On the 3D Rayleigh wave field on an elastic half-space subject to tangential surface loads

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This study is concerned with analysis of the Rayleigh wave field in a 3D isotropic elastic half-space subject to in-plane surface loading. The approach relies on the slow time perturbation of the general representation for the Rayleigh wave eigensolutions in terms of harmonic functions. The resulting hyperbolic-elliptic formulation allows decomposition of the original vector problem of 3D elasticity into a sequence of scalar Dirichlet and Neumann problems for the Laplace equation. The boundary conditions for these are specified through a 2D hyperbolic equation. An example of an impulse tangential load illustrates the efficiency of the derived asymptotic formulation, with the results expressed in terms of elementary functions.

1 Introduction

Surface waves have been a subject of numerous investigations for more than a century since the classical work of Lord Rayleigh [1]. Among the specific formulations for free surface waves, we cite [2], presenting the Rayleigh wave field for a half-plane in terms of two arbitrary plane harmonic functions, [3] expressing the Rayleigh and Stoneley wave eigensolutions in terms of a single harmonic function along with its recent generalization to 3D [4]. Alternative formulations for the Rayleigh wave have been proposed in [5], deriving a membrane equation for case of time-harmonic disturbances in a transversely isotropic half-space, and [6] associating the quasi-particles with the Rayleigh wave.

The approach of this paper is oriented towards extraction of the contribution of the Rayleigh wave to the overall dynamic field excited by a prescribed surface loading, leading to drastic simplification of the analysis, see [7]. This seems to be of particular importance in the near-resonant region when the dynamic phenomena caused by the bulk waves are negligible compared to that arising from the resonant Rayleigh wave field, for example, in applied problems of fast-train operation [8], or in studying the earthquake signals [9].

The idea of approximating the Rayleigh wave field has been used by some of the authors in their previous studies, see e.g. the 2D asymptotic formulation [7], relying on a representation of the Rayleigh wave eigensolution in terms of a single harmonic function [3, 4]. The discussed 2D model contains elliptic equations for the elastic potentials governing the decay over the interior, and a 1D hyperbolic equation at the surface describing wave propagation. The model have been recently extended to mixed problems [10]. The approach was also generalized to a 3D coated elastic half-space [11] subject to normal surface load, with the dispersive effect of the coating leading to a presence of a pseudo-differential operator, singularly perturbing the 2D wave equation at the surface. The described methodology has proved to be efficient for moving load problems [12–14], and has also been developed for other types of localized waves [15].

The aim of the current paper is to develop further the approach of explicit hyperbolic-elliptic models for surface waves to the case of tangentially loaded 3D elastic half-space. The extension seems non-trivial, and also possesses some rather important practical applications, in particular associated with analysis of the ground vibrations of the wind turbines. Indeed, the turbine modeled as an Euler-Bernoulli beam (see [16]) would induce an in-plane loading in the near-surface domain, fading into the area of applicability of the proposed formulation for the Rayleigh wave field.

Similarly to [11], the 3D problem of elastodynamics is reformulated to a 2D problem in terms of the Radon transforms. It is then revealed that excitation of the Rayleigh wave due to an in-plane loading is caused by the gradient part of the load only. Using the standard slow-time perturbation technique, the asymptotic model is then derived. The obtained formulation includes a 2D membrane equation for the shear elastic potentials together with the relation between the shear

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and longitudinal potentials governing surface behaviour, and acting as boundary conditions for the corresponding scaled Laplace equation. Thus the 3D problem of elastodynamics is reduced to a sequence of scalar Dirichlet and Neumann problems for the Laplace equations. Remarkably, the normal displacement on the surface may be expressed through the shear potentials at the surface. The obtained formulation is then applied to a model example of an impulse load acting along one of the axis on the surface. This problem is not straightforward for exact 3D analysis, however application of the developed procedure allows elegant solutions for the shear potentials and normal displacement at the surface expressed in terms of elementary functions.

The paper is organized as follows. In Sect. 2 the governing equations are formulated, Sect. 3 contains the derivation of the hyperbolic-elliptic model for the Rayleigh wave field, with a model example illustrating the approach presented in the final Sect. 4.

2 Statement of the problem

Consider a 3D homogeneous isotropic elastic half-space, occupying the domain $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, $-\infty < x_3 \leq 0$. The equations of motion in elasticity are adopted in conventional form [18]

$$(\lambda + \mu)\text{grad div}\,\mathbf{u} + \mu\Delta\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the 3D Laplace operator, and ρ stands for the volume density of mass.

The constitutive relations are assumed in the usual form

$$\sigma_{ij} = \lambda\delta_{ij}\text{div}\mathbf{u} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2}$$

where λ and μ are the Lamé constants, δ_{ij} is the Kronecker's symbol, and σ_{ij} $(i, j = 1, 2, 3)$ are components of the stress tensor. The imposed boundary conditions on the surface $x_3 = 0$ are specified as

$$\sigma_{i3} = -\mathbf{P} \quad \text{and} \quad \sigma_{33} = 0, \tag{3}$$

where $\mathbf{P}(x_1, x_2, t)$ is a given in-plane force, (see Fig. 1).

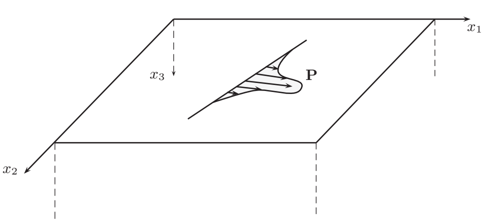

Fig. 1 The profile of the tangential loading on the surface of elastic half-space.

As will be seen from further analysis, it is convenient to decompose the load into the gradient and rotational parts through the Helmholtz theorem, resulting in

$$\mathbf{P} = \left(\frac{\partial P_0}{\partial x_1} + \frac{\partial P_1}{\partial x_2}, \frac{\partial P_0}{\partial x_2} - \frac{\partial P_1}{\partial x_1}, 0\right). \tag{4}$$

3 The Rayleigh wave field

In this section, we derive an approximate formulation for the Rayleigh wave field arising in case of the boundary value problem (1) - (3).

Substituting (2) into (1), and applying the Radon integral transform [17] to the result similarly to [11], we obtain

$$\left[(\lambda + \mu)\cos^2\alpha + \mu\right]\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2} + \mu\frac{\partial^2 u_1^{(\alpha)}}{\partial x_3^2} + (\lambda + \mu)\cos\alpha + \left(\sin\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2} + \frac{\partial^2 u_3^{(\alpha)}}{\partial\chi\,\partial x_3}\right) = \rho\frac{\partial^2 u_1^{(\alpha)}}{\partial t^2},$$

$$\left[(\lambda+\mu)\sin^2\alpha+\mu\right]\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2}+\mu\frac{\partial^2 u_2^{(\alpha)}}{\partial x_3^2}+(\lambda+\mu)\sin\alpha+\left(\cos\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2}+\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi\partial x_3}\right)=\rho\frac{\partial^2 u_2^{(\alpha)}}{\partial t^2}$$

$$(\lambda+\mu)\left(\cos\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi\partial x_3}+\sin\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi\partial x_3}\right)+\mu\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi^2}+(\lambda+2\mu)\frac{\partial^2 u_3^{(\alpha)}}{\partial x_3^2}=\rho\frac{\partial^2 u_3^{(\alpha)}}{\partial t^2}, \tag{5}$$

where the transformed displacements are defined as

$$u_k^{\alpha}(\chi,\alpha,x_3,t)=\int_{-\infty}^{\infty}u_k(\chi\cos\alpha-\zeta\sin\alpha,\,\chi\sin\alpha+\zeta\cos\alpha,\,x_3,\,t)\,d\zeta,$$

and

$$\chi=x_1\cos\alpha+x_2\sin\alpha,\qquad \zeta=-x_1\sin\alpha+x_2\cos\alpha,$$

with the angle α varying over the interval $0\le\alpha<2\pi$. The transformed quantities may then be rewritten in Cartesian frame (χ,ζ) as

$$u_\chi^{\alpha}=u_1^{\alpha}\cos\alpha+u_2^{\alpha}\sin\alpha,\qquad u_\zeta^{\alpha}=-u_1^{\alpha}\sin\alpha+u_2^{\alpha}\cos\alpha. \tag{6}$$

As in [11], we assume $u_\zeta^{\alpha}=0$, since the analyzed surface wave field is not excited by the anti-plane motion. Thus, the original 3D problem of elasticity is reduced to a 2D problem in terms of the Radon transforms.
Let us now introduce the transformed wave potentials $\phi^{(\alpha)}$ and $\psi^{(\alpha)}$ as

$$u_\chi^{(\alpha)}=\frac{\partial\phi^{(\alpha)}}{\partial\chi}-\frac{\partial\psi^{(\alpha)}}{\partial x_3}\quad\text{and}\quad u_3^{(\alpha)}=\frac{\partial\phi^{(\alpha)}}{\partial x_3}+\frac{\partial\psi^{(\alpha)}}{\partial\chi}. \tag{7}$$

Then the boundary value problem (3) and (5) takes the form

$$\frac{\partial^2\phi^{(\alpha)}}{\partial\chi^2}+\frac{\partial^2\phi^{(\alpha)}}{\partial x_3^2}-\frac{1}{c_1^2}\frac{\partial^2\phi^{(\alpha)}}{\partial t^2}=0,$$

$$\frac{\partial^2\psi^{(\alpha)}}{\partial\chi^2}+\frac{\partial^2\psi^{(\alpha)}}{\partial x_3^2}-\frac{1}{c_2^2}\frac{\partial^2\psi^{(\alpha)}}{\partial t^2}=0, \tag{8}$$

and

$$\mu\left(2\frac{\partial^2\phi^{(\alpha)}}{\partial\chi\partial x_3}+\frac{\partial^2\psi^{(\alpha)}}{\partial\chi^2}-\frac{\partial^2\psi^{(\alpha)}}{\partial x_3^2}\right)=\frac{\partial P_0^{(\alpha)}}{\partial\chi},$$

$$(\kappa^2-2)\frac{\partial^2\phi^{(\alpha)}}{\partial\chi^2}+\kappa^2\frac{\partial^2\phi^{(\alpha)}}{\partial x_3^2}+2\frac{\partial^2\psi^{(\alpha)}}{\partial\chi\partial x_3}=0. \tag{9}$$

Here c_1 and c_2 are the longitudinal and transverse wave speeds, respectively, and $\kappa=c_1/c_2$.

A qualitative result should now be pointed out, namely, that the rotational component of the load P_1 does not contribute to excitation of the Rayleigh wave, being a natural implication of the fact that the surface wave is not excited by the anti-plane motion. Indeed, this may be readily observed from (9), containing only the gradient part of the load in the right hand side.

We proceed with the asymptotic scaling

$$\xi=\frac{\chi-c_Rt}{L},\qquad \gamma=\frac{x_3}{L},\qquad \tau=\frac{c_R\varepsilon}{L}t, \tag{10}$$

where $\varepsilon\ll 1$, L is the typical wavelength, and c_R denotes the Rayleigh wave speed. We remark that the physical interpretation of the parameter ε is a small deviation of the phase velocity of studied waves from the Rayleigh wave speed c_R. It should be noted that the boundary value problem (8) and (9) is formally identical to the case of tangential loading treated in [7].

The wave equations (8) are now rewritten in terms of the new variables (10) as

$$\frac{\partial^2\phi^{(\alpha)}}{\partial\gamma^2}+k_1^2\frac{\partial^2\phi^{(\alpha)}}{\partial\xi^2}+2\varepsilon\left(1-k_1^2\right)\frac{\partial^2\phi^{(\alpha)}}{\partial\xi\partial\tau}-\varepsilon^2\left(1-k_1^2\right)\frac{\partial^2\phi^{(\alpha)}}{\partial\tau^2}=0,$$

$$\frac{\partial^2\psi^{(\alpha)}}{\partial\gamma^2}+k_1^2\frac{\partial^2\psi^{(\alpha)}}{\partial\xi^2}+2\varepsilon\left(1-k_1^2\right)\frac{\partial^2\psi^{(\alpha)}}{\partial\xi\partial\tau}-\varepsilon^2\left(1-k_1^2\right)\frac{\partial^2\psi^{(\alpha)}}{\partial\tau^2}=0, \tag{11}$$

where $k_i^2=1-c_R^2/c_i^2,\ i=1,2$.

The two-term asymptotic solutions for Eqs. (11) are found as (cf. [11])

$$\phi^{(\alpha)} = \frac{P_* L^3}{\mu \varepsilon}\left(\phi^{(0)} + \varepsilon \phi^{(1)}\right), \qquad \psi^{(\alpha)} = \frac{P_* L^3}{\mu \varepsilon}\left(\psi^{(0)} + \varepsilon \psi^{(1)}\right), \tag{12}$$

where

$$\phi^{(1)} = \phi^{(1,0)} - \gamma \frac{1-k_1^2}{k_1}\frac{\partial \bar{\phi}^{(0)}}{\partial \tau}, \qquad \psi^{(1)} = \psi^{(1,0)} - \gamma \frac{1-k_2^2}{k_2}\frac{\partial \bar{\psi}^{(0)}}{\partial \tau}.$$

Here P_* is the maximal amplitude of P_0, and $\phi^{(0)} = \phi^{(0)}(\xi, k_1\gamma, \tau)$, $\phi^{(1,0)} = \phi^{(1,0)}(\xi, k_1\gamma, \tau)$, $\psi^{(0)} = \psi^{(0)}(\xi, k_2\gamma, \tau)$, $\psi^{(1,0)} = \psi^{(1,0)}(\xi, k_2\gamma, \tau)$ are arbitrary plane harmonic functions in the first two arguments, with the bar denoting a harmonic conjugate.

On substituting expressions (12) into the boundary conditions (9) and using the Cauchy-Riemann identities, we obtain at leading order

$$\begin{aligned} &2k_1 \frac{\partial^2 \phi^{(0)}}{\partial \xi^2} + \left(1+k_2^2\right)\frac{\partial^2 \bar{\psi}^{(0)}}{\partial \xi^2} = 0, \\ &\left(1+k_2^2\right)\frac{\partial^2 \phi^{(0)}}{\partial \xi^2} + 2k_2 \frac{\partial^2 \bar{\phi}^{(0)}}{\partial \xi^2} = 0, \end{aligned} \tag{13}$$

implying the classical Rayleigh wave equation

$$4k_1k_2 - \left(1+k_2^2\right)^2 = 0, \tag{14}$$

along with the relation

$$\bar{\psi}^{(0)} = -\frac{2k_1}{1+k_2^2}\phi^{(0)}, \quad \gamma = 0, \tag{15}$$

which has seemingly been first noted in [3] within the plane strain framework. At next order, the boundary conditions (9) become

$$\begin{aligned} &2\frac{\partial^2 \phi^{(1)}}{\partial \xi \partial \gamma} + \frac{\partial^2 \psi^{(1)}}{\partial \xi^2} - \frac{\partial^2 \psi^{(1)}}{\partial \gamma^2} = \frac{1}{P_* L^2}\frac{\partial P_0^{(\alpha)}}{\partial \xi}, \\ &(\kappa^2 - 2)\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + \kappa^2 \frac{\partial^2 \phi^{(1)}}{\partial \gamma^2} + 2\frac{\partial^2 \psi^{(1)}}{\partial \xi \partial \gamma} = 0, \end{aligned} \tag{16}$$

leading to

$$\frac{\partial^2 \psi^{(0)}}{\partial \xi \partial \tau} = \frac{\left(1+k_2^2\right)}{B P_* L^2}\frac{\partial P_0^{(\alpha)}}{\partial \xi}, \qquad \gamma = 0. \tag{17}$$

Re-casting Eq. (17) in terms of the dimensional variables (χ, x_3, t) we result in a hyperbolic equation for the transformed potential

$$\frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} - \frac{1}{c_R^2}\frac{\partial^2 \psi^{(\alpha)}}{\partial t^2} = \frac{\left(1+k_2^2\right)}{2\mu B}\frac{\partial P_0^{(\alpha)}}{\partial \chi}, \qquad x_3 = 0, \tag{18}$$

where

$$B = \frac{k_1}{k_2}\left(1-k_2^2\right) + \frac{k_2}{k_1}\left(1-k_1^2\right) - \left(1-k_2^4\right).$$

It may be deduced from relation (15) between the transformed potentials that

$$\frac{\partial \psi^{(\alpha)}}{\partial \chi} = -\frac{2}{1+k_2^2}\frac{\partial \phi^{(\alpha)}}{\partial x_3}, \qquad \text{or} \qquad \frac{\partial \phi^{(\alpha)}}{\partial \chi} = \frac{2}{1+k_2^2}\frac{\partial \psi^{(\alpha)}}{\partial x_3}, \tag{19}$$

serving together with (18) as boundary conditions for the elliptic equations

$$\frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} + k_1^2 \frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} = 0, \tag{20}$$

$$\frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2} + k_2^2 \frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} = 0. \tag{21}$$

Let us now introduce the quantities $\psi^\alpha \cos\alpha$ and $\psi^\alpha \sin\alpha$, and their inverse transforms ψ_1 and ψ_2. Using the properties of Radon transform ([19], see also [17]) the relations (18) - (21) are inverted.

The resulting asymptotic formulation for the Rayleigh wave field in case of an in-plane surface loading is given by pseudo-static (not involving time in explicit form) scaled Laplace equations over the interior of a half-space ($x_3 > 0$)

$$\frac{\partial^2 \phi}{\partial x_3^2} + k_1^2 \Delta_2 \phi = 0, \tag{22}$$

$$\frac{\partial^2 \psi_i}{\partial x_3^2} + k_2^2 \Delta_2 \psi_i = 0, \tag{23}$$

where $\Delta_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, with the boundary conditions on the surface $x_3 = 0$ specified as a 2D hyperbolic equation

$$\Delta_2 \psi_i - \frac{1}{c_R^2} \frac{\partial^2 \psi_i}{\partial t^2} = \frac{2\left(1 + k_2^2\right)}{\mu B} \frac{\partial P_0}{\partial x_i}, \qquad (i = 1, 2). \tag{24}$$

along with relations between the potentials mirroring the representation of the Rayleigh wave field in terms of a single harmonic function [4], given by

$$\frac{\partial \phi}{\partial x_i} = \frac{2}{1 + k_2^2} \frac{\partial \psi_i}{\partial x_3}, \qquad \frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} = -\frac{2}{1 + k_2^2} \frac{\partial \phi}{\partial x_3}, \qquad (i = 1, 2). \tag{25}$$

The displacement components are expressed through the potentials ϕ, ψ_i as

$$\mathbf{u} = \operatorname{grad} \phi + \operatorname{curl} \psi, \qquad \psi = (-\psi_2, \psi_1, 0), \tag{26}$$

see also [15].

Thus, given an in-plane load $\mathbf{P}$, its gradient part may be separated from (4). Equation (24) then provides a boundary condition for the elliptic problem (23). The relations (25) may then be used as boundary conditions for (22).

4 A model example

Let us illustrate the asymptotic formulation for the Rayleigh wave derived in the previous section by a model example for the impulse point load acting along one of the in-plane axis, say Ox_1, namely

$$\mathbf{P} = \left(A\delta(x_1)\delta(x_2)\delta(t); 0; 0\right). \tag{27}$$

The load $\mathbf{P}$ may then be decomposed through (4), giving

$$\Delta P_0 = A\delta'(x_1)\delta(x_2)\delta(t). \tag{28}$$

Using the fundamental solution for the 2D Laplace equation [20]

$$\mathcal{E}(x_1, x_2) = \frac{1}{4\pi} \ln(x_1^2 + x_2^2),$$

the value of P_0 may be obtained as a convolution of the latter with the right hand side $A\delta'(x_1)\delta(x_2)\delta(t)$

$$P_0(x_1, x_2, t) = \mathcal{E}(x_1, x_2) * A\delta'(x_1)\delta(x_2)\delta(t) = \frac{A}{2\pi} \frac{x_1}{x_1^2 + x_2^2} \delta(t). \tag{29}$$

We may now use the obtained value of P_0 to solve the hyperbolic equations (24)

$$\Delta_2 \psi_1 - \frac{1}{c_R^2} \frac{\partial^2 \psi_1}{\partial t^2} = -2A_0 \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2} \delta(t), \tag{30}$$

and

$$\Delta_2 \psi_2 - \frac{1}{c_R^2} \frac{\partial^2 \psi_2}{\partial t^2} = 4A_0 \frac{x_1 x_2}{(x_1^2 + x_2^2)^2} \delta(t), \tag{31}$$

where

$$A_0 = \frac{A\left(1+k_2^2\right)}{\pi\mu B}.$$

Applying a double Fourier transform and Laplace transform to Eq. (30), we obtain

$$\psi_1^{FFL} = \frac{2A\left(1+k_2^2\right)}{\pi\mu B}\frac{p^2}{s^2(p^2+q^2)+c_R^2(p^2+q^2)^2},$$

where (p, q) is the Fourier transform parameter and s is the Laplace transform parameter. Taking inverse Laplace transform gives

$$\psi_1^{FF} = 2A_0\frac{p^2}{(p^2+q^2)}\frac{\sin(c_R\kappa t)}{c_R\kappa}, \tag{32}$$

where $\kappa^2 = p^2 + q^2$. Employing the inverse double Fourier transform we arrive at

$$\psi_1(x_1, x_2, 0, t) = \frac{A_0}{2\pi^2 c_R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p^2}{\kappa(p^2+q^2)}\sin(c_R\kappa t)\mathrm{e}^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\,d\boldsymbol{\kappa},$$

where $\mathbf{r} = (x, y) = (r\cos\theta, r\sin\theta)$ where $|\mathbf{r}| = r$, and $\boldsymbol{\kappa} = (p, q) = (\kappa\cos\phi, \kappa\sin\phi)$ with $|\boldsymbol{\kappa}| = \kappa$. The above integral may then be rewritten as

$$\psi_1(r, \theta, 0, t) = \frac{A_0}{2\pi^2 c_R}\int_0^{\infty}\sin(c_R\kappa t)\left\{\frac{1}{4}\int_0^{2\pi}\mathrm{e}^{2i\phi}e^{ir\kappa\cos(\theta-\phi)}d\phi + \frac{1}{4}\int_0^{2\pi}\mathrm{e}^{-2i\phi}e^{ir\kappa\cos(\theta-\phi)}d\phi + \frac{1}{2}\int_0^{2\pi}\mathrm{e}^{ir\kappa\cos(\theta-\phi)}d\phi\right\}d\kappa. \tag{33}$$

The latter may be simplified through Hankel transforms [21]. The first of the integrals is evaluated as

$$\int_0^{2\pi}\mathrm{e}^{2i\phi}e^{ir\kappa\cos(\theta-\phi)}d\phi = \int_{\theta_0}^{2\pi+\theta_0}\mathrm{e}^{\theta+\alpha+\pi/2}\mathrm{e}^{ir\kappa\cos(\alpha+\pi/2)}d\alpha = -\mathrm{e}^{2i\theta}\int_{\theta_0}^{2\pi+\theta_0}\mathrm{e}^{i(2\alpha-r\kappa\sin\alpha)}d\alpha = -2\pi\mathrm{e}^{2i\theta}J_2(r\kappa), \tag{34}$$

where $\theta - \phi = -(\alpha + \pi/2)$, $\theta_0 = -(\theta + \pi/2)$ and $J_2(r\kappa)$ is a Bessel function of the first kind.

Similarly, the second and third integrals give

$$\int_0^{2\pi}\mathrm{e}^{-2i\phi}\mathrm{e}^{ir\kappa\cos(\theta-\phi)}d\phi = -2\pi\mathrm{e}^{-2i\theta}J_{-2}(r\kappa),$$

and

$$\int_0^{2\pi}\mathrm{e}^{ir\kappa\cos(\theta-\phi)}d\phi = 2\pi J_0(r\kappa),$$

respectively. Hence

$$\psi_1(r, \theta, 0, t) = -\frac{A_0}{2\pi c_R}\int_0^{\infty}\sin(c_R\kappa t)\left\{\cos 2\theta J_2(r\kappa) + J_0(r\kappa)\right\}d\kappa. \tag{35}$$

Using the properties of Bessel functions, the integral (35) results in

$$\psi_1(r,\theta,0,t) = \frac{A_0 c_R}{\pi}\left\{\left[\cos 2\theta\left(\frac{c_R^2 t^2}{r^2}-\frac{1}{2}\right)-\frac{1}{2}\right]\right\}\frac{H(c_R t - r)}{\sqrt{c_R^2 t^2 - r^2}}. \tag{36}$$

Performing similar calculation for ψ_2 gives

$$\psi_2(r,\theta,0,t) = \frac{A_0 c_R}{2\pi}\left\{\sin 2\theta\left(1-\frac{c_R^2 t^2}{r^2}\right)\right\}\frac{H(c_R t - r)}{\sqrt{c_R^2 t^2 - r^2}}.$$

Using (25) and (26), we obtain for normal displacement at the surface $x_3 = 0$

$$u_3\mid_{x_3=0} = \frac{\partial\phi}{\partial x_3}+\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2} = \frac{1-k_2^2}{2}\left(\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2}\right). \tag{37}$$

On employing the polar coordinates and using the dimensionless scaling $\xi = \dfrac{r}{c_R t}$ it is possible to express the scaled displacement as

$$U_3 = f_1(\xi)\cos\theta + f_2(\xi)\cos 3\theta, \qquad 0 \le \xi \le 1, \tag{38}$$

where

$$U_3 = \frac{2\pi r^2 u_3}{A_0 c_R\left(1-k_2^2\right)}, \quad f_1(\xi) = \frac{(1-2\xi^2)\xi}{4(1-\xi^2)^{3/2}}, \quad f_2(\xi) = \frac{-6\xi^4+19\xi^2-12}{4\xi(1-\xi^2)^{3/2}}.$$

Figures 2, 3 contain plots of cross-sections of the scaled normal displacement U_3 defined by (38). On Fig. 2 the dependence of U_3 on the dimensionless variable ξ is shown for $\theta = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi$. We note that the discontinuities at $\xi = 0$ and $\xi = 1$ are clearly associated with impulse point load and the wave front, respectively. It may also be noticed from (38) that $U_3 = 0$ at $\theta = \frac{\pi}{2}$. In other words zero normal displacement of the surface corresponds to the direction which is perpendicular to the direction of the applied surface loading, which may be expected intuitively. Figure 3 shows dependence of U_3 on the angle θ $(0 \le \theta < 2\pi)$ for several values of ξ. Not surprisingly, the magnitude of displacement grows as ξ becomes closer to either the origin or the wave front. It may also be observed from both figures that the increase of the magnitude as $\xi \to 1$ is sharper than that in the vicinity of the origin.

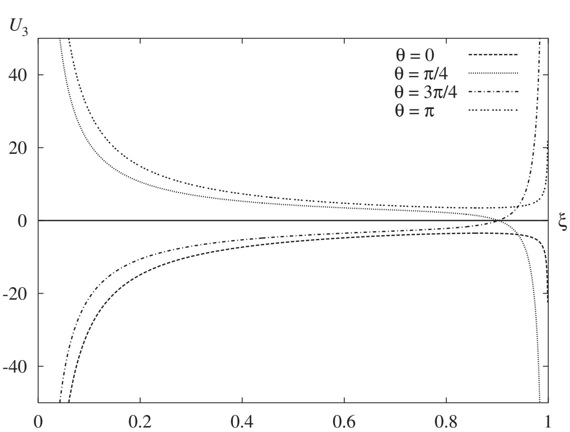

Fig. 2 Dependence of the vertical displacement on the surface on the scaled radial variable.

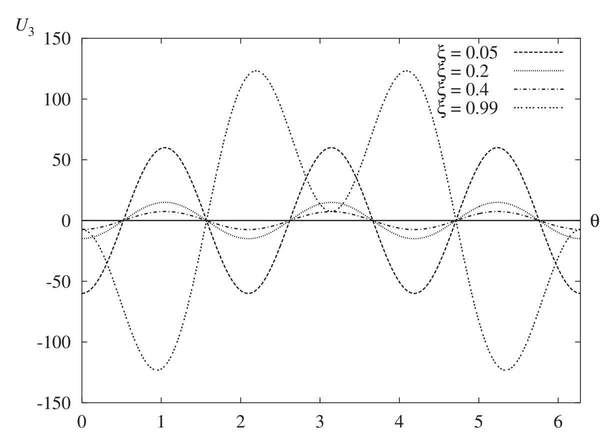

Fig. 3 Dependence of vertical displacement on the surface on the polar angle.

Concluding remarks

In this paper the methodology of asymptotic models for the Rayleigh wave field has been extended to the case of in-plane loading of a 3D isotropic elastic half-space. The results on the boundary containing 2D wave equations (24) for the shear potentials ψ_1 and ψ_2 are complementary to these obtained in [11] for the longitudinal potential ϕ, along with further clarification of the relations between the potentials at the surface. Thus, the case of arbitrary surface loading may now be treated. In addition, a non-trivial result that the rotational part of the in-plane load does not contribute to excitation of the surface wave, has been established. A considered model example of concentrated impulse force acting along one of the in-plane axis, has demonstrated the efficiency of the approach, allowing a rather straightforward analytical solution for the vertical displacement on the surface.

The obtained results open wide prospectives for modelling ground vibrations arising from the wind turbines, when the turbine is modelled as an elastic beam, causing not only vertical, but also an in-plane surface loading. Another direction of further development is associated with the interfacial waves, with recent progress reported in [4, 22], and [15]. The effects of anisotropy and pre-stress may also be incorporated, with some preliminary results presented in [23]. Generalization to dissipative media is not very straightforward in view of the fact that surface wave would decay rapidly and not propagate, however, the case of thin viscoelastic coating bonded to isotropic elastic half-space may be considered. Finally, we note a possible further extension to lateral inhomogeneity.

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