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Maximal linear subspaces of strong self-dual 2-forms and the Bonan 4-form

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ABSTRACT

The notion of self-duality of 2-forms in 4-dimensions plays an eminent role in many areas of mathematics and physics, but although the 2-forms have a genuine meaning related to curvature and gauge-field-strength in higher dimensions also, their “self-duality” is something which is almost avoided above 4-dimensions. We show that self-duality of 2-forms is a very natural notion in higher (even) dimensions also and we prove the equivalence of some scattered and rarely used definitions in the literature. We demonstrate the usefulness of this higher self-duality by studying it in 8-dimensions and we derive a natural expression for the Bonan form in terms of self-dual 2-forms and we give an explicit expression of the local action of $SO(8)$ on the Bonan form.

1. Introduction

Let M be a 4-dimensional, oriented Riemannian manifold and let e^i, $i = 1, \dots, 4$ be a local, positively oriented, orthonormal basis for the cotangent bundle T^*M of M. The local expression of a 2-form is given by

$$\omega = \sum_{i<j} \omega_{ij} e^i \wedge e^j.$$

(We will use e^{ij} for $e^i \wedge e^j$.) In 4-dimensions, ω is Hodge self-dual if $*\omega = \omega$ while it is Hodge anti-self-dual if $*\omega = -\omega$. For any p-form on an n-dimensional manifold, we have $** = (-1)^{p(n-p)}$. In

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particular for $n = 4$ and $p = 2$, $** = 1$ and therefore $*$ has eigenvalues ± 1. Hence in 4-dimensions, the 6-dimensional linear space of 2-forms is the orthogonal direct sum of the 3-dimensional eigenspaces of the Hodge map. This construction fails in higher dimensions as the Hodge dual of a 2-form is no longer a 2-form. There are various definitions of self-duality of 2-forms in higher dimensions, each with some type of a drawback or restrictions. In previous papers [1–3], we have proposed the notion of "strong self-duality" which unifies most of the existing definitions in the literature. In this report we recall some of these results and apply them to obtain new results on the relation of linear subspaces of strong self-dual 2-forms and the Bonan form in 8-dimensions [4], whereby we construct new natural expressions for the Bonan form. This construction allows a very convenient way of obtaining the expression of the action of $SO(8)$ on the Bonan form.

The ω_{ij}'s form a skew-symmetric matrix whose eigenvalues are pure imaginary and occur in conjugate pairs. If these are denoted in 4-dimensions as $\pm i\lambda_1$ and $\pm i\lambda_2$, it can be seen that they satisfy

$$\lambda_1^2 + \lambda_2^2 = \omega_{12}^2 + \omega_{13}^2 + \omega_{14}^2 + \omega_{23}^2 + \omega_{24}^2 + \omega_{34}^2,$$
$$\lambda_1^2\lambda_2^2 = (\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23})^2.$$

Hence

$$\lambda_1 \mp \lambda_2 = \sqrt{(\omega_{12} \mp \omega_{34})^2 + (\omega_{13} \pm \omega_{24})^2 + (\omega_{14} \mp \omega_{23})^2}.$$

Thus for self-duality

$$\lambda_1 = \lambda_2,$$

while for anti-self-duality

$$\lambda_1 = -\lambda_2.$$

In both cases the absolute values of the eigenvalues are equal. Two cases are distinguished by the sign of the Pfaffian of ω:

$$\mathrm{Pf}(\omega) = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23}.$$

Thus in 4-dimensions, the equality of the absolute values of the eigenvalues gives the usual notion of self-duality in the Hodge sense.

This crucial remark is the starting point of our work on strong self-duality. We declare a 2-form in $2n$-dimensions to be strong self-dual or strong anti-self-dual, if the eigenvalues of its matrix with respect to some local orthonormal basis $\{e^1, \ldots, e^{2n}\}$ are equal in absolute value and nonzero. The two cases are again distinguished by the sign of the Pfaffian, or more simply, by the sign of $*(\omega^n)$. We also note that odd dimensional manifolds can be ignored because the 2-forms on them are degenerate.

Denoting the 2-form and its matrix with respect to some orthonormal basis by the same symbol ω, strong self-duality or anti self-duality can also be expressed by the minimal polynomial requirement

$$\omega^2 + \lambda^2 I = 0,$$

where

$$\lambda^2 = -\frac{1}{2n}\mathrm{Tr}\,\omega^2.$$

2. Strong self-duality in higher dimensions

In Section 2.1 we recall the definition of strong self-duality and anti-self-duality in terms of the associated anti-symmetric matrices, in Section 2.2 we discuss some useful inequalities and in Section 2.3 we prove the equivalence of various self-duality notions for 2-forms.

2.1. Strong self-duality (SD) and anti-self-duality (ASD) of 2-forms as an eigenvalue criterion

Let ω be a 2-form on a $2n$-dimensional oriented real vector space with an inner-product. We denote the 2-form ω and the corresponding skew-symmetric matrix consisting of its components with respect to some orthonormal basis by the same symbol. The distinction between the wedge product of forms and the matrix multiplication should be made from the context. Since ω is a $2n \times 2n$ skew-symmetric matrix, its eigenvalues are pure imaginary and pairwise conjugate, i.e. $\pm i\lambda_1, \ldots, \pm i\lambda_n$. Thus there is an orthonormal basis $\{X_k, Y_k\}$ such that

$$\omega X_k = -\lambda_k Y_k, \quad \omega Y_k = \lambda_k X_k$$

and ω takes the block-diagonal form

$$\omega = \begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 0 & \lambda_n \\ & & & & -\lambda_n & 0 \end{pmatrix}.$$

We define strong self-duality and anti-self-duality as follows (see [1]):

Definition 1. Let ω be a real 2-form on a $2n$-dimensional oriented real vector space with an inner product and denote the corresponding $2n \times 2n$ skew-symmetric matrix with respect to some orthonormal basis by the same symbol. Let $\pm i\lambda_1, \ldots, \pm i\lambda_n$ be the eigenvalues of ω. Then ω is said to be strong self-dual (respectively, strong anti-self-dual) if

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n| \tag{1}$$

and $*\omega^n > 0$ (respectively, $*\omega^n < 0$).

Note that this is equivalent to the statement that the distinction is based on the sign of the Pfaffian of ω with respect to a positively oriented orthonormal basis.

If ω is strong self-dual, its matrix with respect to a positively oriented orthonormal basis can be brought to a block diagonal form $K_\lambda = I \otimes \epsilon_\lambda$ where $\epsilon_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, by an orientation preserving orthogonal transformation, while if it is strong anti-self-dual, the same K_λ can be realized by an orientation reversing transformation.

The strong self-duality condition is equivalent to the matrix equation

$$\omega^2 + \lambda^2 I = 0,$$

where I is the identity matrix, and $\lambda^2 = -\frac{1}{2n}\mathrm{Tr}\,\omega^2$. This definition gives quadratic equations for the ω_{ij}'s, hence the strong self-duality condition determines an algebraic variety. This algebraic variety will be denoted by $\mathcal{S}_{2n}$.

In 4-dimensions, the strong self-duality coincides with usual Hodge duality. More precisely, the matrices satisfying $\omega^2 + \lambda^2 I = 0$ consist of the union of the usual self-dual and anti-self-dual 2-forms (including the zero form). Thus the algebraic variety consists of the union of two linear spaces.

2.2. Eigenvalue inequalities

In this section we shall use the well known inequalities between elementary symmetric functions of the eigenvalues of a skew symmetric matrix to obtain inequalities between the norms of the powers of a 2-form.

Lemma 1 [6]. *Let s_k be the kth elementary symmetric function of the numbers $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, with $\alpha_i \in R$ and let the weighted elementary symmetric polynomials, q_k's, be defined by*

$$\binom{n}{k} q_k = s_k. \tag{2}$$

Then

$$q_1 \geq q_2^{1/2} \geq q_3^{1/3} \geq \cdots \geq q_n^{1/n}, \tag{3}$$

$$q_{r-1}q_{r+1} \leq q_r^2, \quad 1 \leq r < n. \tag{4}$$

If all α_i's are equal, then the equalities hold and if any single equality holds, then all α_i's are equal.

Although the inequalites (4) are more convenient to use, the inequalites (3) are more refined in the sense that (4) implies (3).

Remark 1. If A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\det(I + tA) = \sum_{k=0}^{n} \sigma_k t^k = \prod_{k=1}^{n} (1 + t\lambda_k), \tag{5}$$

where σ_k is just the kth elementary symmetric function of the eigenvalues.

If A is a real skew-symmetric $2n \times 2n$ matrix, then the eigenvalues of A are $\pm i\lambda_k$, $k = 1, 2, \ldots, n$. It can be seen that, in this case $\sigma_{2k+1} = 0$ while the σ_{2k}'s are kth elementary symmetric functions of $\{\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2\}$, i.e.

$$\begin{aligned}
\sigma_2 &= \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2, \\
\sigma_4 &= \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \cdots + \lambda_{n-1}^2\lambda_n^2, \\
\sigma_6 &= \lambda_1^2\lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2^2\lambda_4^2 + \cdots + \lambda_{n-2}^2\lambda_{n-1}^2\lambda_n^2, \\
&\vdots \\
\sigma_{2n} &= \lambda_1^2\lambda_2^2 \cdots \lambda_n^2.
\end{aligned} \tag{6}$$

Thus for a skew-symmetric matrix, using (2), we can express σ_{2k}'s as

$$\sigma_{2k} = \frac{n!}{k!(n-k)!} q_k. \tag{7}$$

On the other hand, the σ_{2k}'s are related to the norms of the kth powers of ω as

$$\sigma_{2k} = \frac{1}{(k!)^2} (\omega^k, \omega^k), \tag{8}$$

where the brackets (,) denote the inner product.

Combining these we have the relations

$$\begin{aligned}
\sigma_2 &= (\omega, \omega) = nq_1, \\
\sigma_4 &= \frac{1}{(2!)^2} (\omega^2, \omega^2) = \frac{n(n-1)}{2} q_2, \\
\sigma_6 &= \frac{1}{(3!)^2} (\omega^3, \omega^3) = \frac{n(n-1)(n-2)}{6} q_3,
\end{aligned}$$

$$\vdots$$
$$\sigma_{2k} = \frac{1}{(k!)^2}(\omega^k, \omega^k) = \frac{n!}{k!(n-k)!}q_k,$$
$$\vdots$$
$$\sigma_{2n} = \frac{1}{(n!)^2}(\omega^n, \omega^n) = \frac{1}{(n!)^2}\ |*\omega^n|^2 = q_n.$$

Thus

$$(\omega^k, \omega^k) = \frac{n!k!}{(n-k)!}q_k \tag{9}$$

or

$$q_k = \frac{(n-k)!}{n!k!}(\omega^k, \omega^k). \tag{10}$$

From the inequalities (3),

$$(\omega^k, \omega^k) \leq \frac{n!k!}{n^k(n-k)!}(\omega, \omega)^k \tag{11}$$

and in the case of the equality of the eigenvalues

$$(\omega^k, \omega^k) = \frac{n!k!}{n^k(n-k)!}(\omega, \omega)^k. \tag{12}$$

For $k = n$ this formula gives for a strong SD/ASD form in $2n$-dimensions

$$(\omega^n, \omega^n) = \frac{(n!)^2}{n^n}(\omega, \omega)^n \tag{13}$$

or

$$|*\omega^n| = \frac{n!}{n^{\frac{n}{2}}}|\omega|^n. \tag{14}$$

2.3. Equivalence of strong self-duality with previous definitions of self-duality

We defined the strong SD/ASD of a 2-form as the equality of the absolute values of the eigenvalues of the corresponding matrix. Now we will show that (i) a 2-form ω in $2n$-dimensions is strong SD if and only if ω^{n-1} is proportional to the Hodge dual of ω and (ii) a 2-form ω in $4n$-dimensions is strong SD if ω^n is SD in the Hodge sense. The first condition has been proposed as a definition of self-duality by Trautman [11] while the second one appears in the work of Grossman et al. [7]. We start with the second result which is easier to prove.

Theorem 2. *Let ω be a (non-degenerate) 2-form in $4n$-dimensions. Then ω is strong self-dual (anti self-dual) if and only if ω^n is self-dual (anti self-dual) in the Hodge sense, that is $*\omega^n = \omega^n$ ($*\omega^n = -\omega^n$).*

Proof. If ω is strong SD, we can choose a positive orthonormal basis such that $\omega = \lambda(e^{12} + e^{34} + \cdots + e^{4n-1,4n})$ with respect to this basis and it can be seen that $*\omega^n = \omega^n$. For the ASD case we can choose $\omega = \lambda(-e^{12} + e^{34} + \cdots + e^{4n-1,4n})$ giving $*\omega^n = -\omega^n$.

Conversely, if $*\omega^n = \omega^n$ holds, then

$$*\omega^{2n} = *(\omega^n \wedge \omega^n) = *(\omega^n \wedge *\omega^n) = (\omega^n, \omega^n). \tag{15}$$

By the inequalities (2) we have for $k = n$ in $4n$-dimensions, $q_n^{1/n} \geq q_{2n}^{1/2n}$, i.e. $q_n^2 \geq q_{2n}$. As $q_n = \frac{1}{(2n)!}(\omega^n, \omega^n)$ and $q_{2n} = \frac{1}{(2n)!^2}(\omega^{2n}, \omega^{2n})$ this gives $(\omega^n, \omega^n)^2 \geq (\omega^{2n}, \omega^{2n}) = |*\omega^{2n}|^2$. Thus this inequality is saturated by Eq. (15) and ω is strong SD.

If $*\omega^n = -\omega^n$, then $*\omega^{2n} = -(\omega^n, \omega^n)$, $*\omega^{2n} < 0$ and the same saturation argument shows that ω is strong ASD. □

We will now show that the strong self-duality condition is also equivalent to the self-duality definition used by Trautman [11].

Theorem 3. *Let ω be a strong SD (ASD) 2-form in $2n$-dimensions. Then*

$$\omega^{n-1} = \frac{n!}{n^{n/2}}(\omega, \omega)^{\frac{n}{2}-1} * \omega \quad \left(\omega^{n-1} = -\frac{n!}{n^{n/2}}(\omega, \omega)^{\frac{n}{2}-1} * \omega\right). \tag{16}$$

Conversely, if for a (non-degenerate) 2-form ω the equality $\omega^{n-1} = k * \omega$ holds, then $k = \pm\frac{n!}{n^{n/2}}(\omega, \omega)^{\frac{n}{2}-1}$ and ω is strong SD (ASD) for positive (negative) k.

Proof. If ω is strong SD, we can choose a positive orthonormal basis such that $\omega = \lambda(e^{12} + e^{34} + \cdots + e^{4n-1,4n})$ with respect to this basis. Then $(\omega, \omega) = n\lambda^2$ and it is not difficult to see that ω^{n-1} consists of the products of $2n - 2$ distinct e^i's, with coefficient $(n - 1)!$. Thus $\omega^{n-1} = \lambda^{n-2}(n - 1)! * \omega$ and the result follows by inserting $\lambda = \frac{(\omega,\omega)^{1/2}}{n^{1/2}}$. If ω is strong ASD, then we can take $\omega = \lambda(-e^{12} + e^{34} + \cdots + e^{4n-1,4n})$.

Conversely, if $\omega^{n-1} = k * \omega$ holds, then multiplying it with ω and taking Hodge duals, we obtain, $*\omega^n = k(\omega, \omega)$. Since $(\omega, \omega) = \sigma_2 = nq_1$ and $|*\omega^n| = n!\sigma_{2n}^{1/2} = n!q_n^{1/2}$, we obtain $k = (n - 1)!q_n^{1/2}/q_1$. Then taking inner products of both sides of $\omega^{n-1} = k * \omega$ with themselves, we obtain $(\omega^{n-1}, \omega^{n-1}) = k^2(*\omega, *\omega) = k^2(\omega, \omega)$. Substituting the value of k obtained above, and using $(\omega^{n-1}, \omega^{n-1}) = ((n - 1)!)^2 nq_{n-1}$, we obtain $q_n = q_{n-1}q_1$. But since $q_1 \geq q_n^{1/n}$, we have $q_n \geq q_{n-1}q_n^{1/n}$, which leads to $q_n^{n-1} \geq q_{n-1}^n$. This is just the reverse of the weighted elementary symmetric polynomials q_k's inequality in Section 2.2, hence equality must hold, and all eigenvalues of ω are equal in absolute value. Thus ω is strong SD/ASD. $*\omega^n = k(\omega, \omega)$ and the Eq. (14) gives $|k| = \frac{n!}{n^{n/2}}(\omega, \omega)^{\frac{n}{2}-1}$. For positive (negative) k we have $*\omega^n > 0$ ($*\omega^n \leq 0$), thus ω is strong SD (ASD). □

3. Geometry of 2-forms

In Section 3.1 we recall the manifold structure of strong SD/ASD 2-forms, in Section 3.2 we recall some basics from Clifford algebras, in Section 3.3 we discuss the maximal linear subspaces of strong SD/ASD 2-forms and in Section 3.4 we specialize to 8-dimensions where possibly the richest structures are encountered.

3.1. Manifold structure of strong SD/ASD 2-forms

In this section we describe the geometrical structure of strong self-dual and anti-self-dual 2-forms in arbitrary even dimensions. Let $\mathcal{S}_{2n}$ be the set of SD/ASD 2-forms in $2n$-dimensions. Taking as vector space the standard $\mathbb{R}^{2n}$ (with the usual metric and orientation) this set can be equivalently defined in terms of skew symmetric matrices as follows.

Definition 2. Let $\mathcal{A}_{2n}$ be the set of anti-symmetric matrices in $2n$-dimensions. Then $\mathcal{S}_{2n} = \{A \in \mathcal{A}_{2n} \mid A^2 + \lambda^2 I = 0, \lambda \in \mathbb{R}, \lambda \neq 0\}$.

Including the zero matrix, we denote the closure of $\mathcal{S}_{2n}$ by $\overline{\mathcal{S}}_{2n}$. Note that at each A, $\overline{\mathcal{S}}_{2n}$ contains the line through A, i.e. if $A \in \mathcal{S}_{2n}$, then $\lambda A \in \overline{\mathcal{S}}_{2n}$ for $\lambda \in \mathbb{R}$ and hence the existence of 1-dimensional linear subspaces of the closure is trivial. In the next section we will determine the dimension of maximal linear spaces of $\overline{\mathcal{S}}_{2n}$.

We recall now the manifold structure of $\mathcal{S}_{2n}$.

Proposition 1. *$\mathcal{S}_{2n}$ is diffeomorphic to $(O(2n) \cap \mathcal{A}_{2n}) \times \mathbb{R}^+$.*

Proof.

$$\phi : \mathcal{S}_{2n} \longrightarrow (O(2n) \cap \mathcal{A}_{2n}) \times \mathbb{R}^+$$

given by $\phi(A) = \left(\frac{1}{\kappa}A, \kappa\right)$ with $\kappa = \left[-\frac{1}{2n}\mathrm{Tr}\, A^2\right]^{1/2}$ is a diffeomorphism [2]. □

There is another useful description of $\mathcal{S}_{2n}$:

Proposition 2. *$\mathcal{S}_{2n}$ is diffeomorphic to the homogeneous manifold $(O(2n) \times \mathbb{R}^+)/U(n) \times \{1\}$ (where $\mathbb{R}^+$ is considered as a multiplicative group), and $\dim \mathcal{S}_{2n} = n^2 - n + 1$.*

Proof. $O(2n) \times \mathbb{R}^+$ acts on $\mathcal{S}_{2n}$ transitively by $(P, \alpha)A = \alpha P^t A P$ (where $P \in O(2n)$, $\alpha \in \mathbb{R}^+$, and $A \in \mathcal{S}_{2n}$) with isotropy group $U(n)$ [2]. □

In particular, in 8-dimensions, $\mathcal{S}_8$ is a 13 dimensional manifold (with two connected components, one of them containing $\omega = e^{12} + e^{34} + e^{56} + e^{78}$ and all strongly SD 2-forms, the other containing $\omega = -e^{12} + e^{34} + e^{56} + e^{78}$ and all strongly ASD 2-forms).

3.2. Clifford algebras

We recall very briefly the notion of a (real) Clifford algebra. Let V be a real vector space and q be a (real) quadratic form on V. The *Clifford algebra* $Cl(V, q)$ associated to V and q is a real associative algebra with identity 1, generated by the vector space V and by the identity, subject to the relations $v \cdot v = -q(v)1$ for any vector v in V. The map $\alpha(v) = -v$ for $v \in V$ extends to an involution of the Clifford algebra $Cl(V, q)$ and its ± 1 eigenspaces are called, respectively, the *even* and *odd* parts, denoted by $Cl^{ev}(V, q)$ and $Cl^{odd}(V, q)$. A representation of a Clifford algebra $Cl(V, q)$ on a real vector space W is a homomorphism from $Cl(V, q)$ to $Hom(W, W)$.

The real Clifford algebra associated to $V = \mathbb{R}^n$ and to the quadratic form $q(x) = x_1^2 + \cdots + x_n^2$, is denoted by $Cl(n)$.

If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis for V, the real Clifford algebra $Cl(n)$ is generated by the $\{e_i\}$'s, subject to the relations,

$$e_i^2 = -1, \quad i = 1, \ldots, n \qquad e_i e_j + e_j e_i = 0, \quad i \neq j,$$

and it is a 2^n-dimensional vector space spanned by the set

$$\{1, e_1, e_2, \ldots, e_1 e_2, \ldots, e_1 e_2 e_3, \ldots, e_1 e_2 e_3, \ldots, e_n\}.$$

There is a transpose-antiinvolution on $Cl(n)$, given by reversing the order of generators: $e_{i_1} e_{i_2} \ldots e_{i_k} \mapsto e_{i_k} e_{i_{k-1}} \ldots e_{i_1}$. We denote the image of an element $u \in Cl(n)$ by u^t.

The spin groups are defined by

$$Spin(n) = \{u \in Cl^{ev}(n) \mid u x u^{-1} \in V \text{ for } x \in V \text{ and } u u^t = 1\}. \tag{17}$$

Clifford algebras have the following fundamental property: if $f : V \to A$ is a linear map into an associative algebra with unit such that $f(v)^2 = -q(v) \cdot 1$ holds for all $v \in V$, then f can be uniquely extended to an algebra homomorphism from $Cl(V, q)$ to A.

To give an example relevant for us, let $V = \mathbb{R}^7$ with the standard quadratic form and $A = End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O})$ where $\mathbb{O}$ denotes the octonions. To give a map $f : V \to A$, we understand $\mathbb{R}^7$ as $Im(\mathbb{O})$ (imaginary octonions) and define $f(v) = (R_v, -R_v)$ where R_v denotes the octonion multiplication from the right with the imaginary octonion v. This map can be seen to possess the required property and thus extends to an algebra homomorphism $Cl(7) \to End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O})$, which can be seen to be an isomorphism (for this and generally for Clifford algebras we refer to [8,10]).

Under this isomorphism $Cl^{ev}(7)$ is embedded as the diagonal of $End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O})$, $Spin(7) \subset Cl^{ev}(7)$ is embedded diagonally into $O(8) \oplus O(8)$ and we can also understand $Spin(7)$ as a subgroup of $O(8)$ (in fact, $SO(8)$) by projecting into a factor; it can be shown that $Spin(7)$ is the subgroup generated by the right-multiplication maps R_v for $v \in Im(\mathbb{O})$, $\|v\| = 1$. We note for later reference, that in this model and according to the Eq. (17), $Spin(7)$ consists exactly of those elements $P \in O(8)$ for which any R_v $(v \in Im(\mathbb{O}))$ is transformed under PR_vP^{-1} into another right-multiplication map for some $w \in Im(\mathbb{O})$: $PR_vP^{-1} = R_w$.

3.3. Maximal linear subspaces of $\overline{\mathcal{S}}_{2n}$

In this section we will show that the dimension of maximal linear subspaces of $\overline{\mathcal{S}}_{2n}$ is equal to the number of linearly independent vector fields on S^{2n-1}. The maximal number of pointwise linearly independent vector fields on the sphere S^N is given by the Radon–Hurwitz number associated to $N+1$: If $N + 1 = (2a + 1)2^{4d+c}$ with $c = 0, 1, 2$ or 3, then the R–H-number associated to $N + 1$ equals $8d + 2^c - 1$ [10].

Using this formula it can be seen that there are three pointwise linearly independent vector fields on S^3, seven on S^7, three on S^{11} and so on. In particular this number is one for the spheres S^{2n-1} for odd n.

Let L^k_{2n} be a k-dimensional linear subspace of $\overline{\mathcal{S}}_{2n}$. We will show that the maximum of the numbers k is equal to the Radon–Hurwitz number associated to $2n$.

Proposition 3. *The dimension of the maximal linear subspaces of $\overline{\mathcal{S}}_{2n}$ is equal to the number of linearly independent vector fields on S^{2n-1}.*

Proof. Let L^k_{2n} be a k-dimensional linear subspace of $\overline{\mathcal{S}}_{2n}$, and choose an orthogonal basis $\{A_1, A_2, \ldots, A_k\}$ consisting of orthogonal and anti-symmetric matrices for this linear subspace (note that a suitable multiple of any nonzero matrix in $\overline{\mathcal{S}}_{2n}$ is orthogonal). As $(A_i + A_j) \in L^k_{2n}$, $(A_i + A_j)^2$ is a scalar matrix, consequently $A_iA_j + A_jA_i$ is a scalar matrix and $(A_i, A_j) = \mathrm{Tr}(A_i^tA_j) = 0$ implies that $A_iA_j + A_jA_i = 0$. This means that the assignment $e_i \mapsto A_i$ $(i = 1, 2 \ldots, k)$ gives a representation of $Cl(k)$ on $\mathbb{R}^{2n}$.

Conversely, if for some k, there is a representation of $Cl(k)$ on $\mathbb{R}^{2n}$, then there is an orthogonal representation also and the relations $e_i^2 = -1$, $e_ie_j + e_je_i = 0$ imply that the images A_i of e_i under this representation are anti-symmetric and anti-commuting. This means that the matrices $\{A_1, A_2, \ldots, A_k\}$ span a k-dimensional linear subspace of $\overline{\mathcal{S}}_{2n}$.

Thus, the maximal dimension of a linear subspace of $\overline{\mathcal{S}}_{2n}$ is the maximal k, for which $Cl(k)$ acts on $\mathbb{R}^{2n}$. This the Radon–Hurwitz number associated to $2n$ [10]. □

This property shows that there is an intimate relationship between strong self-duality and Clifford-algebras. Namely, $\overline{\mathcal{S}}_{2n}$ admits a k-dimensional linear subspace (i.e. including the zero-form, there exists a k-dimensional linear space of strongly SD/ASD 2-forms on $\mathbb{R}^{2n}$) if and only if there is a representation of $Cl(k)$ on $\mathbb{R}^{2n}$.

Remark 2. The 7-dimensional plane of 2-forms on $\mathbb{R}^8$ given by the linear self-duality equations of Corrigan et al. [5] is one of these planes in $\overline{\mathcal{S}}_8$.

We now prove directly that for odd n there are no linear subspaces other than the 1-dimensional ones.

Proposition 4. *Let $L = \{A \in \overline{\mathcal{S}}_{2n} \mid (A + J) \in \overline{\mathcal{S}}_{2n}\}$ where $J = \epsilon_1 \otimes I$ is a reference matrix. Then $L = \{kJ \mid k \in \mathbb{R}\}$ for odd n.*

Proof. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^t & A_{22} \end{pmatrix}$, where $A_{11} + A_{11}^t = 0$, $A_{22} + A_{22}^t = 0$. Since $(A + J) \in \overline{\mathcal{S}}_{2n}$, $AJ + JA$ is proportional to the identity. This gives $A_{11} + A_{22} = 0$ and the symmetric part of A_{12} is proportional to identity. Therefore $A = kJ + \begin{pmatrix} A_{11} & A_{12o} \\ A_{12o} & -A_{11} \end{pmatrix}$, where A_{12o} denotes the antisymmetric part of A_{12} and $k \in \mathbb{R}$. Then the requirement that $A \in \overline{\mathcal{S}}_{2n}$ gives

$$[A_{11}, A_{12o}] = 0, \quad A_{11}^2 + A_{12o}^2 + k'I = 0, \quad k' \in \mathbb{R}.$$

As A_{11} and A_{12o} commute, they can be simultaneously diagonalised, hence for odd n they can be brought to the form

$$A_{11} = \mathrm{diag}(\lambda_1\epsilon, \ldots, \lambda_{(n-1)/2}\epsilon, 0),$$
$$A_{12o} = \mathrm{diag}(\mu_1\epsilon, \ldots, \mu_{(n-1)/2}\epsilon, 0),$$

up to the permutation of blocks, where $\epsilon = \epsilon_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and 0 denotes a 1×1 block. If the blocks occur as shown, clearly $A_{11}^2 + A_{12o}^2$ cannot be proportional to the identity unless $\lambda_i = \mu_i = 0$ for all i. It can also be seen that the same is the case for any permutation of the blocks. □

Note that these structures refer to local constructions on a manifold. Existence of k-dimensional strong SD/ASD sub-bundles of the bundle of 2-forms is another matter. If there exists a section ω of strong SD/ASD 2-forms on the manifold, then ω can be normalized to have constant norm and it defines an almost complex structure. Conversely, almost complex manifolds provide examples of manifolds admitting a (global) section of strong self-dual 2-forms. In this case $*\omega = \kappa\omega^{n-1}$, where κ is constant. Then if $d\omega = 0$ it follows that $d * \omega = 0$, hence if ω is closed and has constant norm, then ω is harmonic.

3.4. Maximal linear spaces of strong AS/ASD 2-forms in 8-dimensions

By Theorem 3, the maximal linear spaces of strong SD/ASD 2-forms on $\mathbb{R}^8$ are 7-dimensional. By the proof of Theorem 3, to produce one such space, it is enough to take a representation of $Cl(7)$ on $\mathbb{R}^8 = \mathbb{O}$ and take the span of the images of the generators $e_1, e_2, \ldots, e_7$ of $Cl(7)$.

Let us take the representation (implicit in Section 3.2) given by $e_i \mapsto R_{e_i}$. The corresponding strong SD 2-forms are the following (see the Appendix for the multiplication table we use):

$$\begin{aligned}
\omega_1 &= -e^{12} + e^{34} + e^{56} - e^{78},\\
\omega_2 &= -e^{13} - e^{24} + e^{57} + e^{68},\\
\omega_3 &= -e^{14} + e^{23} + e^{58} - e^{67},\\
\omega_4 &= -e^{15} - e^{26} - e^{37} - e^{48},\\
\omega_5 &= -e^{16} + e^{25} - e^{38} + e^{47},\\
\omega_6 &= -e^{17} + e^{28} + e^{35} - e^{46},\\
\omega_7 &= -e^{18} - e^{27} + e^{36} + e^{45}.
\end{aligned} \tag{18}$$

We will denote the span of these 2-forms by $\mathbb{L}^7$ and use it as a reference 7-plane inside $\overline{\mathcal{S}}_8 \subset End_{\mathbb{R}}(\mathbb{O}) = End(\mathbb{R}^8)$. $\mathbb{L}^7$ is the first projection of the image of $\mathbb{R}^7 = Im(\mathbb{O})$ under the map $f : Im(\mathbb{O}) \to End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O})$, $f(v) = (R_v, -R_v)$. $\mathbb{L}^7$ is invariant under the matrix-conjugation action of $Spin(7)$.

Let us denote the set of maximal (7-dimensional) linear subspaces of $\overline{\mathcal{S}}_8$ by $\mathcal{L}_8^7$. $O(8)$ acts on $\mathcal{L}_8^7$ (by conjugation on the level of elements, which maps a maximal plane onto another maximal plane) and we now show that this action is transitive:

Lemma 4. *The conjugation action of $O(8)$ on $\mathcal{L}_8^7$ is transitive.*

Proof. Let L^7 and $L'^7 \in \mathcal{L}_8^7$ be given and choose matrices $A_i \in L^7$, $B_i \in L'7$ $(i = 1, \ldots, 7)$ with $A_i^2 = -I$, $A_iA_j + A_jA_i = 0$, $B_i^2 = -I$, $B_iB_j + B_jB_i = 0$. The assignments $e_i \mapsto A_i$ and $e_i \mapsto B_i$ determine two representations ρ and σ of $Cl(7)$ on $\mathbb{R}^8$. The assignment $e_i \mapsto -A_i$ determines another representation ρ^- of $Cl(7)$ on $\mathbb{R}^8$, which is inequivalent to ρ. It is a fact that [10] there are exactly two inequivalent representations of $Cl(7)$ on $\mathbb{R}^8$, so σ is either equivalent to ρ or to ρ^-. Let us assume σ to be equivalent to ρ (the other case being similar) which means that for a certain $P \in GL(\mathbb{R}, 8)$ it holds $B_i = PA_iP^{-1}$. Anti-symmetry of A_i and B_i gives $P^tPA_i = A_iP^tP$, which necessitates P^tP to be a scalar matrix, since it commutes with all A_i (and the A_i generate the algebra $End(\mathbb{R}^8)$). So, if $P^tP = \lambda I$, then $Q = \frac{1}{\sqrt{\lambda}}P$ is orthogonal and $B_i = QA_iQ^{-1}$. Thus $L'7$ is obtained from L^7 by the action of $Q \in O(8)$. □

Remark 3. *Spin*(7) is the isotropy group of $\mathbb{L}^7$ under this action of $O(8)$ on $\mathcal{L}_8^7$.

Remark 4. All planes $L^7 \in \mathcal{L}_8^7$ are “pure" in the sense that, the non-zero 2-forms in a plane are either all strong SD, or all strong ASD, because $\mathbb{L}^7$ is pure (SD), so any other is pure (SD or ASD) depending on whether the conjugation is by matrix from $SO(8)$ or $O(8)\backslash SO(8)$. (This could be seen by a connectivity argument also.)

Remark 5. If ω is a 2-form with the associated matrix A and $P \in End(\mathbb{R}^8)$, then the pull-back $\eta = P^*\omega$ has the matrix $B = P^tAP$, so if $P \in O(8)$ then $B = P^{-1}AP$, the conjugation by P^{-1}.

Before describing the Bonan form we want to note a few useful properties in 8-dimensions. The basic eigenvalue inequalities reduce to

$$(\omega, \omega)^2 \geq \frac{2}{3}(\omega^2, \omega^2) \geq \frac{2}{3}|*\omega^4|. \tag{19}$$

When ω is strong self-dual we have the equalities

$$(\omega, \omega)^2 = \frac{2}{3}(\omega^2, \omega^2) = \frac{2}{3} *\omega^4, \quad \omega^3 = \frac{3}{2}(\omega, \omega) *\omega, \tag{20}$$

while when ω is strongly anti self-dual

$$(\omega, \omega)^2 = \frac{2}{3}(\omega^2, \omega^2) = -\frac{2}{3} *\omega^4, \quad \omega^3 = -\frac{3}{2}(\omega, \omega) *\omega. \tag{21}$$

Lemma 5. *Let ω and η be two orthogonal strong SD/ASD 2-forms with commuting matrices. Then*

$$\omega^2\eta = -\frac{1}{2}(\omega, \omega) * \eta. \tag{22}$$

Proof. Since the matrices of ω and η are commuting, they can be simultaneously diagonalized and without loss of generality we can choose

$$\omega = e_{12} + e_{34} + e_{56} + e_{78}, \quad \eta = e_{12} + e_{34} - e_{56} - e_{78}.$$

The result follows by simple computation. □

Lemma 6. *Let ω and η be two orthogonal strong SD/ASD 2-forms whose matrices anti-commute. Then*

$$\omega^2\eta = \frac{1}{2}(\omega, \omega) * \eta. \tag{23}$$

Proof. By anti-commutativity $\omega \pm \eta$ are also strong SD/ASD and by connectivity reasons all are SD or ASD. Applying (19) for the SD-case and (20) for the-ASD case to both $\omega \pm \eta$ and using orthogonality gives the result by simple computation. □

4. The Bonan-form

In Section 4.1 we give an expression for the Bonan-form in terms of strong self-dual 2-forms and in Section 4.2 we compute explicitly the eigenvalues and eigenspaces of the self-adjoint map on 2-forms associated with the Bonan-form.

4.1. The relationship of the Bonan-form with strong self-duality

The Bonan-form, which is a *Spin*(7)-invariant 4-form in 8-dimensions [4,10,9], and which plays an important role in special Spin-geometry, can be constructed in an elegant way in terms of strong self-dual 2-forms.

Consider the 7-plane $\mathbb{L}^7 \in \mathcal{L}_8^7$ with an orthonormal basis $\{\eta_1, \eta_2, \ldots, \eta_7\}$ and define

$$\Phi = \eta_1^2 + \eta_2^2 + \cdots + \eta_7^2 \in \Lambda^4(\mathbb{R}^8)^*. \tag{24}$$

If $\{\theta_1, \theta_2, \ldots, \theta_7\}$ is another orthonormal basis of $\mathbb{L}^7$ it is easily seen that

$$\eta_1^2 + \eta_2^2 + \cdots + \eta_7^2 = \theta_1^2 + \theta_2^2 + \cdots + \theta_7^2.$$

The operation of pullback of a 2-form $\eta \in \mathbb{L}^7$ by an element $P \in Spin(7)$ is expressed at the matrix level by the operation of conjugation by P^{-1}, so the map $\mathbb{L}^7 \to \mathbb{L}^7$, $\eta \mapsto P^*\eta$ is orthogonal and consequently,

$$P^*\Phi = (P^*\eta_1)^2 + (P^*\eta_2)^2 + \cdots + (P^*\eta_7)^2 = \Phi, \tag{25}$$

i.e. the 4-form Φ (which is self-dual by Theorem 1) is *Spin*(7)-invariant. Thus, the Bonan-form is expressed in terms of strong self-dual 2-forms.

To give an explicit expression we can use the ortonormal basis

$$\left\{\eta_1 = \frac{1}{2}\omega_1, \eta_2 = \frac{1}{2}\omega_2, \ldots, \eta_7 = \frac{1}{2}\omega_7\right\}, \tag{26}$$

where the ω_i's are given by Eq. (18) and find

$$\Phi = -\frac{3}{2}(e^{1234} + e^{1256} - e^{1278} + e^{1357} + e^{1368} + e^{1458} - e^{1467}$$
$$-e^{2358} + e^{2367} + e^{2457} + e^{2468} - e^{3456} + e^{3478} + e^{5678}). \tag{27}$$

The same construction can naturally be applied to any $L^7 \in \mathcal{L}_8^7$ and in this way a Bonan-form Φ_L is associated with every maximal 7-plane of strong SD/ASD 2-forms. If $Q \in O(8)$, then $Q^*\Phi$ is the Bonan-form associated with the plane $Q^{-1}\mathbb{L}^7 Q$ and the isotropy group of $Q^*\Phi$ is $Q^{-1}Spin(7)Q$.

4.2. Eigenspaces of the Bonan-form

We want to compute the eigenspaces and eigenvalues of the self-adjoint map T_Φ on $\Lambda^2(\mathbb{R}^8)^*$ defined by $T_\Phi(\eta) = *(\Phi\eta)$. (The method explained below works for any Φ_L.) We start with an orthonormal basis of $\mathbb{L}^7$ and denote it by $\{\omega_{12}, \omega_{13}, \ldots, \omega_{18}\}$ (We abuse the notation: double-indices denote no more the coefficients of a 2-form, but a 2-form itself; it will be convenient in the sequel.) Let A_{1i} $(i = 2, \ldots, 8)$ denote the matrices corresponding to ω_{1i}. Let $A_{ij} = A_{1i}A_{1j}$ for $1 < i < j$. The matrices A_{ij} for $i < j$ will constitute a basis for skew-symmetric matrices in $End(\mathbb{R}^8)$ and will represent strong SD/ASD forms (their squares are scalar matrices). Now, it can easily be seen that two distinct A_{ij} and A_{kl} commute iff $\{i, j\}$ and $\{k, l\}$ are disjoint; A_{ij} and A_{kl} anti-commute iff $\{i, j\}$ and $\{k, l\}$ have one index common. We denote the 2-form corresponding to A_{ij} by ω_{ij}. The set $\{\omega_{12}, \ldots, \omega_{78}\}$ is a basis of 2-forms.

Let

$$\Phi = \omega_{12}^2 + \cdots + \omega_{18}^2, \quad \Phi' = \omega_{23}^2 + \cdots + \omega_{78}^2.$$

By symbolic computation with REDUCE, it can be seen that $\Phi' = -\Phi$, hence the sum of the squares of the basis elements is zero. Since $\{\omega_{12}, \ldots, \omega_{18}\}$ is an orthogonal basis of a 7-plane $\mathbb{L}^7$, Φ is a Bonan form. We will show that the linear maps T_Φ and $T_{\Phi'}$ have both 7- and 21-dimensional eigenspaces with eigenvalues ± 3 and ∓ 1 (after scaling), and $\mathbb{L}$ is the eigenspace corresponding to the eigenvalue ± 3. Thus each Bonan form determines an $\mathbb{L}$. Hence maximal linear subspaces of strong self-dual 2-forms and the Bonan forms are in 1–1 correspondence.

Proposition 5. *$T_\Phi : \Lambda^2(\mathbb{R}^8)^* \to \Lambda^2(\mathbb{R}^8)^*$ defined by $T_\Phi(\eta) = *(\Phi\eta)$ where $\Phi = \omega_{12}^2 + \cdots + \omega_{18}^2$, has eigenvalues $9/2$ and $-3/2$, with 7- and 21-dimensional eigenspaces, respectively. (Scaling Φ we can get eigenvalues 3 and -1.)*

Proof. Applying the Lemmas 3 and 4, one gets,

$$T_\Phi(\omega_{1j}) = T_{\omega_{1j}^2}(\omega_{1j}) + \sum_{k \neq j} T_{\omega_{1k}^2}(\omega_{1j}) = \frac{3}{2}\omega_{1j} + 6 \times \frac{1}{2}\omega_{1j}.$$

$$T_\Phi(\omega_{jk}) = T_{\omega_{1j}^2}(\omega_{jk}) + T_{\omega_{1k}^2}(\omega_{jk}) + \sum_{l \neq j,k} T_{\omega_{1l}^2}(\omega_{jk}) = 2 \times \frac{1}{2}\omega_{jk} - 5 \times \frac{1}{2}\omega_{jk}. \quad \square$$

We will now show that Φ' has eigenvalues $-9/2$ and $3/2$ with, respectively, 7- and 21-dimensional eigenspaces, hence prove that it is equal to $-\Phi$.

Proposition 6. *$T_{\Phi'} : \Lambda^2(\mathbb{R}^8)^* \to \Lambda^2(\mathbb{R}^8)^*$ defined by $T_{\Phi'}(\eta) = *(\Phi'\eta)$ where $\Phi' = \omega_{23}^2 + \cdots + \omega_{28}^2$, has eigenvalues $-9/2$ and $3/2$, with 7- and 21-dimensional eigenspaces, respectively. (Scaling Φ' we can get eigenvalues -3 and 1.)*

Proof. We start by computing $T_{\Phi'}$ on ω_{1k}. Φ' is a sum of 21 terms and there are exactly 6 terms ω_{ij}^2 where $i = k$ or $j = k$ and 15 terms with $i \neq k$ and $j \neq k$. By Lemmas 4 and 3, these first and the second group of terms lead to positive and negative contributions, hence

$$T_{\Phi'}(\omega_{1k}) = T_{\omega_{ik}^2}(\omega_{1k}) + \sum_{i,j \neq k} T_{\omega_{ij}}(\omega_{1k}) = 6 \times \frac{1}{2}\omega_{1k} - 15 \times \frac{1}{2}\omega_{1k} = -\frac{9}{2}\omega_{1k}.$$

By similar counting arguments,

$$T_{\phi'}(\omega_{jk}) = \left[\frac{3}{2} + (5+5) \times \frac{1}{2} - (5+5) \times \frac{1}{2}\right]\omega_{jk} = \frac{3}{2}. \quad \square$$

We have thus seen that any maximal linear subspace $\mathbb{L}^7$ of strong self-dual 2-forms determines a Bonan form as the sum of squares of any orthogonal basis and conversely, given a Bonan form Φ, an $\mathbb{L}^7$ is determined as a 7-dimensional eigenspace of the linear map T_Φ determined by Φ. We will now show that given an $\mathbb{L}^7$, the Bonan form is uniquely determined by a linear transformation that acts as identity on $\mathbb{L}^7$. The following proposition is proved by directly checking the claim by symbolic computation with REDUCE.

Proposition 7. *Let Ψ be a 4-form such that*

$$T_\Psi(\omega_{1i}) = k\omega_{1i}, \quad i = 2, \ldots, 8.$$

Then

$$\Psi = k\frac{2}{9}\Phi.$$

We note that in the characterization above, Ψ is not assumed to be self-dual, but its self-duality follows from the invariance of the ω_{1i}'s. Furthermore, if an $\mathbb{L}^7$ has a basis ω_{1i} with anti self-dual ω_{1i}^2's, then the form Ψ defined above is anti self-dual.

4.3. The $SO(8)$ action on the Bonan form

In this section we give the explicit expression of the Bonan form under the action of $SO(8)$. Let ω_{ij} be as in the previous section and let A_{ij} be the corresponding matrix. Thus the A_{ij}'s span the Lie algebra of $SO(8)$. It is easy to see that the exponential of matrix A_{ij} is

$$e^{tA_{ij}} = \cos(t)I + sin(t)A_{ij}.$$

The action of $SO(8)$ on 2-forms is obtained by its action on matrices by conjugation. If $R \in SO(8)$, and $R \cdot \omega$ denotes its action on the 2-form ω, then

$$R \cdot \omega = RAR^t.$$

We will show that $e^{tA_{ij}}$ for $i \geq 2$ leaves the Bonan form invariant. For simplicity of notation let

$$t_{ij} = e^{tA_{ij}} = (cI + sA_{ij}),$$

where

$$c = \cos(t), \quad s = \sin(t).$$

The 2-form $t_{ij} \cdot \omega_{1k}$ is the 2-form corresponding to the matrix $(cI + sA_{ij})A_{1k}(cI - sA_{ij})$. Then,

$$\begin{aligned} t_{ij} \cdot \omega_{1i} &= (c^2 - s^2)\omega_{1i} + 2cs\omega_{1j}, \\ t_{ij} \cdot \omega_{1j} &= (c^2 - s^2)\omega_{1j} - 2cs\omega_{1i}, \\ t_{ij} \cdot \omega_{1k} &= \omega_{ik}, \quad k \neq i, j. \end{aligned} \tag{28}$$

It is easy to see that

$$(t_{ij} \cdot \omega_{1i}) \wedge t_{ij} \cdot \omega_{1i} + (t_{ij} \cdot \omega_{1j}) \wedge t_{ij} \cdot \omega_{1j} = \omega_{1i}^2 + \omega_{1j}^2.$$

It follows that t_{ij} leaves the Bonan form invariant hence generate $Spin(7)$.

Identifying the Lie algebra of $SO(8)$ with 2-forms, if X is any vector in $SO(8)$,

$$e^{tX} \cdot \Phi = e^{tY} \cdot \Phi,$$

where Y belongs to the span of the $A1i$'s. Thus the orbit of the Bonan form is locally generated by t_{1j}, for $j = 2, \ldots, 8$. Let $y_i(t)$, $i = 1, \ldots, 8$ be a 1-parameter family of functions such that

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2 + y_8^2 = 1.$$

Then the matrix R defined by

$$R = y_1 I + y_2\omega_{12} + y_3\omega_{13} + y_4\omega_{14} + y_5\omega_{15} + y_6\omega_{16} + y_7\omega_{17} + y_8\omega_{18}$$

is a one parameter family of orthogonal matrices. The image of the Bonan form Φ under the action of R is obtained by taking the wedge products of the images of the 2-forms ω_{1j}'s and summing up. The result is

$$\tilde{\Phi} = \sum_{i<j<k} c_{ijk}(e^{1ijk} + *e^{1ijk}),$$

where

$$
\begin{aligned}
c_{234} &= \ \ 1 - 8y_1^2\left(y_5^2 + y_6^2 + y_7^2 + y_8^2\right),\\
c_{256} &= \ \ 1 - 8y_1^2\left(y_3^2 + y_4^2 + y_7^2 + y_8^2\right),\\
c_{278} &= -1 + 8y_1^2\left(y_3^2 + y_4^2 + y_5^2 + y_6^2\right),\\
c_{357} &= \ \ 1 - 8y_1^2\left(y_2^2 + y_4^2 + y_6^2 + y_8^2\right),\\
c_{368} &= \ \ 1 - 8y_1^2\left(y_2^2 + y_4^2 + y_5^2 + y_7^2\right),\\
c_{458} &= \ \ 1 - 8y_1^2\left(y_2^2 + y_3^2 + y_6^2 + y_7^2\right),\\
c_{467} &= -1 + 8y_1^2\left(y_2^2 + y_3^2 + y_5^2 + y_8^2\right),
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
c_{235} &= -4y_1y_8 + 8y_1^2(y_1y_8 + y_2y_7 - y_3y_6 + y_4y_5),\\
c_{246} &= 4y_1y_8 + 8y_1^2(-y_1y_8 - y_2y_7 - y_3y_6 + y_4y_5),\\
c_{347} &= 4y_1y_8 + 8y_1^2(-y_1y_8 + y_2y_7 + y_3y_6 + y_4y_5),\\
c_{567} &= 4y_1y_8 + 8y_1^2(-y_1y_8 + y_2y_7 - y_3y_6 - y_4y_5),
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
c_{236} &= 4y_1y_7 + 8y_1^2(-y_1y_7 + y_2y_8 + y_3y_5 + y_4y_6),\\
c_{245} &= 4y_1y_7 + 8y_1^2(-y_1y_7 + y_2y_8 - y_3y_5 - y_4y_6),\\
c_{348} &= -4y_1y_7 + 8y_1^2(y_1y_7 + y_2y_8 - y_3y_5 + y_4y_6),\\
c_{568} &= -4y_1y_7 + 8y_1^2(y_1y_7 + y_2y_8 + y_3y_5 - y_4y_6),
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
c_{237} &= -4y_1y_6 + 8y_1^2(y_1y_6 - y_2y_5 + y_3y_8 + y_4y_7),\\
c_{248} &= -4y_1y_6 + 8y_1^2(y_1y_6 - y_2y_5 - y_3y_8 - y_4y_7),\\
c_{345} &= -4y_1y_6 + 8y_1^2(y_1y_6 + y_2y_5 + y_3y_8 - y_4y_7),\\
c_{578} &= 4y_1y_6 + 8y_1^2(-y_1y_6 - y_2y_5 + y_3y_8 - y_4y_7),
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
c_{238} &= 4y_1y_5 + 8y_1^2(-y_1y_5 - y_2y_6 - y_3y_7 + y_4y_8),\\
c_{247} &= -4y_1y_5 + 8y_1^2(y_1y_5 + y_2y_6 - y_3y_7 + y_4y_8),\\
c_{346} &= 4y_1y_5 + 8y_1^2(-y_1y_5 + y_2y_6 - y_3y_7 - y_4y_8),\\
c_{678} &= -4y_1y_5 + 8y_1^2(y_1y_5 - y_2y_6 - y_3y_7 - y_4y_8),
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
c_{257} &= 4y_1y_4 + 8y_1^2(-y_1y_4 + y_2y_3 + y_5y_8 + y_6y_7),\\
c_{268} &= 4y_1y_4 + 8y_1^2(-y_1y_4 + y_2y_3 - y_5y_8 - y_6y_7),\\
c_{356} &= -4y_1y_4 + 8y_1^2(y_1y_4 + y_2y_3 - y_5y_8 + y_6y_7),\\
c_{378} &= 4y_1y_4 + 8y_1^2(-y_1y_4 - y_2y_3 - y_5y_8 + y_6y_7),
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
c_{258} &= -4y_1y_3 + 8y_1^2(y_1y_3 + y_2y_4 - y_5y_7 + y_6y_8),\\
c_{267} &= 4y_1y_3 + 8y_1^2(-y_1y_3 - y_2y_4 - y_5y_7 + y_6y_8),\\
c_{456} &= 4y_1y_3 + 8y_1^2(-y_1y_3 + y_2y_4 + y_5y_7 + y_6y_8),\\
c_{478} &= -4y_1y_3 + 8y_1^2(y_1y_3 - y_2y_4 + y_5y_7 + y_6y_8),
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
c_{358} &= 4y_1y_2 + 8y_1^2(-y_1y_2 + y_3y_4 + y_5y_6 + y_7y_8),\\
c_{367} &= -4y_1y_2 + 8y_1^2(y_1y_2 - y_3y_4 + y_5y_6 + y_7y_8),\\
c_{457} &= -4y_1y_2 + 8y_1^2(y_1y_2 + y_3y_4 - y_5y_6 + y_7y_8),\\
c_{468} &= -4y_1y_2 + 8y_1^2(y_1y_2 + y_3y_4 + y_5y_6 - y_7y_8).
\end{aligned} \tag{36}
$$

5. Conclusion

In order to conclude, we wish to re-emphasize that a Bonan 4-form is associated with every maximal 7-plane of strongly self-dual (or anti-self-dual) 2-forms. The square of any strong self-dual 2-form in 8-dimensions as given above yields a Bonan 4-form that is a *Spin*(7) invariant form and plays an important role in the construction of special spin geometries [11]. We also give explicitly the action of *SO*(8) on a given Bonan 4-form.

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Appendix

We use the following multiplication table for the octonions:

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

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