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Spaces of skew-symmetric matrices satisfying $A^3 = \lambda A$

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ABSTRACT

We calculate the maximum dimension of a space of $m \times m$ real skew-symmetric matrices of corank 1 satisfying $A^3 = \lambda_A A$ for some real $\lambda_A > 0$.

1. Introduction

Each real skew-symmetric matrix is orthogonally similar to a matrix

$$\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{bmatrix} \oplus 0_k \text{ with nonzero } \lambda_1, \ldots, \lambda_m \in \mathbb{R}. \tag{1}$$

We prove the following theorem.

Theorem 1. *The maximum dimension of a space V of $(2n+1) \times (2n+1)$ real skew-symmetric matrices, in which every $A \in V$ is orthogonally similar to a matrix of the form*

$$\begin{bmatrix} 0 & \lambda_A \\ -\lambda_A & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \lambda_A \\ -\lambda_A & 0 \end{bmatrix} \oplus 0_1 \text{ with nonzero } \lambda_A \in \mathbb{R} \tag{2}$$

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is equal to $\rho(2n) - 1$ *if n is even and* $\rho(2n+2) - 1$ *if n is odd.*

Here $\rho(m)$ is the Radon–Hurwitz number of a natural number m and is defined as follows: if m is presented in the form $m = (2a+1)2^{4b+c}$ with $c = \{0, 1, 2, 3\}$ and non-negative integer a, b, then $\rho(m) = 2^c + 8b$.

The Radon–Hurwitz numbers appear in differential topology, coding theory, theoretical physics, and linear algebra. In particular, the following results are close to Theorem 1:

- $\rho(m)$ counts the maximum size of a linear subspace of the real $m \times m$ matrices, for which each nonzero matrix is a product of an orthogonal matrix and a scalar matrix; see [5].
- Let $\mathbb{F}$ be $\mathbb{R}$, $\mathbb{C}$ or the skew-field of real quaternions $\mathbb{H}$. Let $\mathbb{F}(m)$ be the maximum number of matrices $A_1, A_2, \ldots \in \mathbb{F}^{m\times m}$ such that each linear combination $\alpha_1 A_1 + \alpha_2 A_2 + \cdots$ with real coefficients is nonsingular except when all α_i are zero. Then

  $$\mathbb{R}(m) = \rho(m), \quad \mathbb{C}(m) = 2b + 2, \quad \mathbb{H}(m) = \rho(m/2) + 4;$$

  see [1,2].
- The maximum numbers of Hermitian, skew-Hermitian, symmetric, or skew-symmetric matrices $A_1, A_2, \ldots \in \mathbb{F}^{m\times m}$ such that each linear combination $\alpha_1 A_1 + \alpha_2 A_2 + \cdots$ with real coefficients is nonsingular except when all α_i are zero, are equal to

  $$\mathbb{F}(m/2), \quad \mathbb{F}(m) - 1, \quad \rho(m/2) + d_{\mathbb{F}}, \quad \rho(2^{d_{\mathbb{F}}-1}m) - d_{\mathbb{F}},$$

  respectively, in which $d_{\mathbb{F}} = \dim_{\mathbb{R}} \mathbb{F}$; see [3].

This work was inspired by Bilge, Dereli, and Koçak's article [4] about spaces of real skew-symmetric matrices that are orthogonally similar to matrices of the form

$$\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}, \quad 0 \neq \lambda \in \mathbb{R}. \tag{3}$$

2. Proof of the theorem

We denote the rank, trace, image, kernel and orthogonal complement of the kernel of A by r_A, $\mathrm{Tr}\,A$, $\mathrm{Im}\,A$, $\mathrm{Ker}\,A$, and W_A, respectively. We also use the notation $A \perp B$ for the orthogonality of A and B, that is $\mathrm{Tr}(AB^T) = 0$.

Let S_{2n} be the set of $2n \times 2n$ real skew-symmetric matrices for which each matrix is orthogonally similar to a matrix of the form (3) and S_{2n+1} be the set of $(2n+1) \times (2n+1)$ real skew-symmetric matrices for which each matrix is orthogonally similar to a matrix of the form (2). It can be easily shown that $A \in S_{2n}$ if and only if

$$A^2 + \lambda_A^2 I_{2n} = 0 \tag{4}$$

and $A \in S_{2n+1}$ if and only if

$$A^3 + \lambda_A^2 A = 0 \quad \text{and} \quad r_A = 2n \tag{5}$$

where $\lambda_A^2 = -\frac{\mathrm{Tr}(A^2)}{r_A}$.

Lemma 1. *If* $A \in S_{2n+1}$, *then* $W_A = \mathrm{Ker}\left(A^2 + \lambda_A^2 I_{2n+1}\right)$.

Proof. Let $A \in S_{2n+1}$. Then $\operatorname{Im} A \perp \operatorname{Ker} A$ since for each $x \in \mathbb{R}^{2n+1}$ and $y \in \operatorname{Ker} A$ we have $\langle Ax, y\rangle = \langle x, A^T y\rangle = \langle x, -Ay\rangle = \langle x, 0\rangle = 0$.

Let $x \in W_A$. By (5), $y := \left(A^2 + \lambda_A^2 I_{2n+1}\right) x \in \operatorname{Ker} A$, and so $\langle y, y\rangle = \langle A^2x, y\rangle + \langle \lambda_A^2 x, y\rangle = 0$. Hence $y = 0$ and $x \in \operatorname{Ker}(A^2 + \lambda_A^2 I_{2n+1})$.

Conversely, let $x \in \operatorname{Ker}(A^2 + \lambda_A^2 I_{2n+1})$. Then for every $y \in \operatorname{Ker} A$ we have $\lambda_A^2\langle x, y\rangle = \langle A^2x, y\rangle + \lambda_A^2\langle x, y\rangle = \langle (A^2 + \lambda_A^2 I_{2n+1})x, y\rangle = \langle 0, y\rangle = 0$, and so $x \in W_A$. □

Lemma 2. *Let $\mathcal{L} \subset S_{2n+1}$ be a subspace containing the matrices A and B. If $A \perp B$, then*

$$A^2B + ABA + BA^2 + \lambda_A^2 B = 0, \tag{6}$$

$$AB^2 + BAB + B^2A + \lambda_B^2 A = 0. \tag{7}$$

Proof. Since A and B lie in the same subspace, by (5),

$$(A + kB)^3 + \lambda_{A+kB}^2 \,(A + kB) = 0 \tag{8}$$

for all $k \in \mathbb{R}$ where $\lambda_{A+kB}^2 = -\frac{1}{2n}\operatorname{Tr}((A + kB)^2)$. By orthogonality of A and B, $\operatorname{Tr}(AB) = \operatorname{Tr}(BA) = 0$. Then

$$\lambda_{A+kB}^2 = -\frac{1}{2n}\operatorname{Tr}\left((A + kB)^2\right) = -\frac{1}{2n}\operatorname{Tr}\left(A^2\right) - \frac{k^2}{2n}\operatorname{Tr}\left(B^2\right) = \lambda_A^2 + k^2\lambda_B^2. \tag{9}$$

Substituting (9) in (8), we obtain

$$\begin{aligned} 0 = {} & A^3 + \lambda_A^2 A + \left(A^2B + ABA + BA^2 + \lambda_A^2 B\right) k \\ & + \left(AB^2 + BAB + B^2A + \lambda_B^2 A\right) k^2 + \left(B^3 + \lambda_B^2 B\right) k^3 \end{aligned}$$

for all k which gives the Eqs. (6) and (7) since $A^3 + \lambda_A^2 A = 0$ and $B^3 + \lambda_B^2 B = 0$. □

Write

$$F := \begin{bmatrix} I_{2n+1} \\ 0 \cdots 0 \end{bmatrix}_{(2n+2)\times(2n+1)}.$$

Note that for any $(2n+2) \times (2n+2)$ skew-symmetric real matrix B, F^TBF is the $(2n+1) \times (2n+1)$ real skew-symmetric matrix formed by removing from B its last column and row. Also note that for any $(2n + 1) \times (2n + 1)$ real skew-symmetric matrix A, $FAF^T = A \oplus 0_1$.

Lemma 3. *If $B \in S_{2n+2}$, then $F^TBF \in S_{2n+1}$.*

Proof. Let $B \in S_{2n+2}$ and $\overline{B} := FF^TBFF^T$. Note that $\overline{B} = F^TBF \oplus 0_1$. More clearly, $\overline{B}$ is the $(2n + 2) \times (2n + 2)$ real skew-symmetric matrix formed by changing the last column and row of B with the zero column and row:

$$B = \left[\, b_{ij} \,\right]_{1 \leqslant i,j \leqslant 2n+2} \Rightarrow \overline{B} = \left[\, b_{ij} \,\right]_{1 \leqslant i,j \leqslant 2n+1} \oplus 0_1.$$

Since $B \in S_{2n+2}$, B is of the form $\alpha \cdot B_0$ for some real number α where B_0 is an orthogonal matrix by (4). The columns (and the rows) of B are mutually perpendicular since B_0 is orthogonal, so the last column and row of the matrices $B\overline{B}$ and $\overline{B}B$ are zero, which can be seen easily by a simple calculation. On the other hand, other corresponding elements of the matrices $B\overline{B}$ and $\overline{B}B$ are obviously equal by the definition of the skew-symmetric matrix $\overline{B}$. Then $B\overline{B} = \overline{B}B = \overline{B}^2$. Hence

$$(F^TBF)^3 = F^TBFF^TBFF^TBF = F^TB\overline{B}BF = F^T\overline{B}B^2F.$$

It is clear that $F^T\overline{B}F = F^TBF$. Then

$$(F^TBF)^3 = F^T\overline{B}B^2F = -\lambda_B^2F^T\overline{B}F = -\lambda_B^2F^TBF,$$

since $B^2 = -\lambda_B^2 I_{2n+2}$ by (4).

On the other hand, it is known that $r_{M_1M_2} + r_{M_2M_3} \leqslant r_{M_1M_2M_3} + r_{M_2}$ for any multiplying-allowed matrices M_1, M_2, M_3 [6, Example 2]. Substituting $M_1 = F^T, M_2 = B$ and $M_3 = F$, we have $2n + 1 + 2n + 1 \leqslant r_{F^TBF} + 2n + 2$, that is $2n \leqslant r_{F^TBF}$. Then $r_{F^TBF} = 2n$ since F^TBF is an $(2n + 1) \times (2n + 1)$ real skew-symmetric matrix, and so $F^TBF \in S_{2n+1}$ by (5). □

Lemma 4. *If $A \in S_{2n+1}$, then there exists $B \in S_{2n+2}$ such that $A = F^TBF$.*
(In fact there exist only two matrices B^1, B^2 such that $A = F^TB^jF, j = 1, 2$.)

Proof. Let $A \in S_{2n+1}$. There exists a unique orthogonal matrix Q such that Q^TAQ is of the form

$$Q^TAQ = M \oplus 0_1 \quad \text{where } M = \begin{bmatrix} 0 & \lambda_A \\ -\lambda_A & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \lambda_A \\ -\lambda_A & 0 \end{bmatrix}.$$

Obviously, there exist only two skew-symmetric matrices $\widetilde{B}_1$ and $\widetilde{B}_2$ in S_{2n+2} such that $Q^TAQ = F^T\widetilde{B}_jF$ $(j = 1, 2)$ which are of the form

$$\widetilde{B}_1 = M \oplus \begin{bmatrix} 0 & \lambda_A \\ -\lambda_A & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{B}_2 = M \oplus \begin{bmatrix} 0 & -\lambda_A \\ \lambda_A & 0 \end{bmatrix}.$$

Consider the orthogonal matrices $P_1 := Q \oplus 1_1$ and $P_2 := Q \oplus (-1)_1$. An easy calculation shows that $P_1\widetilde{B}_1P_1^T = P_2\widetilde{B}_2P_2^T =: B_1 \in S_{2n+2}$ and $P_1\widetilde{B}_2P_1^T = P_2\widetilde{B}_1P_2^T =: B_2 \in S_{2n+2}$. By the definitions, $QF^TP_1^T = F^T$ and $P_1FQ^T = F$. Then

$$A = QF^T\widetilde{B}_1FQ^T = QF^TP_1^TB_1P_1FQ^T = F^TB_1F$$

and similarly $A = F^TB_2F$, which completes the proof.

We also note that there does not exist any matrix P satisfying $QF^TP^T = F^T$ or $PFQ^T = F$ except for $P = P_1$ or $P = P_2$ which means there does not exist any matrix B satisfying $A = F^TBF$ except for $B = B_1$ or $B = B_2$. □

Lemma 5. *Let $\{A, B\}$ be a basis for a 2-dimensional subspace in S_{2n+1} such that $A \perp B$.*

(i) *If* $\operatorname{Ker} A = \operatorname{Ker} B$, *then* $AB + BA = 0$.
(ii) *If* $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$, *then* $\operatorname{Ker} A \perp \operatorname{Ker} B$.

Proof. *(i)* Let $W := W_A = W_B$ and let $x \in W$. By (7), $AB^2x + BABx + B^2Ax + \lambda_B^2Ax = 0$, and so $BABx + B^2Ax = 0$ by Lemma 1. Then $BABx = \lambda_B^2Ax$ since $Ax \in W$ (note that $Mx \perp x$ for any skew-symmetric matrix M). Multiplying each side of the last equality by B, $-\lambda_B^2ABx = \lambda_B^2BAx$ by Lemma 1 since $ABx \in W$, and so $(AB+BA)x = 0$. In the case $x \in \mathbb{R}^{2n+1}$, $(AB+BA)x = 0$ since x can be expressed as $x = x_1 + x_2$ where $x_1 \in \operatorname{Ker} A = \operatorname{Ker} B$ and $x_2 \in W_A = W_B$.

(ii) Let $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$ and let $\lambda_A^2 = \lambda_B^2 = 1$ for simplicity. Suppose that $\operatorname{Ker} A$ does not orthogonal to $\operatorname{Ker} B$.

Let $0 \neq w \in \operatorname{Ker} B$. By (5), there exists a nonzero $z \in \operatorname{Ker} A$ such that $B^2z+z = w$ and $A^2w+w = bz$ for some nonzero $b \in \mathbb{R}$ since $\operatorname{Ker} A$ and $\operatorname{Ker} B$ are 1-dimensional subspaces (In the case $\operatorname{Ker} A \perp \operatorname{Ker} B$, it cannot be found such a vector $z \in \operatorname{Ker} A$. Note that if $z \in W_B$, then $w = B^2z + z = -z + z = 0$ by Lemma 1). By multiplying each side of $B^2z + z = w$ by A and $A^2w + w = bz$ by B, respectively, we get

$$AB^2z = Aw \quad \text{and} \quad BA^2w = bBz. \tag{10}$$

By (7) and (6), $AB^2z + BABz = 0$ and $BA^2w + ABAw = 0$ since $z \in \operatorname{Ker} A$ and $w \in \operatorname{Ker} B$. Then

$$BABz = -Aw \quad \text{and} \quad ABAw = -bBz \tag{11}$$

by (10). From the fact that $\langle Mu, v\rangle = \langle u, M^Tv\rangle$ and $M^TM = MM^T = -M^2$ for any skew-symmetric matrix M, we obtain

$$\begin{aligned}\langle Aw, Bz\rangle &= -\langle BABz, Bz\rangle = -\langle ABz, B^TBz\rangle = \langle ABz, B^2z\rangle \\ &= -\langle Bz, AB^2z\rangle = -\langle Bz, Aw\rangle,\end{aligned}$$

which means $\langle Aw, Bz\rangle = 0$. Then $w \perp ABz$ and $z \perp BAw$, i.e. $ABz \in W_B$ and $BAw \in W_A$. By Lemma 1, we obtain $B^2ABz = -ABz$ and $A^2BAw = -BAw$. Then $BAw = ABz$ and $BAw = bABz$ by (11), and so $b = 1$.

On the other hand, $B^2z = (w - z) \perp w$ and $A^2w = (bz - w) \perp z$ by Lemma 1. Using these orthogonalities, we get $\langle w, w\rangle = \langle w, z\rangle = b\langle z, z\rangle$ and by $b = 1$, $\langle w, w\rangle = \langle w, z\rangle = \langle z, z\rangle$. Then

$$0 < \langle w - z, w - z\rangle = \langle w, w\rangle - 2\langle w, z\rangle + \langle z, z\rangle = 0,$$

which is a contradiction. □

Lemma 6. *Let $\{A_1, A_2, \ldots, A_k\}$ be an orthogonal basis for a subspace in S_{2n+1}. Either the matrices $A_1, A_2, \ldots, A_k$ have common kernel or the kernels of the matrices $A_1, A_2, \ldots, A_k$ intersect pairwise in the zero vector.*

Proof. Let $\{A, B, C\}$ be an orthogonal basis for a subspace in S_{2n+1} and suppose that $\operatorname{Ker} A = \operatorname{Ker} C$ and $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$.

Let $0 \neq x \in \operatorname{Ker} A$, $y := Bx$ and $z := Ay = ABx$. Note that $y \neq 0$ since $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$. By Lemma 1, we have $Az = A^2y = -\lambda_A^2y$ since $y = Bx \in W_A$, and so $z \neq 0$ since $y \neq 0$. By Lemma 5, $x \in W_B$. Thus by Lemma 1, $B^2x = -\lambda_B^2x$, and so $By = -\lambda_B^2x$. By (6) and (7),

$$\begin{aligned}0 &= (AB^2 + BAB + B^2A + \lambda_B^2A)x = A(-\lambda_B^2x) + BABx = Bz \\ 0 &= (B^2C + BCB + CB^2 + \lambda_B^2C)x = BCBx + C(-\lambda_B^2x) = BCBx = BCy,\end{aligned}$$

which means $z, Cy \in \operatorname{Ker} B$. $Cy = \theta z$ for some nonzero $\theta \in \mathbb{R}$ since $z \in \operatorname{Ker} B$ and $\dim \operatorname{Ker} B = 1$. A and C anticommute on $W_A = W_C$ by Lemma 5. Then

$$0 = ACy + CAy = A(\theta z) + Cz = -\theta\lambda_A^2y + Cz,$$

and so $Cz = \theta\lambda_A^2y$. Then $C^2y = \theta Cz = \theta^2\lambda_A^2y$ which is a contradiction since $\theta^2 > 0$, $C^2y = -\lambda_C^2y$ and $y \neq 0$. □

Let $\mathcal{L}$ be a subspace of S_{2n+1} and $\{A_1, A_2, \ldots, A_k\}$ be an orthogonal basis of $\mathcal{L}$. If $A_1, A_2, \ldots, A_k$ have common kernel, then we call $\mathcal{L}$ is of the first type and $\{A_1, A_2, \ldots, A_k\}$ is a first type basis. Similarly, if the kernels of $A_1, A_2, \ldots, A_k$ intersect pairwise in the zero vector, then we call $\mathcal{L}$ is of the second type and $\{A_1, A_2, \ldots, A_k\}$ is a second type basis. We note that any subspace of S_{2n+1} must be either of the first type or of the second type.

Remark 1. Let $\{A, B\}$ be a basis for a 2-dimensional subspace of the second type in S_{2n+1} such that $\lambda_A^2 = \lambda_B^2 = 1$ and let $\{f_1, f_2, \ldots, f_{2n+1}\}$ be an orthonormal basis of $\mathbb{R}^{2n+1}$ such that $Af_1 = 0$ and $Bf_2 = 0$ (it is possible since Ker $A \perp$ Ker B). Let $x \in$ Ker A and $y \in$ Ker B. By (7) and Lemma 1,

$$AB^2x + BABx = 0 \Rightarrow BABx = 0 \Rightarrow ABx = \gamma y \Rightarrow Bx = -\gamma Ay$$

for some $\gamma \in \mathbb{R}$. In the case $x = f_1$ and $y = f_2$, we have $\gamma = \pm 1$ since A and B preserve the distance on W_A and W_B, respectively (recall that $\lambda_A^2 = \lambda_B^2 = 1$), and so $Af_2 = \pm Bf_1$. Hence, for $\alpha, \beta \in \mathbb{R}$, $\mathrm{Ker}(\alpha A + \beta B)$ is a 1-dimensional subspace spanned by the vectors $(\alpha f_1 + \beta f_2)$ or $(\alpha f_1 - \beta f_2)$.

Remark 2. Let $A \in S_{2n+1}$ and $B \in S_{2n+2}$ such that $B = [b_1\ b_2\ \cdots\ b_{2n+2}]$ and $A = F^TBF$ where each b_i is a $(2n+2) \times 1$ column vector. The column vector b_{2n+2} is of the form $(v, 0)^T$ for some $v \in \mathbb{R}^{2n+1}$. Since B is orthogonal, $(v, 0) \perp b_j$ for each $j = 1, \ldots, 2n+1$ and thus v perpendicular to each column vector of A (recall that $A = F^TBF$), which implies $v \in$ Ker A.

Lemma 7. *Let $\mathcal{K} \subset S_{2n}$ and $\mathcal{M} \subset S_{2n+2}$ be subspaces.*

(i) $\{A \oplus 0_1 \mid A \in \mathcal{K}\} \subset S_{2n+1}$ *is a subspace with dimension* $\dim \mathcal{K}$.
(ii) $\{F^TBF \mid B \in \mathcal{M}\} \subset S_{2n+1}$ *is a subspace with dimension* $\dim \mathcal{M}$.

Proof. *(i)* Let $\mathcal{K} \subset S_{2n}$ be a subspace and $\{A_1, \ldots, A_k\}$ be an orthogonal basis of $\mathcal{K}$. For $j = 1, \ldots, k$, $A_j \oplus 0_1 \in S_{2n+1}$ since it satisfies (5) and has rank $2n$. Obviously, the matrices $A_1 \oplus 0_1, A_2 \oplus 0_1, \ldots, A_k \oplus 0_1$ span k-dimensional subspace in S_{2n+1} since for all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$

$$\alpha_1(A_1 \oplus 0_1) + \cdots + \alpha_k(A_k \oplus 0_1) = (\alpha_1 A_1 + \cdots + \alpha_k A_k) \oplus 0_1.$$

(ii) Let $\mathcal{M} \subset S_{2n+2}$ be a subspace and $\{A_1, \ldots, A_k\}$ be an orthogonal basis of $\mathcal{M}$. For $j = 1, \ldots, k$, $B_j := F^TA_jF \in S_{2n+1}$ by Lemma 3. The set of matrices $\{B_1, \ldots, B_k\}$ is a linearly independent set in S_{2n+1} since $\{A_1, \ldots, A_k\}$ is in S_{2n+2}. Also note that since

$$\alpha_1 B_1 + \cdots + \alpha_k B_k = F^T(\alpha_1 A_1 + \cdots + \alpha_k A_k)F$$

for all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, we have $\alpha_1 B_1 + \cdots + \alpha_k B_k \in S_{2n+1}$ by Lemma 3. Hence $\{B_1, \ldots, B_k\}$ spans k-dimensional subspace in S_{2n+1}. $\square$

We remark that, since there is a subspace of dimension $\rho(2n) - 1$ in S_{2n} (in fact a maximal one, [4, Proposition 3]), there is also a subspace of dimension $\rho(2n) - 1$ in S_{2n+1}. Similarly, since there is a subspace of dimension $\rho(2n+2) - 1$ in S_{2n+2} (in fact a maximal one, [4, Proposition 3]), there is also a subspace of dimension $\rho(2n+2) - 1$ in S_{2n+1}.

Lemma 8. *Let $\mathcal{L}$ be a subspace in S_{2n+1}.*

(i) *If $\mathcal{L}$ is of the first type, then there exists a subspace $\mathcal{K} \subset S_{2n}$ such that* $\dim(\mathcal{L}) = \dim(\mathcal{K})$.
(ii) *If $\mathcal{L}$ is of the second type, then there exists a subspace $\mathcal{M} \subset S_{2n+2}$ such that* $\dim(\mathcal{L}) = \dim(\mathcal{M})$.

Proof. *(i)* Let $\mathcal{L}$ be a subspace of the first type in S_{2n+1} and $\{A_1, A_2, \ldots, A_k\}$ be a first type basis of $\mathcal{L}$ (i.e. $A_1, A_2, \ldots, A_k$ have common kernel). There exists an orthogonal matrix P such that P^TA_jP is of the form (2):

$$P^TA_jP =: B_j \oplus 0_1 \quad (B_j = F^T(P^TA_jP)F \in S_{2n}).$$

Let $\mathcal{K}$ be the subspace in S_{2n} spanned by the matrices $B_1, B_2, \ldots, B_k$. Then $\dim(\mathcal{K}) = \dim(\mathcal{L}) = k$.

(ii) Let $\mathcal{L}$ be a subspace of the second type in S_{2n+1} and $\{A_1, A_2, \ldots, A_k\}$ be a second type basis of $\mathcal{L}$ such that $\lambda_{A_j}^2 = 1$ for all $j = 1, 2, \ldots, k$. By Lemma 5, we can assume that $A_j f_j = 0$ for some corresponding orthonormal basis $\{f_1, f_2, \ldots, f_{2n+1}\}$ of $\mathbb{R}^{2n+1}$. Let $\alpha, \beta \in \mathbb{R}$ and $A := \alpha_1 A_1 + \alpha_2 A_2$. By Lemma 4, there exist $B_1^1, B_1^2, B_2^1, B_2^2$ and B^1, B^2 in S_{2n+2} such that

$$A_1 = F^T B_1^i F, \quad A_2 = F^T B_2^i F, \quad A = F^T B^i F \qquad (i = 1, 2).$$

Combining Remarks 1 and 2, these matrices have to be of the form

$$B_1^1 = \begin{bmatrix} A_1 & -f_1^T \\ f_1 & 0 \end{bmatrix}, \quad B_2^1 = \begin{bmatrix} A_2 & -f_2^T \\ f_2 & 0 \end{bmatrix}, \quad B^1 = \begin{bmatrix} A & -v^T \\ v & 0 \end{bmatrix},$$
$$B_1^2 = \begin{bmatrix} A_1 & f_1^T \\ -f_1 & 0 \end{bmatrix}, \quad B_2^2 = \begin{bmatrix} A_2 & f_2^T \\ -f_2 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} A & v^T \\ -v & 0 \end{bmatrix},$$

where $v = \alpha_1 f_1 - \alpha_2 f_2$ or $v = \alpha_1 f_1 + \alpha_2 f_2$ (it depends on the relation between $A_1 f_1$ and $A_2 f_2$ as remarked in Remark 1).

Let $v := \alpha_1 f_1 + \alpha_2 f_2$. Defining $B_1 := B_1^1$ and $B_2 := B_2^1$, we obtain $\alpha_1 B_1 + \alpha_2 B_2 = B^1 \in S_{2n+2}$. Thus B_1 and B_2 span 2-dimensional subspace in S_{2n+2}. (One can define $B_1 := B_1^2$ and $B_2 := B_2^2$ and obtain $\alpha_1 B_1 + \alpha_2 B_2 = B^2 \in S_{2n+2}$.)

Let $v := \alpha_1 f_1 - \alpha_2 f_2$. Defining $B_1 := B_1^1$ and $B_2 := B_2^2$, we obtain $\alpha_1 B_1 + \alpha_2 B_2 = B^1 \in S_{2n+2}$. Thus B_1 and B_2 span 2-dimensional subspace in S_{2n+2}. (Similarly, one can define $B_1 := B_1^2$ and $B_2 := B_2^1$ and obtain $\alpha_1 B_1 + \alpha_2 B_2 = B^2 \in S_{2n+2}$.)

Applying the same argument to A and $\alpha_3 A_3$ for $\alpha_3 \in \mathbb{R}$, we obtain B_3 such that $A_3 = F^T B_3 F$ and $\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 \in S_{2n+2}$. In this way, one can obtain $\{B_1, B_2, \ldots, B_k\}$ as a basis of k-dimensional subspace $\mathcal{M}$ in S_{2n+2} such that $A = F^T B F$ where $A = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_k A_k$ and $B = \alpha_1 B_1 + \alpha_2 B_2 + \cdots + \alpha_k B_k$. $\square$

Proof of Theorem 1. From Lemmas 7 and 8, it follows obviously that a maximal subspace of S_{2n+1} has dimension $\max\{\rho(2n) - 1, \rho(2n+2) - 1\}$. Using the fact that $2 = \rho(2n) < 4 \leqslant \rho(2n+2)$ for the case n is odd and $\rho(2n) \geqslant 4 > \rho(2n+2) = 2$ for the case n is even, we complete the proof. $\square$

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