - 2\rho_1\rho_2\kappa\left(e^{i\theta_1}\right)\kappa\left(e^{i\theta_2}\right)\cos(\theta_1-\theta_2)$$
$$\leqslant s^2\left[\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2)\right],$$

i.e., denoting $\kappa(e^{i\theta_1})$ and $\kappa(e^{i\theta_2})$ by κ_1 and κ_2,

$$\rho_1^2\left(s^2-\kappa_1^2\right)+\rho_2^2\left(s^2-\kappa_2^2\right)-2\rho_1\rho_2\left(s^2-\kappa_1\kappa_2\right)\cos(\theta_1-\theta_2)\geqslant 0.$$

For the expression on the left-hand side to be non-negative, it is enough to show that the discriminant is non-positive:

$$\left(s^2-\kappa_1\kappa_2\right)^2\cos^2(\theta_1-\theta_2)-\left(s^2-\kappa_1^2\right)\left(s^2-\kappa_2^2\right)\leqslant 0.$$

Using the identity

$$\left(s^2-\kappa_1\kappa_2\right)^2 = s^2(\kappa_1-\kappa_2)^2+\left(s^2-\kappa_1^2\right)\left(s^2-\kappa_2^2\right),$$

it will suffice to show

$$\frac{(s^2-\kappa_1\kappa_2)^2}{(s^2-\kappa_1^2)(s^2-\kappa_2^2)} = 1+\frac{s^2(\kappa_1-\kappa_2)^2}{(s^2-\kappa_1^2)(s^2-\kappa_2^2)} \leqslant \frac{1}{\cos^2(\theta_1-\theta_2)} = 1+\tan^2(\theta_1-\theta_2),$$

i.e.,

$$\frac{s^2(\kappa_1-\kappa_2)^2}{(s^2-\kappa_1^2)(s^2-\kappa_2^2)}\leqslant\tan^2(\theta_1-\theta_2).$$

Now, by assumptions of the theorem the bound

$$(\kappa_1-\kappa_2)^2\leqslant\frac{(s^2-r^2)^2}{s^2}d(\theta_1,\theta_2)^2$$

holds. Hence

$$\frac{s^2(\kappa_1-\kappa_2)^2}{(s^2-\kappa_1^2)(s^2-\kappa_2^2)}\leqslant\frac{(s^2-r^2)^2 d(\theta_1,\theta_2)^2}{(s^2-\kappa_1^2)(s^2-\kappa_2^2)}$$
$$\leqslant d(\theta_1,\theta_2)^2\leqslant\tan^2(\theta_1-\theta_2)\quad\left(\text{by } d(\theta_1,\theta_2)\leqslant\frac{\pi}{2}\right).$$

To settle the case $d(\theta_1,\theta_2)\geqslant\pi/2$, again apply the cosine theorem:

$$\left\|f\left(\rho_1e^{i\theta_1}\right)-f\left(\rho_2e^{i\theta_2}\right)\right\|^2=\rho_1^2\kappa_1^2+\rho_2^2\kappa_2^2-2\rho_1\rho_2\kappa_1\kappa_2\cos(\theta_1-\theta_2)$$
$$\leqslant s^2\left(\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2)\right)=s^2\left\|\rho_1e^{i\theta_1}-\rho_2e^{i\theta_2}\right\|^2$$

because $\kappa_1^2\leqslant s^2$, $\kappa_2^2\leqslant s^2$ and $\kappa_1\kappa_2\leqslant s^2$ and $\cos(\theta_1-\theta_2)\leqslant 0$. □

Corollary 3. *Let $\kappa : S^1 \to \mathbb{R}^+$ be a continuous function with*

$$r = \max_{e^{i\theta} \in S^1} \kappa\big(e^{i\theta}\big) < 1.$$

Assume the inequality $|\kappa(e^{i\theta_1}) - \kappa(e^{i\theta_2})| < \frac{1-r}{2} d(\theta_1, \theta_2)$ holds for $d(\theta_1, \theta_2) < \frac{\pi}{2}$. Then the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(\rho e^{i\theta}) = \rho\kappa(e^{i\theta})e^{i\theta}$ is a contraction on the plane with s being a contractivity coefficient.

Proof. Let us apply Theorem 2 for $s = \frac{1+r}{2}$. It will suffice to show $\frac{1-r}{2} < \frac{s^2-r^2}{s}$ which is immediate. □

Remark 4. Theorem 2 can be generalized to $\mathbb{R}^n$ in obvious terms with the use of a function $\kappa : S^{n-1} \to [0, 1)$ measuring the contractions on the rays.

End of proof of Theorem 1. Using Corollary 3, it is enough to show

$$\big|k(\theta_1) - k(\theta_2)\big| < \frac{1-r}{2} d(\theta_1, \theta_2).$$

By periodicity it suffices to consider the case $d(\theta_1, \theta_2) = |\theta_1 - \theta_2|$. By applying the mean-value theorem to the piece-wise differentiable function k, one gets

$$\big|k(\theta_1) - k(\theta_2)\big| \leqslant M|\theta_1 - \theta_2| \quad \text{where } M = \max_{0\leqslant\theta\leqslant 2\pi} \big|k'(\theta)\big|.$$

So it will be enough to show $M \leqslant \frac{1-r}{2}$. By easy computation, $M = \frac{3}{8}(1 - \frac{3}{\sqrt{55}})$ and $r = -\frac{3}{8} + \frac{\sqrt{55}}{8}$ and $M \leqslant \frac{1-r}{2}$ holds. □

Remark 5. If one considers the $\pi/2$-periodic function

$$k(\theta) = -a(\sin\theta + \cos\theta) + \sqrt{1 - a^2(\sin\theta - \cos\theta)^2}$$

within the meaningful range $0 < a < \frac{1}{\sqrt{2}}$, then it satisfies the hypothesis of Corollary 3 for example in the interval $\frac{3}{8} \leqslant a < \frac{\sqrt{2}}{2}$:

In this case, maximum of $|k'(\theta)|$ is $|k'(\pi/2)| = a(1 - \frac{a}{\sqrt{1-a^2}})$, maximum of $k(\theta)$ is $k(0) = -a + \sqrt{1 - a^2}$ and thus it is enough to show $a(1 - \frac{a}{\sqrt{1-a^2}}) < \frac{1+a-\sqrt{1-a^2}}{2}$ or $1 - 3a^2 < (1 - a)\sqrt{1 - a^2}$ which holds certainly for $\frac{3}{8} \leqslant a < \frac{1}{\sqrt{2}}$ as easily can be seen. (For the range $\frac{1}{\sqrt{3}} \leqslant a < \frac{1}{\sqrt{2}}$ it is obvious and for $\frac{3}{8} \leqslant a < \frac{1}{\sqrt{3}}$ the inequality amounts to $5a^3 - a^2 - 3a + 1 < 0$.)

We now give the promised counterexample, that a function on the plane, which contracts the rays emanating from origin linearly, does not need to be a contraction on the plane:

Example 6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(\rho e^{i\theta}) = \rho k(\theta)e^{i\theta}$ with

$$k(\theta) = \frac{7}{10} + \frac{2}{10}\cos 4\theta. \quad \big(\text{See Fig. 5 for } \rho = k(\theta).\big)$$

f contracts a ray with angle θ with $k(\theta) \leqslant \frac{9}{10}$.

But f is not a contraction on $\mathbb{R}^2$: For $z_1 = 5$ and $z_2 = 2\sqrt{2}e^{\frac{\pi}{4}i}$ we have $d(f(z_1), f(z_2)) = \frac{\sqrt{53}}{2} > \sqrt{13} = d(z_1, z_2)$.

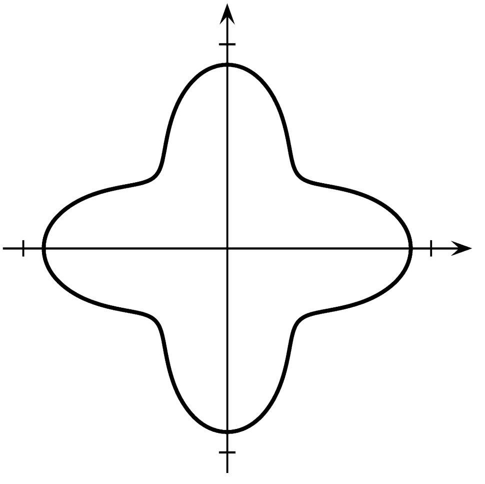

Fig. 5. Polar graph of $\rho = k(\theta)$ of the counterexample.

As a final remark, we would like to note that even the circle itself (without the inside) can be realized as the attractor of an iterated function system of contractions with the same means as above, but this time the contractions being not one-to-one.

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