Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 687951, 16 pages
doi:10.1155/2010/687951

*Research Article*

# On Stability of Parametrized Families of Polynomials and Matrices

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Received 1 June 2010; Revised 2 August 2010; Accepted 13 September 2010

Academic Editor: Ülle Kotta



The Schur and Hurwitz stability problems for a parametric polynomial family as well as the Schur stability problem for a compact set of real matrix family are considered. It is established that the Schur stability of a family of real matrices $\mathcal{A}$ is equivalent to the nonsingularity of the family $\{A^2 - 2tA + I : A \in \mathcal{A}, t \in [-1,1]\}$ if $\mathcal{A}$ has at least one stable member. Based on the Bernstein expansion of a multivariable polynomial and extremal properties of a multilinear function, fast algorithms are suggested.

## 1. Introduction

Let $\mathbb{R}^n$ ($\mathbb{R}$) be the set of real $n$ vectors (numbers), $\mathbb{C}$ the set of complex numbers. Let a polynomial family be defined by

$$p(z,\mathbf{q}) = a_0(\mathbf{q}) + a_1(\mathbf{q})z + \cdots + a_n(\mathbf{q})z^n, \tag{1.1}$$

where the uncertainty vector $\mathbf{q}$ belongs to a box $Q$

$$Q = \left\{(q_1, q_2, \ldots, q_l) \in \mathbb{R}^l : q_i^- \le q_i \le q_i^+,\ i = 1, 2, \ldots, l\right\}. \tag{1.2}$$

Denote the set of all polynomials $p(z,\mathbf{q})$ by $\mathcal{P}$, that is,

$$\mathcal{P} = \{p(z,\mathbf{q}) : \mathbf{q} \in Q\}. \tag{1.3}$$

The family $\mathcal{P}$ is said to be Schur (Hurwitz) stable if every polynomial in this family is Schur (Hurwitz) stable, that is, all roots lie in the open unit disc (open left half plane). A similar definition is valid for a matrix family where the word "roots" is replaced by "eigenvalues".

If $a_i(\mathbf{q}) = q_i$ $(i = 0, 1, \ldots, n)$, then the family (1.3) is called an interval polynomial family. The Hurwitz stability problem of interval polynomials is solved by Kharitonov Theorem [1]. The Schur stability problem of interval polynomials has been studied in many works (see [2–5] and references therein). In [3, 5] using techniques from complex analysis, necessary conditions for the Schur stability of interval polynomials are obtained.

A function $a(\cdot) : Q \to \mathbb{R}^k$ is said to be multilinear if it is affine-linear with respect to each component of $\mathbf{q} \in Q$. The polynomial (1.1) is called multilinear if all coefficient functions $a_i(\mathbf{q})$ $(i = 0, 1, \ldots, n)$ are multilinear. The family (1.1) is called polynomially parameter dependent if all coefficient functions are depending polynomially on parameters $q_i$, $(i = 1, 2, \ldots, l)$.

In [6] an algorithm for the robust Schur stability verification of polynomially parameter dependent families is given. This algorithm relies on the Bernstein expansion of a multivariable polynomial and is based on the decomposition of a polynomial into its symmetric and antisymmetric parts, and on the Chebyshev polynomials of the first and second kinds. In this paper we investigate the robust Schur stability of polynomially dependent families without employing Chebyshev polynomials (cf. [6]) (see Sections 2 and 4).

The following theorems express the well-known properties of a multilinear function defined on a box.

**Theorem 1.1** (see [2]). *Suppose that $Q \subset \mathbb{R}^l$ is a box with extreme points $\{\mathbf{q}^i\}$, $(i = 1, 2, \ldots, k)$ and $f(\cdot) : Q \to \mathbb{R}$ is multilinear. Then both the maximum and the minimum of $f(\cdot)$ are attained at extreme points of $Q$. That is,*

$$\begin{aligned} \max_{\mathbf{q}\in Q} f(\mathbf{q}) &= \max_i f\left(\mathbf{q}^i\right), \\ \min_{\mathbf{q}\in Q} f(\mathbf{q}) &= \min_i f\left(\mathbf{q}^i\right). \end{aligned} \tag{1.4}$$

Theorem 1.1 leads to the following sufficient condition for stability of the multilinear family (1.3).

**Theorem 1.2** (see [2]). *Consider a family of polynomials $\mathcal{P}$ (1.3) with invariant degree, multilinear uncertainty structure, and at least one Schur (Hurwitz) stable member $p(z, \mathbf{q}^0)$. In addition, $Q$ is the box with extreme points $\{\mathbf{q}^i\}$, $(i = 1, 2, \ldots, k)$ given by (1.2). Then the family of multilinear polynomials $\mathcal{P}$ is Schur stable (Hurwitz stable) if the Zero Exclusion Condition*

$$0 \notin \operatorname{conv}\left\{p\left(z, \mathbf{q}^i\right) : i = 1, 2, \ldots, k\right\} \tag{1.5}$$

*is satisfied for all $z \in \partial\mathfrak{D}$, where $\partial\mathfrak{D}$ denotes the boundary of the unit disc (imaginary axis).*

It is well known that a multilinear polynomial family appears quite frequently in practical applications. The characteristic polynomial of an interval matrix is a multilinear polynomial. The mapping between the coefficient vector of a polynomial and its reflection vector is also multilinear. In [7], using this multilinearity, as well as Theorem 1.1 and a splitting procedure of the box of reflection coefficients, new conditions for the Schur stability are given.

The application of Theorem 1.2 for determining the stability of polynomials with multilinear coefficients yield conservative results. In [8, 9] sufficient conditions are given for ensuring that the image of a multilinear function over the box $Q$ is a convex polygon whose edges are images of the edges of the box $Q$. In this case the stability can be tested by the Edge Theorem [2]. In [10], the notion of the principal point of $Q$ is introduced, and it is shown that for a multilinear mapping $f : Q \to \mathbb{C}$, the boundary of $f(Q)$ is a subset of $f(P)$, where $P$ is the set of all principal points. Application of the results obtained to stability of one important subclass of multilinear systems is given in [10]. In [11], using the notion of generalized principal point the characterization of the smallest set of regions in the complex plane within which the roots of (1.1) lie, is given.

As far as the recent works on the stability of multilinear families are concerned, we can refer the reader to [12, 13]. In [13], a multilinear family which can be expressed as the product of independent linear structures is considered. The paper suggests an elimination approach which eliminates the vertices $\mathbf{q}^i \in Q$ that are not useful for the construction of the boundary of the value set $f(Q)$. In [12] a sufficient condition for the zero inclusion of the value set $f(Q)$ is given, where $f : Q \to \mathbb{C}$ is multilinear. On the basis of this condition a numerical procedure for testing the whether or not $f(Q)$ includes the origin is given. The procedure uses the iterative subdivision of the box $Q$.

In this paper we suggest a new simple algorithm for testing Schur stability of a multilinear family. This algorithm is based on Theorem 1.1 and is sufficiently fast.

The Schur stability (rather than Hurwitz stability) has the following advantage. In the obtained results, the segment $[-2, 2]$ arises naturally (see Theorem 1.1, Algorithm 3.1, Theorem 5.1) whereas the cutoff frequencies should be calculated in the Hurwitz stability problems. On the other hand the Hurwitz stability can also be tested by this algorithm, since by using the well-known transformation $s = (z+1)/(z-1)$, the Hurwitz stability problem can be transformed into Schur stability problem (see Example 3.3 taken from [14]).

In the second part of the paper, we consider the application of our approach to matrix Schur stability problem. Stability problem of matrix families has been studied in many works (see [2, 4, 15–18] and references therein). Naturally, a great deal of research has been devoted to interval matrices. Interval matrix structures are ubiquitous in nature and engineering. In [15, 18] extreme point results for Hurwitz stability are obtained which expresses the stability conditions in terms of extreme matrices. In [17], using the notion of a block P-matrix a characterization of the Schur stability of all convex combinations of Schur stable matrices is derived.

We consider the Schur stability problem for a family $\mathcal{A}$, which is a pathwise connected real matrix family. We show that Schur stability of $\mathcal{A}$ is equivalent to the nonsingularity of an extended family. A similar problem for the Hurwitz stability is considered in [16] where $\mathcal{A}$ is a polytope.

As pointed out above this paper addresses the following points:

(1) Robust Schur stability of polynomially dependent polynomials without involving Chebyshev polynomials (see (2.3)).

(2) A new algorithm (multilinearization) for a multilinear family (Algorithm 3.1).

(3) Robust Schur stability criteria for a real matrix family via nonsingularity of an extended family.

In the computational procedures, we use Theorem 1.1 and the Bernstein expansion of a multivariable polynomial developed in [19–21].

The examples were run on a 2.53 GHz Intel Core2 Duo with 4 GB of RAM.

## 2. Stability of a Polynomially Dependent Family

Consider the family (1.1), where $a_i(\mathbf{q})$ are polynomials. Here we give two polynomial equations defined on a box and show that the Schur stability is equivalent to the nonexistence of common solutions to these equations (Theorem 2.1).

Suppose that the points $z = \pm 1$ are not the roots of $\mathcal{P}$ and $\mathcal{P}$ has at least one stable member. Suppose that the family (1.3) is not Schur stable. Then, by continuity of roots (cf. page 52 in [2]), there exists $\theta \in (0, \pi)$ such that $z = e^{j\theta}$ is a root of $\mathcal{P}$, where $j^2 = -1$. Then $z = e^{-j\theta}$ is also a root, and there exist $b_0, b_1, \dots, b_{n-2}$ such that

$$\begin{aligned} a_0(\mathbf{q}) + a_1(\mathbf{q})z + \cdots + a_n(\mathbf{q})z^n &= \left(z - e^{j\theta}\right)\left(z - e^{-j\theta}\right) \\ &\quad\cdot \left(b_0 + b_1 z + \cdots + b_{n-2} z^{n-2}\right) \\ &= \left(z^2 - 2\cos\theta z + 1\right) \\ &\quad\cdot \left(b_0 + b_1 z + \cdots + b_{n-2} z^{n-2}\right) \end{aligned} \tag{2.1}$$

is valid. Taking $t = 2\cos\theta$ in (2.1), it follows that the equalities

$$\begin{aligned} b_0 &= a_0(\mathbf{q}) \\ b_1 - t b_0 &= a_1(\mathbf{q}) \\ b_2 - t b_1 + b_0 &= a_2(\mathbf{q}) \\ b_3 - t b_2 + b_1 &= a_3(\mathbf{q}) \\ &\vdots\vdots\vdots \\ b_k - t b_{k-1} + b_{k-2} &= a_k(\mathbf{q}) \\ &\vdots\vdots\vdots \\ b_{n-2} - t b_{n-3} + b_{n-4} &= a_{n-2}(\mathbf{q}) \\ b_{n-3} - t b_{n-2} &= a_{n-1}(\mathbf{q}) \\ b_{n-2} &= a_n(\mathbf{q}) \end{aligned} \tag{2.2}$$

are satisfied.

Eliminating $b_0, b_1, \dots, b_{n-2}$ in (2.2) reduces the system of equations (2.2) to the system

$$\begin{aligned} f_1(t, \mathbf{q}) &= 0, \\ f_2(t, \mathbf{q}) &= 0. \end{aligned} \tag{2.3}$$

Thus we obtain the following result.

**Theorem 2.1.** *Assume that the points $z = \pm 1$ are not the roots of $\mathcal{P}$ and $\mathcal{P}$ has at least one stable member. Then $\mathcal{P}$ is Schur stable if and only if the system* (2.3) *has no solution on* $[-2, 2] \times Q \subset \mathbb{R}^{l+1}$.

## 3. Special Case: Stability of a Multilinear Family, Multilinearization

Let the family (1.1) be given, where the functions $a_i(\mathbf{q})$ are multilinear ($i = 0, 1, 2, \dots, n$). In the case of a multilinear family, using Theorem 1.1, we can easily check whether the family (1.1) has roots $z = \pm 1$.

Consider the system (2.3). In (2.3) both $f_1$ and $f_2$ are multilinear on $\mathbf{q}$ and polynomially depend on $t$. More precisely, the greatest powers of $t$ in $(f_1, f_2)$ are

$$\begin{aligned} &(t, t) \quad \text{for } n = 3, \\ &\left(t^2, t\right) \quad \text{for } n = 4, \\ &\left(t^2, t^2\right) \quad \text{for } n = 5, \\ &\left(t^3, t^2\right) \quad \text{for } n = 6, \\ &\left(t^3, t^3\right) \quad \text{for } n = 7, \end{aligned} \tag{3.1}$$

and so on. The advantage of system (2.3) is that it is "almost" multilinear and the variables $(t, \mathbf{q})$ vary on the box $[-2, 2] \times Q$. This system can be transformed into a multilinear system by introducing new variables. Indeed, if system (2.3) contains $t^k$ as the greatest power of $t$, we can replace $t^k$ by the product $t_1 t_2 \cdots t_k$ and add new equations $t_2 - t_1 = 0$, $t_3 - t_1 = 0, \dots, t_k - t_1 = 0$ to (2.3) (we set $t = t_1$) (cf. [22]). This extended system will then be a multilinear system defined on a box, and Theorem 1.1 will be applicable. For example, assume that $n = 4$. Then system (2.3) becomes

$$\begin{aligned} a_0(\mathbf{q}) - a_2(\mathbf{q}) + a_4(\mathbf{q}) - t a_1(\mathbf{q}) - t^2 a_0(\mathbf{q}) &= 0, \\ a_1(\mathbf{q}) - a_3(\mathbf{q}) + t a_0(\mathbf{q}) - t a_4(\mathbf{q}) &= 0, \end{aligned} \tag{3.2}$$

and the extended system will be

$$\begin{aligned} a_0(\mathbf{q}) - a_2(\mathbf{q}) + a_4(\mathbf{q}) - t_1 a_1(\mathbf{q}) - t_1 t_2 a_0(\mathbf{q}) &= 0, \\ a_1(\mathbf{q}) - a_3(\mathbf{q}) + t_1 a_0(\mathbf{q}) - t_1 a_4(\mathbf{q}) &= 0, \\ t_2 - t_1 &= 0, \end{aligned} \tag{3.3}$$

where $(t_1, t_2, \mathbf{q}) \in [-2,2] \times [-2,2] \times Q$. The family $\mathcal{P}$ is Schur stable if and only if the system of multilinear equations (3.3) has no solution on the box $\widetilde{Q} = [-2,2] \times [-2,2] \times Q$. By Theorem 1.1 the range of each function from (3.3) over the box $\widetilde{Q}$ can be easily and exactly calculated. If one of these ranges does not contain zero, then system (3.3) has no solution on this box. Otherwise, the initial box $\widetilde{Q}$ should be divided into small subboxes, and the new ranges over these small boxes should be calculated. A subbox on which at least one range does not contain zero will be eliminated since the system (3.3) has no solution on this subbox.

If the family $\mathcal{P}$ is Schur stable, then all subboxes will be eliminated after a finite number of steps.

Using the above procedure, we get the following algorithm for checking the Schur stability of a multilinear family.

*Algorithm 3.1.* Let the multilinear family $\mathcal{P}$ be given, where $Q \subset \mathbb{R}^l$.

(1) Using Theorem 1.1, check the nonexistence of the roots $z = \pm 1$ (according to this theorem, only extreme points are sufficient for this checking). Otherwise, $\mathcal{P}$ is unstable.

(2) Obtain the equations

$$f_1(t, \mathbf{q}) = 0, \qquad f_2(t, \mathbf{q}) = 0. \tag{3.4}$$

(3) Multilinearize this system by replacing $t = t_1$ and introducing new variables $t_2, t_3, \dots, t_k$ and new equations

$$\begin{gathered} f_1(t_1, \dots, t_k, \mathbf{q}) = 0, \ f_2(t_1, \dots, t_k, \mathbf{q}) = 0, \\ t_2 - t_1 = 0, \ t_3 - t_1 = 0, \dots, t_k - t_1 = 0, \end{gathered} \tag{3.5}$$

where $(t_1, t_2, \dots, t_k, \mathbf{q}) \in \widetilde{Q} = \underbrace{[-2,2] \times \cdots \times [-2,2]}_{k} \times Q \subset \mathbb{R}^{k+l}$.

(4) Check for stability of (1.3) in all extreme points of the box $\widetilde{Q}$. If there is an extreme point $\mathbf{q}^i$ such that the polynomial $p(z, \mathbf{q}^i)$ is not Schur stable, then stop. The family $\mathcal{P}$ is unstable. Otherwise, apply the next step.

(5) Using Theorem 1.1, calculate the ranges of all functions in (3.5). If at least one range does not contain zero, then stop. The family of $\mathcal{P}$ is stable. Otherwise, apply the next step.

(6) Divide the box $\widetilde{Q}$ into two boxes in the chosen coordinate direction. For each subbox repeat the steps (2)–(5). Eliminate a subbox on which one range does not contain zero.

The algorithm is finished if

(a) all subboxes are eliminated

or

(b) for a given $\varepsilon > 0$ the size of all remaining boxes are less then $\varepsilon$.

In case (a), the family is stable. In case (b), it is expected that the family is unstable, and check for unstability of the family for $q = q^*$, where $q^*$ is the center of a remaining box.

We have solved a number of examples using Algorithm 3.1. These examples show that this algorithm is sufficiently fast.

*Example 3.2.* Consider the family

$$\begin{aligned} p(z,\mathbf{q}) = {} & (30q_1 + 40q_1q_2 + 65)z^5 + (5.1q_1 + 0.01q_2 + q_1q_2 + 26.1)z^4 \\ & + (-22 + 0.2q_1 + 4q_1q_2)z^3 + (6q_1q_2 - 0.02q_2 - 10q_1 - 10.2)z^2 \\ & + (-18 + 4q_1q_2 - 0.2q_1)z + 0.01q_2 + 4.9q_1 + q_1q_2 + 24.1, \quad q_1 \in [1,2], \; q_2 \in [1,2]. \end{aligned} \tag{3.6}$$

The system (3.5), in this case, becomes

$$\begin{gathered} t_1a_5(q_1,q_2) + a_4(q_1,q_2) - t_1t_2a_0(q_1,q_2) - t_1a_1(q_1,q_2) + a_0(q_1,q_2) - a_2(q_1,q_2) = 0, \\ t_1a_0(q_1,q_2) + a_1(q_1,q_2) - t_1t_2a_5(q_1,q_2) - t_1a_4(q_1,q_2) - a_3(q_1,q_2) + a_5(q_1,q_2) = 0, \\ t_2 - t_1 = 0, \end{gathered} \tag{3.7}$$

and $(t_1,t_2,q_1,q_2) \in \widetilde{Q} = [-2,2] \times [-2,2] \times [1,2] \times [1,2] \subset \mathbb{R}^4$. Algorithm 3.1 reports after 0.078 sec in 11 steps that this family is Schur stable.

The solution processes show that if a family is unstable, the division of the box $\widetilde{Q}$ in all directions is more effective. In this case the division process will lead to a such value of $\mathbf{q} \in Q$, for which the polynomial $p(z,\mathbf{q})$ is unstable.

The following example is taken from [14].

*Example 3.3* (see [14]). Consider, Hurwitz stability problem for the interval matrix family

$$A(\mathbf{q}) = \begin{pmatrix} q_1 & -12.06 & -0.06 & 0 \\ -0.25 & -0.03 & 1 & 0.5 \\ 0.25 & -4 & -1.03 & 0 \\ 0 & 0.5 & 0 & q_2 \end{pmatrix}, \quad q_1 \in [-1.5,-0.5], \; q_2 \in [-4,-1]. \tag{3.8}$$

The characteristic polynomial of $A(\mathbf{q})$ is

$$\begin{aligned}\tilde{p}(s,\mathbf{q}) &= s^4 + (-q_1 - q_2 + 1.06)s^3 + (-1.06q_2 + 0.7809 - 1.06q_1 + q_1q_2)s^2 \\ &\quad + (-3.7809q_1 - 1.0309q_2 - 0.2875 + 1.06q_1q_2)s \\ &\quad + 0.2575q_1 - 0.00375 + 0.03q_2 + 4.0309q_1q_2.\end{aligned} \tag{3.9}$$

We apply the transformation $s = (z+1)/(z-1)$ to the polynomial (3.9) and obtain

$$\begin{aligned}p(z,\mathbf{q}) &= (z-1)^4\tilde{p}\left(\frac{z+1}{z-1},\mathbf{q}\right) \\ &= (-5.5834q_1 - 3.0609q_2 + 6.0909q_1q_2 + 2.54965)z^4 \\ &\quad + (4.5318q_1 - 0.0582q_2 - 18.2436q_1q_2 + 6.71)z^3 \\ &\quad + (22.1854q_1q_2 + 4.4157 + 3.665q_1 + 2.3q_2)z^2 \\ &\quad + (1.32 - 0.1818q_2 - 14.0036q_1q_2 - 6.5918q_1)z \\ &\quad + 3.9784q_1 + 1.0009q_2 + 3.9709q_1q_2 + 1.00465.\end{aligned} \tag{3.10}$$

Therefore, Hurwitz stability of the family (3.9) is equivalent to the Schur stability of the family (3.10) (see [2], page 221). For this example, the box $\tilde{Q}$ is $[-2,2] \times [-2,2] \times [-1.5,-0.5] \times [-4,-1] \subset \mathbb{R}^4$. By dividing the box $\tilde{Q}$ in all directions, Algorithm 3.1 reports after 0.7 sec that this family is not Schur stable. The polynomial $p(z,\mathbf{q})$ (3.10) becomes Schur unstable for an extreme point in which $q_1 = -1$, $q_2 = -2.5$. Therefore the interval matrix family $A(\mathbf{q})$ is not Hurwitz stable. Note that, in [14], this unstability has been established through the solutions of at least 1200 linear programming problems.

## 4. Bernstein Expansion

One of the methods for checking the positivity of a multivariable polynomial on a box is the method of Bernstein expansion developed in [19–21, 23]. Let us briefly describe this algorithm.

Let $L = (i_1, i_2, \ldots, i_m)$ be an $m$-tuple of nonnegative integers, and for $\mathrm{x} = (x_1, x_2, \ldots, x_m)$

$$\mathrm{x}^L = x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}. \tag{4.1}$$

For $N = (n_1, \ldots, n_m)$,

$$L \le N \Longleftrightarrow 0 \le i_k \le n_k \quad (k = 1, 2, \ldots, m). \tag{4.2}$$

An $m$-variate polynomial $p(\mathrm{x})$ is defined as

$$p(\mathrm{x}) = \sum_{L \le N} a_L \mathrm{x}^L, \quad (\mathrm{x} \in \mathbb{R}^m). \tag{4.3}$$

Here $N$ is called the degree of the polynomial $p(\mathrm{x})$.

The $i$th Bernstein polynomial of degree $d$ is defined as

$$b_{d,i}(x) = \binom{d}{i} x^i (1-x)^{d-i}, \quad 0 \le i \le d. \tag{4.4}$$

In the multivariate case, the $L$th Bernstein polynomial of degree $N$ is defined by

$$B_{N,L}(\mathrm{x}) = b_{n_1,i_1}(x_1) \cdots b_{n_m,i_m}(x_m) \quad (\mathrm{x} \in \mathbb{R}^m). \tag{4.5}$$

The transformation of a polynomial from its power form (4.3) into its Bernstein form results in

$$p(\mathrm{x}) = \sum_{L \le N} p_L(U) B_{N,L}(\mathrm{x}), \tag{4.6}$$

where the Bernstein coefficients $p_L(U)$ of $p$ over the $m$-dimensional unit box $U = [0,1] \times \cdots \times [0,1]$ are given by

$$p_L(U) = \sum_{J \le L} \frac{\binom{L}{J}}{\binom{N}{J}} a_J, \quad (L \le N). \tag{4.7}$$

Here $\binom{N}{L}$ is defined as the product $\binom{n_1}{i_1} \cdots \binom{n_m}{i_m}$. In [20], a matrix method for computing the Bernstein coefficients efficiently is described.

Denote

$$\begin{aligned} \underline{m} &= \min\{p(\mathrm{x}) : \mathrm{x} \in U\}, & \overline{m} &= \max\{p(\mathrm{x}) : \mathrm{x} \in U\}, \\ \alpha &= \min\{p_L(U) : L \le N\}, & \beta &= \max\{p_L(U) : L \le N\}. \end{aligned} \tag{4.8}$$

**Theorem 4.1** (see [19]). *The inequalities*

$$\alpha \le \underline{m} \le \overline{m} \le \beta \tag{4.9}$$

*are satisfied.*

Theorem 4.1 gives the bounds for the range of the multivariate polynomial (4.3) over the unit box $U$. In order to obtain the Bernstein coefficients and bounds over an arbitrary box $D$, the box $D$ should be affinely mapped onto $U$. As a result a new polynomial is obtained, and its Bernstein coefficients are the Bernstein coefficients of the initial polynomial $p(\mathrm{x})$ (4.3) over $D$.

In order to obtain convergent bounds for the range of polynomial (4.3) over a box $D$, the box $D$ should be divided into two boxes. If the division is continued and one calculates the minimal and maximal Bernstein coefficients in each subdivision step, the calculated bounds converge to the exact bounds (provided that the diameter of subboxes tends to zero). Note that by the sweep procedure the explicit transformation of the subboxes generated by sweeps back to $D$ is avoided.

It is known that the number of Bernstein coefficients is often very large. In [23], a new technique which avoids the computation of all the Bernstein coefficients is represented for evaluating $\underline{m}$ and $\overline{m}$ from Theorem 4.1. The computational complexity of this technique is only nearly linear in the number of the terms of the polynomial.

Bernstein expansion can also be applied in order to check whether two polynomial equations (for example, system (2.3)) have a common zero in a box $D$ [6]. Indeed, consider system (2.3) on the box $\widetilde{Q} = [-2,2] \times Q$. By expanding $f_1$ and $f_2$ simultaneously into their Bernstein forms, we obtain a set of points $(b_I(f_1,\widetilde{Q}), b_I(f_2,\widetilde{Q}))$ in the plane, denoted by $b_I(\widetilde{Q})$. Then

$$\operatorname{conv} p\left(\widetilde{Q}\right) \subseteq \operatorname{conv} B\left(\widetilde{Q}\right), \tag{4.10}$$

where $p(\widetilde{Q}) = \{(f_1(\mathrm{x}), f_2(\mathrm{x})) : \mathrm{x} \in \widetilde{Q}\}$, $B(\widetilde{Q}) = \{b_I(\widetilde{Q}) : I \in S\}$. By any standard convex hull algorithm, it is easy to check whether the origin belongs to $\operatorname{conv} B(\widetilde{Q})$. If it is outside, then the family (1.1) is stable. Otherwise the sweep procedure is applied splitting the domain $D$ into two subdomains on which we proceed as before.

In all examples below the Bernstein coefficients have been evaluated on the basis of the results in [20].

*Example 4.2.* Consider the Schur stability problem for the following polynomially parameter dependent family:

$$\begin{aligned} p(z,\mathbf{q}) = {} & \left(q_2^2 - 0.8q_1\right)z^4 + \left(-0.07q_1q_2^3 - 0.05q_1^5 - 0.25q_2\right)z^3 \\ & + \left(-0.8q_1 + 0.57q_2 - 0.548\right)z^2 + \left(0.1q_1^2 + 0.18q_1q_2 - 0.1\right)z + 0.425 \\ & \qquad q_1 \in [-0.7, 0.5], \quad q_2 \in [1,2]. \end{aligned} \tag{4.11}$$

This family is stable for $q_1 = 0$, $q_2 = 1$. System (2.3) now reduces to

$$\begin{aligned} f_1(t,q_1,q_2) &= -0.425t^2 - t\left(0.1q_1^2 + 0.18q_1q_2 - 0.1\right) + q_2^2 - 0.57q_2 + 0.973 = 0, \\ f_2(t,q_1,q_2) &= t\left(-q_2^2 + 0.8q_1 + 0.425\right) + 0.05q_1^5 + 0.1q_1^2 + 0.18q_1q_2 + 0.07q_1q_2^3 + 0.25q_2 - 0.1 = 0 \end{aligned} \tag{4.12}$$

and $(t,q_1,q_2) \in \widetilde{Q} = [-2,2] \times [-0.7,0.5] \times [1,2] \subset \mathbb{R}^3$. We have to test for the nonexistence of a solution for this system.

After calculating the Bernstein coefficient on $\tilde{Q}$ for $f_1$ and $f_2$, we see that zero is contained in the convex hull of the Bernstein coefficients (see (4.10)). Therefore the bisection procedure must be applied to this problem. After 8 bisections in 0.421 sec, we conclude that the family is Schur stable.

On the other hand, the algorithm from [6] (by using Chebyshev polynomials) gives result after 0.51 sec in 10 steps.

*Example 4.3.* Consider the following multilinear family:

$$
\begin{aligned}
p(z,\mathbf{q}) = {} & (4q_2q_3 + 26 + 5q_1q_2 - 5q_1)z^8 + (q_1q_2 + 3q_1)z^7 + (-4q_1 + 3 + q_2)z^6 \\
& + (-2q_3 + q_1 - 6q_2q_3 + 4)z^5 + (2q_3 - 5 - 8q_1 - 3q_1q_3)z^4 + (-4q_1q_2 + q_1q_3)z^3 \\
& + (3q_1q_2 - 1 - 4q_2 - q_3)z^2 + (-2q_1q_2 + 2 + 12q_2q_3)z - 2q_1q_2q_3 + q_1q_2 - 4q_2q_3 + 2, \\
& \qquad q_1 \in [-1,0], \quad q_2 \in [-0.5,0], \; q_3 \in [-1,0]
\end{aligned}
\tag{4.13}
$$

which is Schur stable for $q_1 = q_2 = q_3 = 0$. For this family, system (2.3) becomes

$$
\begin{aligned}
f_1(t,q_1,q_2,q_3) = {} & (2q_1q_2q_3 - q_1q_2 + 4q_2q_3 - 2)t^4 + (2q_1q_2 - 12q_2q_3 - 2)t^3 \\
& + (-6q_1q_2q_3 + 5q_1q_2 - 8q_2q_3 - 5q_1 + 4q_2 + q_3 + 33)t^2 \\
& + (q_1q_2 + 24q_2q_3 - q_1q_3 + 3q_1 + 4)t \\
& + 2q_1q_2q_3 - 3q_1q_2 + 3q_1q_3 + 9q_1 - 3q_2 - 3q_3 - 21, \\
f_2(t,q_1,q_2,q_3) = {} & (-2q_1q_2q_3 - 4q_1q_2 - 8q_2q_3 + 5q_1 - 24)t^3 \\
& + (12q_2q_3 - 3q_1q_2 - 3q_1 + 2)t^2 \\
& + (4q_1q_2q_3 + 11q_1q_2 + 16q_2q_3 - 6q_1 - 5q_2 - q_3 + 44)t \\
& - q_1q_2 - 6q_2q_3 + q_1q_3 + 2q_1 + 2q_3 - 6,
\end{aligned}
\tag{4.14}
$$

and $(t,q_1,q_2,q_3) \in \tilde{Q} = [-2,2] \times [-1,0] \times [-0.5,0] \times [-1,0] \subset \mathbb{R}^4$. We have to test for the nonexistence of a solution for this system.

After calculating the Bernstein coefficient on $\tilde{Q}$ for $f_1$ and $f_2$, we see that zero is contained in the convex hull of the Bernstein coefficients (see (4.10)). After 255 bisections in 8.892 sec, we conclude that the family is Schur stable.

On the other hand, the multilinearization algorithm gives result after 92.758 sec in 2143 steps.

*Example 4.4.* The family

$$
\begin{aligned}
p(z,\mathbf{q}) = {} & z^6 + (-0.2 - q_2 - q_3)z^5 + (0.2q_3 - 0.1q_1 + 0.2q_2 + q_2q_3)z^4 \\
& + (0.1q_1q_2 + 0.1q_1q_3 - 0.01q_3 - 0.1q_1q_4 - 0.2q_2q_3 + 0.001)z^3 \\
& + (-0.001q_2 + 0.01q_1q_4 - 0.1q_1q_2q_3 + 0.01q_2q_3 + 0.001q_1 + 0.1q_1q_2q_4)z^2 \\
& + (-0.01q_1q_2q_4 - 0.001q_1q_2 - 0.001q_1q_3)z + 0.001q_1q_2q_3, \quad q_i \in [0.1, 0.2] \ (i = 1,2,3,4)
\end{aligned}
\tag{4.15}
$$

is Schur stable.

$$
\begin{aligned}
\text{Results. By} \quad & \text{multilinearization in 19 steps, after 1.15 sec,} \\
& \text{Bernstein expansion in 31 steps, after 3.744 sec,} \\
& \text{using Chebyshev polynomials in 527 steps, after 69.717 sec.}
\end{aligned}
\tag{4.16}
$$

*Example 4.5.* The family

$$
\begin{aligned}
p(z,\mathbf{q}) = {} & (200q_1q_2 + q_3 - q_4q_6q_7)z^6 + (30q_1 + 40q_1q_2 - q_7 + 65)z^5 \\
& + (5.1q_1q_4 + 0.01q_2 + q_1q_2 - 2q_6 + 26.1)z^4 + (0.2q_1 + 4q_1q_2q_3 - 22)z^3 \\
& + (6q_1q_2 - 0.02q_2 - 10q_1 - q_7 - 10.2)z^2 + (4q_1q_2 - 0.2q_1q_5q_6 - 18)z + 0.01q_2q_3 \\
& + 4.9q_1 + q_4 + q_5 + q_1q_2 + 24.1, \quad q_1 \in [1.8, 2], \ q_2 \in [1.5, 2], \ q_3 \in [-0.5, 0], \\
& \qquad q_4 \in [-0.5, 0], \quad q_5 \in [0.5, 1], q_6 \in [0.5, 1], \quad q_7 \in [0, 0.5]
\end{aligned}
\tag{4.17}
$$

is Schur stable.

$$
\begin{aligned}
\textit{Results}\text{: By} \quad & \text{multilinearization in 13 steps, after 6.188 sec,} \\
& \text{Bernstein expansion in 255 steps, after 289.797 sec.}
\end{aligned}
\tag{4.18}
$$

Increase in the number of parameters $q_i$ essentially increases the computational time for the Bernstein approach in comparison with multilinearization.

## 5. Stability of a Matrix Family

In this section, we apply our approach (i.e., the approach which has been applied in Sections 2, 3, 4 to polynomial stability) to the matrix stability problem. Let $\mathscr{A}$ be a pathwise connected set of real $n \times n$ matrices and contain at least one Schur stable member. In this section, we obtain a criterion for the Schur stability of $\mathscr{A}$. A similar Hurwitz stability criterion of $\mathscr{A}$ is obtained in [16], where $\mathscr{A}$ is a polytope.

**Theorem 5.1.** *Let $\mathcal{A}$ be a pathwise connected set of real $n \times n$ matrices and contain at least one Schur stable member $A_0$. Then $\mathcal{A}$ is Schur stable if and only if*

$$\det\left(A^2 - 2tA + I\right) \neq 0 \tag{5.1}$$

*for all $A \in \mathcal{A}$, $t \in [-1, 1]$.*

*Proof.* $\Rightarrow$) Assume that $\mathcal{A}$ is Schur stable and $A \in \mathcal{A}$. Then

$$\det\left(A - e^{j\theta} I\right)\left(A - e^{-j\theta} I\right) \neq 0 \tag{5.2}$$

for all $\theta \in [0, \pi]$, where $j^2 = -1$, gives

$$\det\left(A^2 - 2tA + I\right) \neq 0, \tag{5.3}$$

where $t = \cos\theta$.

$\Leftarrow$) Assume that (5.1) is satisfied. By contradiction if $\mathcal{A}$ contains a matrix which is not Schur stable, then by continuity there exists $A_* \in \mathcal{A}$ such that $A_*$ has an eigenvalue $\lambda_*$ with $|\lambda_*| = 1$.

If $\lambda_* = 1$ ($\lambda_* = -1$), then $\det(A_* - I) = 0 \quad (\det(A_* + I) = 0)$ which contradicts (5.1).

If $\lambda_* \neq \pm 1$, then $\lambda_* = e^{j\theta_*}$, $\theta_* \in (0, \pi)$, and we have

$$\begin{aligned} 0 &= \det\left(\left(A_* - e^{j\theta_*} I\right)\left(A_* - e^{-j\theta_*} I\right)\right) \\ &= \det\left(A_*^2 - 2\cos\theta_* A_* + I\right) \end{aligned} \tag{5.4}$$

which contradicts (5.1). □

Let a family $\mathcal{A}$ be defined as

$$\mathcal{A} = \{A(\mathbf{q}) : \ \mathbf{q} \in Q\}, \tag{5.5}$$

where $Q$ is a box and $A(\mathbf{q})$ polynomially depends on $\mathbf{q}$. In this case by Theorem 5.1 Schur stability of $\mathcal{A}$ can be tested via positivity (or negativity) of the scalar multivariate polynomial

$$f(t, \mathbf{q}) = \det\left(A^2(\mathbf{q}) - 2tA(\mathbf{q}) + I\right) \tag{5.6}$$

on $\widetilde{Q} = [-1, 1] \times Q$. For this problem, Theorem 4.1 and the splitting procedure described in the previous section can be applied.

The following example is taken from [24].

*Example 5.2* (see [24]). Consider the interval matrix family

$$\mathcal{A} = \left\{ \begin{pmatrix} 0.6 & q_1 \\ q_2 & q_3 \end{pmatrix} : q_1 \in [0, 0.2],\ q_2 \in [-0.78, 0],\ q_3 \in [-0.6, 0.6] \right\}. \tag{5.7}$$

The matrix $A_0 = \left(\begin{smallmatrix} 0.6 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ belongs to $\mathcal{A}$ and is Schur stable. The determinant function (5.1) is

$$\begin{aligned} f(t, q_1, q_2, q_3) &= \det\left(A^2 - 2tA + I\right) \\ &= 2q_1q_2 + 1.36q_3^2 - 2.72tq_3 + 1.36 + q_1^2q_2^2 + 2tq_1q_2q_3 \\ &\quad + 1.2tq_1q_2 - 1.2tq_3^2 + 2.4t^2q_3 - 1.2t - 1.2q_1q_2q_3 - 4t^2q_1q_2. \end{aligned} \tag{5.8}$$

After 22 bisections and eliminations in 1.1 sec, we decide that $f(t, q_1, q_2, q_3) > 0$ on $[-1, 1] \times [0, 0.2] \times [-0.78, 0] \times [-0.6, 0.6] \subset \mathbb{R}^4$, and by Theorem 5.1 this family is Schur stable.

*Example 5.3.* Consider the following quadratic family:

$$\mathcal{A} = \left\{ A_0 + \lambda A_1 + \lambda^2 A_2 : \lambda \in [0, 1] \right\}, \tag{5.9}$$

where

$$A_0 = \begin{pmatrix} -0.1 & 0 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 1 & 0 & 0.3 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & -2 & 1 \\ -1 & 0 & -2 \\ -1 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0.2 & 0.1 \\ 0.1 & 0.2 & -0.5 \\ 0.3 & 0.2 & -0.4 \end{pmatrix}. \tag{5.10}$$

The matrix $A_0$ is Schur stable. The determinant function (5.1)

$$f(t, \lambda) = \det\left(A^2(\lambda) - 2tA(\lambda) + \mathrm{I}\right) \tag{5.11}$$

is a two-variable polynomial having 40 terms. After 17 bisections and eliminations in 1.75 sec we conclude that $f(t, \lambda)$ is positive on $[-1, 1] \times [0, 1] \subset \mathbb{R}^2$ and this family is Schur stable.

The investigation of Schur stability of a matrix family by the guardian maps [25] requires calculating of eigenvalues of matrices having high dimensions. In Example 5.3, if the determinant of Kronecker sum $A \to \det(A \oplus B)$ is used as a guardian map, then a matrix whose eigenvalues should be calculated would have dimension $36 \times 36$.

Theorem 5.1 shows that the map

$$A \longrightarrow \min_{t \in [-1, 1]} \left| \det\left(A^2 - 2tA + I\right) \right| \tag{5.12}$$

is a semiguardian map for the family of $n \times n$ dimensional Schur stable matrices.

Note that an alternative method for checking the positivity of a multivariable polynomial on a box is given in [26].

## 6. Conclusions

We consider stability problems for a multilinear and polynomially dependent polynomial families. Two algorithms such as multilinearization and the Bernstein expansion algorithm are suggested. If the number of parameters is increased then multilinearization gives a better result. In the case of unstability, the multilinearization algorithm leads to an unstable point.

A new result on Schur stability of an compact matrix family is obtained. Based on this result and the Bernstein expansion, a fast algorithm for Schur stability is given.

## Acknowledgment

The authors would like to thank the reviewers for their helpful suggestions to the improvement of this paper.

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