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A NEW METHOD OF APPROXIMATION FOR FUZZY MEMBERSHIP FUNCTION WITH APPLICATION

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ABSTRACT

In the present study we have formulated a new Maximum Fuzzy Entropy Problem (Max(F)EntP) for fuzzy membership function and proposed sufficient conditions for existence of its solution. Mentioned problem consists of approximation fuzzy membership function by maximizing Maximum Fuzzy Entropy (Max(F)Ent) measure with respect to membership functions with finite number of the fuzzy values subject to constraints generated by given moment functions. The existence of solution of mentioned problem is proved by virtue of convexity property of Max(F)Ent measure, the implicit function theorem and Lagrange multipliers method. Moreover, by using MATLAB programme one application of suggested method on fuzzy data analysis is given.

Keywords: Fuzzy entropy measure, Maximum fuzzy entropy problem, Membership function, Fuzzy data analysis

BULANIK ÜYELİK FONKSİYONUNUN YAKLAŞIMI İÇİN BİR YENİ METOT VE UYGULAMA

ÖZET

Bu çalışmada, tarafımızdan bulanık üyelik fonksiyonu için yeni bir maksimum bulanık entropi problemi (Max(F)EntP) geliştirilmiş ve bu problemin çözümünün varlığı için yeterli koşullar belirlenmiştir. Söz konusu problem sonlu sayıda bulanık değerlere sahip üyelik fonksiyonunu bulanık entropi ölçümünü verilmiş moment fonksiyonları yardımıyla üretilen moment koşulları altında maksimize etmekle yaklaşık olarak elde etme problemidir. Söz konusu problemin çözümünün varlığı Max(F)Ent ölçümünün konvekslik özelliği, kapalı fonksiyon teoremi ve Lagrange çarpanları yardımıyla ispatlanmıştır. Buna ek olarak, bulanık veri analizi üzerine sunulmuş yöntemin bir uygulaması MATLAB programı kullanılarak verilmiştir.

Anahtar Kelimeler: Bulanık entropi ölçümü, Maksimum bulanık entropi problemi, Üyelik fonksiyonu, Bulanık veri analizi

1. INTRODUCTION

The concept of entropy and many problems concerned with its applications are given in [1]. Zadeh [2] has introduced the concept of fuzzy sets and developed his own theory to measure the uncertainity of a fuzzy set. It is known that a fuzzy set A is defined in the universal set X by a membership function $\mu_A(x)$ and represented as

$$A = \{\, x_i \mid \mu_A(x_i) : i = 0,1,2,\dots,n\}.$$

μ_A in crisp set maps whole members in universal set X to $\{0,1\}$, $\mu_A: X \to \{0,1\}$. However, in fuzzy sets, each element is mapped to $[0,1]$ by membership function, $\mu_A: X \to [0,1]$. For this reason, fuzzy set can be described as “vague boundary” set comparing with crisp set in [3,4].

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By starting from the concept of fuzzy sets De Luca and Termini [5] suggested that corresponding to Shannon's [6] probabilistic entropy the fuzzy entropy measure $H(A)$ for fuzzy set A containing finite number elements can be expressed by formula

$$H(A) = -\sum_{i=0}^{n}[\mu_A(x_i)log\mu_A(x_i) + (1-\mu_A(x_i))\,log\big(1-\mu_A(x_i)\big)]\ ,$$

where A is a fuzzy set, $\mu_A(x)$ is membership function and $\mu_A(x_i)$ are the fuzzy values.

After the development of fuzzy entropy measure formula $H(A)$ given in [5], a large number of measures of fuzzy entropy were discussed, characterized and generalized by various authors. Some other interesting findings are related with theoretical measures of fuzzy entropy and their applications have been provided by Kapur [7], Parkash and Sharma [8], Yager [9], Bhandari and Pal [10], Parkash, Sharma and Kumar [11] etc. In [12], Parkash, Sharma and Mahajan introduced new measures of weigted fuzzy entropy including two moment conditions.

The aim of this study consists of developing a new Generalized Maximum Fuzzy Entropy Methods in the form of MinMax(F)EntM and MaxMax(F)EntM on the basis of entropy optimization theory [13-15] and present an application.

The suggested paper is formed as follows. In Section 2, a complete definition of Maximum Fuzzy Entropy Problem, Method and Distribution are proposed. In Section 3, the convexity of Max(F)Ent measure is introduced. In Section 4, the existence of Maximum Fuzzy Entropy Problem (Max(F)EntP) is proved by using Lagrange multipliers methods [16] and implicit function theorem [17]. In Section 5, the evaluation of Lagrange multipliers for Maximum Fuzzy Entropy Problem (Max(F)EntP) is formulated by Existence theorem. In Section 6, the maximum fuzzy entropy value is achieved. In Section 7, new Generalized Maximum Fuzzy Entropy Problems $(\text{MinMax(F)Ent})_\text{m}$, $(\text{MaxMax(F)Ent})_\text{m}$ and methods of solving these problems are developed [18-20]. In Section 8, it is given an application about GMax(F)EntM. Finally, the main results obtained in this study are summarized.

2. MAXIMUM FUZZY ENTROPY PROBLEM (MAX(F)ENTP)

On the basis of entropy optimization theory we have suggested a new Generalized Maximum Fuzzy Entropy Methods (GMax(F)EntM) in the form of MinMax(F)EntM and MaxMax(F)EntM. These methods are based on primary maximizing Max(F)Ent measure for fixed moment vector function in order to define the special functional with maximum values of Max(F)Ent measure and secondary optimization of mentioned functional with respect to moment vector functions. Distribution, in other words a set of successive values of estimated membership function closest to (furthest from) the given membership function in the sense of Max(F)Ent measure, for the first time, obtained by mentioned methods is defined as $(\text{MinMax(F)Ent})_\text{m}$ ($(\text{MaxMax(F)Ent})_\text{m}$) distribution which can be applied in many problems of fuzzy data analysis. One of mentioned applications is approximately obtaining fuzzy membership function according to given fuzzy data.

In formula for fuzzy set A containing finite number elements suggested by De Luca and Termini [5] we write "ln" instead of "log" in order to simply operations. Therefore, we shall consider the following formula

$$H(A) = -\sum_{i=0}^{n}[\mu_A(x_i)ln\mu_A(x_i) + (1-\mu_A(x_i))\,ln\big(1-\mu_A(x_i)\big)]. \tag{1}$$

Maximum Fuzzy Entropy Problem (Max(F)EntP) consists of maximizing Max(F)Ent measure (1) with respect to membership functions $\mu_A(x)$ with finite number of the fuzzy values $\mu_A(x_i), i = 0,1,...,n$ subject to constraints

$$\sum_{i=0}^{n}\mu_A(x_i)\,g_j(x_i) = \mu_j,\ j = 0,1,2,\dots,m \tag{2}$$

where $g_0(x) \equiv 1$; μ_j, $j = 0,1,2, \dots, m$ are moment values of $\mu_A(x_i)$, $i = 0,1,\dots,n$ with respect to moment functions $g_j(x)$, $j = 0,1,2, \dots, m; m < n$.

The distribution of fuzzy values $\left(\mu_A(x_0), \mu_A(x_1), \dots, \mu_A(x_n)\right)$ maximizing function (1) subject to constraints (2) (briefly stated problem (1),(2)) we call Maximum Fuzzy Entropy Distribution (Max(F)EntD) just as Maximum Entropy Distribution (MaxEntD) of probabilistic entropy optimization problem.

Fuzzy Entropy Optimization Problem (1),(2) is a conditional extremum problem. The solvability of this problem requires to fulfillment of several conditions. Mentioned conditions are following:

1) Moment functions $g_j(x)$, $j = 0,1,2, \dots, m$ are linearly independent;
2) The inequality $n > m$ is satisfied;
3) Moment values $\tilde{\mu}_j$, $j = 0,1,2, \dots, m$ are obtained by virtue of given fuzzy values $\tilde{\mu}_A(x_i)$, $i = 0,1, \dots, n$ and moment functions $g_j(x)$, $j = 0,1, \dots, m$ in the form of equalities

$$\sum_{i=0}^{n} g_j(x_i)\, \tilde{\mu}_A(x_i) = \tilde{\mu}_j, \; j = 0,1, \dots, m. \qquad (2_1)$$

Remark. (2_1) means that there are linear dependency between the column $\tilde{\mu} = (\tilde{\mu}_0, \tilde{\mu}_1, \dots, \tilde{\mu}_m)^T$ and all columns of matrix $A = \left(g_j(x_i)\right)_{\substack{j=0,1,..,m \\ i=0,1,\dots,n}}$. Consequently, $rankA$ is equal to $rank(A{:}\,\tilde{\mu})$, where $(A{:}\,\tilde{\mu})$ is augmented matrix for system (2) as the matrix A with column $\tilde{\mu}$ added to it. Therefore, system (2) with respect to $\mu_A(x_i)$, $i = 0,1,\dots,n$ has a solution. Note that from condition 1) follows that $rankA = m + 1$.

3. CONVEXITY OF MAX(F)ENT MEASURE

In order to simplify mathematical operations constrained with (1) we use conventional signs $\mu_A(x_i) = X_i$, $i = 0,1,\dots,n$ and write (1) in the form

$$H = -\sum_{i=0}^{n} [X_i ln X_i + (1 - X_i)\, ln(1 - X_i)]. \qquad (1')$$

From $(1')$ follows that

$$\frac{\partial H}{\partial X_i} = ln\left(\frac{1-X_i}{X_i}\right);$$

$$\frac{\partial^2 H}{\partial X_i \partial X_j} = \begin{cases} -\frac{1}{X_j(1-X_j)}, & i = j \\ 0, & i \neq j. \end{cases}$$

Consequently, Hessian matrix $\mathcal{H}$ is defined in the following form

$$\mathcal{H} = \left(\frac{\partial^2 H}{\partial X_i \partial X_j}\right)_{\substack{j=0,1,..,n \\ i=0,1,\dots,n}}.$$

Since eigenvalues of $\mathcal{H}$ matrix are negative : $-\frac{1}{X_i(1-X_i)} < 0$, $i = 0,1, \dots, n$, therefore this matrix is negative defined. This result shows that function H is convex and at critical point $(X_0^0, X_1^0, \dots, X_n^0)$, $X_i^0 = \frac{1}{2}$ for which $\frac{\partial H}{\partial X_i} = 0$ function H reaches maximum value.

4. THE EXISTENCE OF SOLUTION OF MAX(F)ENTP

Maximum Fuzzy Entropy Problem (1),(2) is a conditional extremum problem and can be solved by Lagrange multipliers method. According to Lagrange multipliers method firstly the new auxiliary function U is constructed:

$$U=-\sum_{i=0}^{n}[\mu_A(x_i)ln\mu_A(x_i)+(1-\mu_A(x_i))\,ln\big(1-\mu_A(x_i)\big)-\sum_{j=0}^{m}\lambda_j\big(\sum_{i=0}^{n}\mu_A(x_i)\,g_j(x)-\mu_j\big)],\quad (3)$$

where λ_j are certain constant factors and the function U is now investigated for an unconditional extremum; we form a system of equations $\frac{\partial U}{\partial \lambda_j}=0,\ j=0,1,\dots,m$ supplemented by the constraint equations (2) from which all the $n+m+2$ unknowns $\mu_A(x_i),\ i=0,1,\dots,n$ and $\lambda_j,\ j=0,1,\dots,m$ are determined.

From (3) follows that

$$\frac{\partial U}{\partial \mu_A(x_i)}=\Big[-ln\mu_A(x_i)+\mu_A(x_i)\frac{1}{\mu_A(x_i)}-ln\big(1-\mu_A(x_i)\big)+(1-\mu_A(x_i))\frac{1}{\mu_A(x_i)}(-1)\Big]-$$
$$-\sum_{j=0}^{m}\lambda_j g_j(x_i)=0,$$

$$\Big[-ln\mu_A(x_i)+1-ln\big(1-\mu_A(x_i)\big)-1\Big]-\sum_{j=0}^{m}\lambda_j g_j(x_i)=0;$$

$$-ln\frac{\mu_A(x_i)}{(1-\mu_A(x_i))}=\sum_{j=0}^{m}\lambda_j g_j(x_i);$$

$$\frac{1-\mu_A(x_i)}{\mu_A(x_i)}=e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)},\quad i=0,1,\dots,n$$

$$\frac{1}{\mu_A(x_i)}=1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)},$$

$$\mu_A(x_i)=\frac{1}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}},\quad i=0,1,\dots,n. \quad (4)$$

Also, from (1) follows that

$$\frac{\partial U}{\partial \lambda_j}=-\big(\sum_{j=0}^{m}\mu_A(x_i)\,g_j(x)-\mu_j\big)=0,\quad j=0,1,\dots,m.$$

If we take (4) into account in (2), then

$$\sum_{i=0}^{n}\frac{1}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}}g_j(x_i)=\mu_j,\quad j=0,1,2,\dots,m. \quad (5)$$

If denote the left-hand of (5) by $f_j(\lambda_0,\lambda_1,\dots,\lambda_m)$, then (5) can be written as

$$f_j(\lambda_0,\lambda_1,\dots,\lambda_m)=\sum_{i=0}^{n}\frac{1}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}}g_j(x_i)=\mu_j,\quad j=0,1,\dots,m. \quad (6)$$

From (2_1) follows that

$$\sum_{i=0}^{m}g_j(x_i)\,\tilde{\mu}_A(x_i)=\tilde{\mu}_j-\sum_{i=m+1}^{n}g_j(x_i)\,\tilde{\mu}_A(x_i),\quad j=0,1,\dots,m\,. \quad (7)$$

(7) shows that there are linear correlations between $\tilde{\mu}_A(x_i), i = 0,1,\dots,m$ and $\tilde{\mu}_A(x_i), i = m+1,\dots,n$. Consequently, from the assumption 1), that moment functions $g_j(x)$, $j = 0,1,\dots,m$ are linearly indepedent, then the following condition is satisfied

$$det\left(g_j(x_i)\right)_{\substack{j=0,1,..,m \\ i=0,1,\dots,m}} \neq 0. \qquad (2_2)$$

In (7), $\tilde{\mu}_A(x_i)$, $i = 0,1,\dots,m$ by Cramer method of solving linear nonhomojenus algebrical equations can be expressed via $\tilde{\mu}_j$, $j = 0,1,\dots,m$ and $\tilde{\mu}_A(x_i)$, $i = m+1,\dots,n$ in the form

$$\tilde{\mu}_A(x_i) = F\left(\tilde{\mu}_0, \tilde{\mu}_1, \dots, \tilde{\mu}_m, \tilde{\mu}_A(x_{m+1}), \dots, \tilde{\mu}_A(x_n)\right),\ i = 0,1,\dots,m. \qquad (2_3)$$

From (4) follows that

$$\sum_{j=0}^{m} \lambda_j g_j(x_i) = \ln\left(\frac{1-\mu_A(x_i)}{\mu_A(x_i)}\right),\ \ i = 0,1,\dots,m. \qquad (8)$$

If subsitute (2_3) in (8) and solve the getting equations with respect to $\lambda_0, \lambda_1, \dots, \lambda_m$, then it is possible to find $\tilde{\lambda}_0, \tilde{\lambda}_1, \dots, \tilde{\lambda}_m$ satisyfing (6) and the equations

$$f_j\left(\tilde{\lambda}_0, \tilde{\lambda}_1, \dots, \tilde{\lambda}_m\right) = \tilde{\mu}_j,\ \ j = 0,1,\dots,m \qquad (9)$$

are arised. Therefore, subject to assumption (2_2) by solving (7) with respect to $\tilde{\mu}_A(x_i), i = 0,1,\dots,m$ and substituting (2_3) in (8) the relations (9) are appeared.

Note that relations (9) are one of the important conditions to solve equations (6) with respect to $\lambda_0, \lambda_1, \dots, \lambda_m$. The other condition to solve equations (6) with respect to $\lambda_0, \lambda_1, \dots, \lambda_m$ in the some neighbourhood of $\left(\tilde{\lambda}_0, \tilde{\lambda}_1, \dots, \tilde{\lambda}_m\right)$ satisfiying (9) is the condition

$$J = \frac{D(f_0, f_1, \cdots, f_m)}{D(\lambda_0, \lambda_1, \cdots, \lambda_m)} \neq 0. \qquad (10)$$

Now, we prove the fulfillment of (10). From (6), it follows that

$$\frac{\partial f_j}{\partial \lambda_k} = \sum_{i=0}^{n} \mu_A(x_i)(1-\mu_A(x_i))\, g_j(x_i) g_k(x_i),\ \ j = 0,1,\dots,m, k = \ 0,1,\dots,m. \qquad (11)$$

Let us $\sum_{i=0}^{n} \mu_A(x_i)(1-\mu_A(x_i)) = \alpha$. Then,

$$\sum_{i=0}^{n} \frac{\mu_A(x_i)(1-\mu_A(x_i))}{\alpha} = 1. \qquad (12)$$

In (12), if the ratio $\frac{\mu_A(x_i)(1-\mu_A(x_i))}{\alpha}$ is considered as probability measure, then

$$\frac{\mu_A(x_i)(1-\mu_A(x_i))}{\alpha} = P_i,\ \ i = 0,1,\dots,n;\ \sum_{i=0}^{n} P_i = 1$$

and from (11) it follows that

$$\frac{\partial f_j}{\partial \lambda_k} = \alpha \sum_{i=0}^{n} \frac{\mu_A(x_i)(1-\mu_A(x_i))}{\alpha} g_j(x_i) g_k(x_i) = \alpha \sum_{i=0}^{n} P_i\, g_j(x_i) g_k(x_i) = \alpha E\left[g_j g_k\right]$$

and

$$R=\left(\frac{\partial f_j}{\partial\lambda_k}\right)_{j,k=0,1,\dots,m}=\alpha\begin{pmatrix} E[g_0g_0] & E[g_0g_1] & \dots & E[g_0g_m] \\ E[g_1g_0] & E[g_1g_1] & \dots & E[g_1g_m] \\ \vdots & \vdots & \ddots & \vdots \\ E[g_mg_0] & E[g_mg_1] & \dots & E[g_mg_m] \end{pmatrix}.$$

Since R is correlation matrix of random variables $g_0(x),\dots,g_m(x)$ each of which has $n+1$ number of values, then

$$J=\frac{D(f_0,f_1,\dots,f_m)}{D(\lambda_0,\lambda_1,\dots,\lambda_m)}=det(R)\neq 0.$$

Note that the satisfiability of last condition can be also proved as following:

$$0\leq E\{|a_0g_0+a_1g_1+\dots+a_mg_m|^2\}=E\left\{\sum_{j=0}^m\sum_{k=0}^m a_ja_kg_j\,g_k\right\}=$$

$$=\sum_{j=0}^m\sum_{k=0}^m a_ja_kE\left\{g_jg_k\right\}=a\,Ra^T,\ a\neq 0,\ a=(a_0,\dots,a_m). \quad (13)$$

It is seen that random variables $g_j(x), j=0,1,\dots,m$ are linearly independent according to assumption 1), the left-hand of (13) is equal to zero if and only if at $a=0$, consequently R is positive defined matrix, therefore $detR\neq 0$ and condition (10) is satisfied.

The satisfability of (9) and (10) indicates that the implicit function theorem [16] can be applied to solvability of (6) with respect to $\lambda_0,\lambda_1,\dots,\lambda_m$.

The obtained results for solvability of (6) can be expressed in the following theorem.

Existence Theorem. Let us the conditions 1), 2) and 3) are satisfied.

Then, Maximum Fuzzy Entropy Problem (Max(F)EntP) which consists of maximizing Max(F)Ent measure (1) with respect to membership functions $\mu_A(x)$ with finite number of the fuzzy values $\mu_A(x_i)$, $i=0,1,\dots,n$ subject to constraints (2) has a solution $\left(\mu_A(x_0),\mu_A(x_1),\dots,\mu_A(x_n)\right)$.

5. EVALUATION OF LAGRANGE MULTIPLIERS FOR MAX(F)ENTP

From the proof of Existence theorem, it is indicated that evaluation of Lagrange multipliers occupies very important place. For this reason, it is required to consider this problem in more detail. One of basic stages of application of numerical methods is the choice of any initial point.

The mean problem consists of solving system of equations (6) with respect to $\lambda_0,\lambda_1,\dots,\lambda_m$ by starting any initial point $(\tilde{\lambda}_0,\tilde{\lambda}_1,\dots,\tilde{\lambda}_m)$. Mentioned point is obtained by following way. From (2_1) follows (7) and from (2_2) $\tilde{\mu}_A(x_i),\ i=0,1,\dots,m$ are obtained in the form of (2_3), later (2_3) is taken into account in (8). Solving (8) with respect to $\lambda_0,\lambda_1,\dots,\lambda_m$ and showing obtained values as $\tilde{\lambda}_0,\tilde{\lambda}_1,\dots,\tilde{\lambda}_m$ it is seen that these values satisfy (9). Consequently to solve system (6) by some numerical methods $(\tilde{\lambda}_0,\tilde{\lambda}_1,\dots,\tilde{\lambda}_m)$ can be taken as initial point.

6. MAXIMUM FUZZY ENTROPY VALUE

In Section 4, maximization of Max(F)Ent measure (1) is realized by Lagrange multipliers method and membership function $\mu_A(x)$ which gives maximum value to (1) is expressed by formula (4). By virtue of formula (4) from (1) follows that

$$maxH_A = -\sum_{i=0}^{n} ln\frac{e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}} + \sum_{j=0}^{m}\lambda_j\mu_j. \quad (14)$$

The formula (14) represents maximum value of Max(F)Ent measure (1). It is seen that $maxH_A$ depends on Lagrange multipliers $\lambda_0, \lambda_1, \ldots, \lambda_m$, moment functions $g_0(x), g_1(x), \ldots, g_m(x)$ and moment fuzzy values $\mu_0, \mu_1, \ldots, \mu_m$.

7. GENERALIZED MAXIMUM FUZZY ENTROPY PROBLEMS

Before it is showed that maximum value of Max(F)Ent measure H_A is represented by formula (14) in the dependency of moment functions $g_0(x), g_1(x), \ldots, g_m(x)$, Lagrange multipliers $\lambda_0, \lambda_1, \ldots, \lambda_m$, and moment fuzzy values $\mu_0, \mu_1, \ldots, \mu_m$. Let us $g = (g_0, g_1, \ldots, g_m)$ be vector moment functions with components $g_0, g_1, \ldots, g_m$. If we take into account that according to condition (2) both Lagrange multipliers $\lambda_0, \lambda_1, \ldots, \lambda_m$ and moment fuzzy values $\mu_0, \mu_1, \ldots, \mu_m$ are generated by moment vector fuction g and given fuzzy values of $\mu_A(x_i)$, $i = 0,1, \ldots, n$, then the $maxH_A$ defined by formula (14) can be expressed as a functional $U(g)$ depended on moment vector fuction g. That is to say

$$U(g) = \max_{g} H_A = -\sum_{i=0}^{n} ln\frac{e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}} + \sum_{j=0}^{m}\lambda_j\mu_j. \quad (15)$$

According to [13-15], let K be the compact set of moment vector functions $g(x)$. $U(g)$ reaches its least and greatest values in this compact set, because of its continuity property. For this reason,

$$\min_{g\in K} U(g) = U\left(g^{(1)}\right) \quad ; \quad \max_{g\in K} U(g) = U\left(g^{(2)}\right).$$

Consequently,

$$U\left(g^{(1)}\right) \le U\left(g^{(2)}\right).$$

Distributions $\mu^{(1)} = \left(\mu^{(1)}(x_0), \mu^{(1)}(x_1), \ldots, \mu^{(1)}(x_n)\right)$ and $\mu^{(2)} = \left(\mu^{(2)}(x_0), \mu^{(2)}(x_1), \ldots, \mu^{(2)}(x_n)\right)$ corresponding to the moment functions $g^{(1)}(x)$ and $g^{(2)}(x)$ respectively, we call as MinMax(F)Ent and MaxMax(F)Ent distributions. Methods obtaining distributions MinMax(F)Ent and MaxMax(F)Ent we call as MinMax(F)EntM and MaxMax(F)EntM, respectively.

Now, MinMax(F)EntM and MaxMax(F)EntM for a finite set of characterizing moment functions can be defined in the following form.

Let $K_0 = \{g_1, \ldots, g_r\}$ be the set of characterizing moment vector functions and all combinations of r elements of K_0 taken m elements at a time be $K_{0,m}$. We note that each element of $K_{0,m}$ is vector g with m components. Note that the number of vectors g is equal to $\binom{r}{m}$.

Solving the MinMax(F)Ent and MaxMax(F)Ent problems require to find vector functions $\left(g_0, g^{(1)}(x)\right)$ and $\left(g_0, g^{(2)}(x)\right)$, where $g_0(x) \equiv 1$, $g^{(1)} \in K_{0,m}$, $g^{(2)} \in K_{0,m}$ minimizing and maximizing functional $U(g)$ defined by (15). It should be noted that $U(g)$ reaches its minimum (maximum) value subject to constraints (2) generated by function $g_0(x)$ and all m −dimensional vector functions $g(x)$, $g \in K_{0,m}$. In other words, minimum (maximum) value of $U(g)$ is least (greatest) value of values $U(g)$ corresponding to $g(x)$, $g \in K_{0,m}$. In other words, MinMax(F)Ent (MaxMax(F)Ent) is distribution giving minimum (maximum) value to functional $U(g)$ along of all distributions generated by $\binom{r}{m}$ number of moment vector functions $g(x), g \in K_{0,m}$. Mentioned distributions can be denoted by $(\mathrm{MinMax(F)Ent})_\mathrm{m}$ and $(\mathrm{MaxMax(F)Ent})_\mathrm{m}$.

If $\left(g_0\,, g^{(1)}(x)\right)$ gives the minimum value to $U(g)$, then distribution $\mu^{(1)} = \left(\mu^{(1)}(x_0), \mu^{(1)}(x_1), \ldots, \mu^{(1)}(x_n)\right)$ is called $(\text{MinMax(F)Ent})_\text{m}$ distribution. In a similar way, if $\left(g_0\,, g^{(1)}(x)\right)$ gives the maximum value to $U(g)$, then distribution of $\mu^{(2)} = \left(\mu^{(2)}(x_0), \mu^{(2)}(x_1), \ldots, \mu^{(2)}(x_n)\right)$ is called $(\text{MaxMax(F)Ent})_\text{m}$ distribution. MinMax(F)Ent and MaxMax(F)Ent methods represent maximum fuzzy entropy distributions in the form of $(\text{MinMax(F)Ent})_\text{m}$ and $(\text{MaxMax(F)Ent})_\text{m}$ distributions. It should be noted that both distributions can be applied in solving proper problems in fuzzy data analysis.

8. APPLICATION OF MINMAX(F)ENT AND MAXMAX(F)ENT METHODS

In this section, $(\text{MinMax(F)Ent})_\text{m}$ and $(\text{MaxMax(F)Ent})_\text{m}$ distributions are obtained for the following membership function values in fuzzy data given by Table 1. It should be noted that mentioned distributions are calculated by using MATLAB.

Table 1. Membership function values $\mu(x_i) = \mu_i,\ \ i = 1,2,\ldots,18$

x_i	μ_i
0.1000	0.0004
0.6000	0.0011
1.1000	0.0030
1.6000	0.0082
2.1000	0.0219
2.6000	0.0573
3.1000	0.1419
3.6000	0.3100
4.1000	0.5498
4.6000	0.7685
5.1000	0.9002
5.6000	0.9608
6.1000	0.9852
6.6000	0.9945
7.1000	0.9980
7.6000	0.9993
8.1000	0.9997
8.6000	0.9999

In order to calculate $(\text{MinMax(F)Ent})_\text{m}$ and $(\text{MaxMax(F)Ent})_\text{m}$ distributions for fuzzy data given in Table 1, it is required to realize the following steps:

1. Determine Max(F)Ent characterizing moments $E\{g_0(x)\}, E\{g_1(x)\}, \ldots, E\{g_m(x)\}$ according to fuzzy data.

2. Calculate Max(F)Ent distributions subject to each of Max(F)Ent characterizing moments $E\{g_0(x)\}, E\{g_1(x)\}, \ldots, E\{g_m(x)\}$.

3. Determine $(\text{MinMax(F)Ent})_\text{m}$ and $(\text{MaxMax(F)Ent})_\text{m}$ distributions corresponding to selected Max(F)Ent characterizing moments $E\{g_0(x)\}, E\{g_1(x)\}, \ldots, E\{g_m(x)\}$

4. Along obtained distributions choose the accepted Generalized Maximum Fuzzy Entropy distributions.

It is noted that selection of moment functions set is important in the application of Max(F)Ent method. In our investigation, Max(F)Ent characterizing moments $E\{\sqrt{x}\}, E\{lnx\}, E\{ln(1+x)\}, E\{ln(1+x^2)\}$ are acquired by experimental way.

In order to obtain $(\text{MinMax(F)Ent})_\text{m}$ and $(\text{MaxMax(F)Ent})_\text{m}$ $(m = 1,2)$ distributions, we should choose the moment vector functions giving the maximum and minimum values to the Max(F)Ent functional $U(g)$. Here, we used the moment functions

$$g_0(x) = 1, g_1(x) = \sqrt{x}\,, g_2(x) = lnx,\ \ g_3(x) = ln(1+x)\,, g_4(x) = ln(1+x^2).$$

According to suggested method, $K_0 = \{g_0, g_1, g_2, g_3, g_4\}$ and all combinations of r elements of K_0 taken m elements at a time are denoted by $K_{0,m}$. Max(F)Ent values subject to moment constraints generated by elements of $K_{0,m}$, m=1,2 is given in Tables 2,3.

Table 2. Entropy of calculated Max(F)Ent values subject to two moment functions

Moments Functions	Fuzzy Entropy
$(1, \sqrt{x})$	**0.0375**
$(1, lnx)$	**0.0896**
$(1, ln(1+x))$	0.0379
$(1, ln(1+x^2))$	0.0464

Table 3. Entropy of calculated Max(F)Ent values subject to three moment functions

Moment Functions	Fuzzy Entropy
$(1, \sqrt{x}\,, lnx)$	**0.0351**
$(1, \sqrt{x}, ln(1+x))$	**0.0182**
$(1, \sqrt{x}\,, ln(1+x^2))$	0.0277
$(1, lnx, ln(1+x))$	0.0268
$(1, lnx, ln(1+x^2))$	0.0281
$(1, ln(1+x)\,, ln(1+x^2))$	0.0280

For $m = 1, K_{0,1} = \{(1, \sqrt{x}), (1, lnx), (1, ln(1+x)), (1, ln(1+x^2))\}$.

From Table 2, it is shown that $(g_0\,, g^{(1)}) = (1, \sqrt{x})$, $g^{(1)} \in K_{0,1}$ gives to least value to $U(g)$, consequently corresponding distribution is $(\text{MinMax(F)Ent})_1$, and $(g_0\,, g^{(2)}) = (1, lnx)$, $g^{(2)} \in K_{0,1}$ gives to greatest value to $U(g)$, consequently corresponding distribution is $(\text{MaxMax(F)Ent})_1$. In a similar way,

For $m = 2$, $K_{0,2} = \{(1, \sqrt{x}\,, lnx), (1, \sqrt{x}\,, \ln(1+x)), (1, \sqrt{x}\,, ln(1+x^2)), (1, lnx, ln(1+x)), (1, lnx, ln(1+x^2)\}$.

From Table 3, it is shown that $(g_0\,, g^{(1)}) = (1, \sqrt{x}, ln(1+x))$, $g^{(1)} \in K_{0,2}$ gives to least value to $U(g)$, consequently corresponding distribution is $(\text{MinMax(F)Ent})_2$ and $(g_0\,, g^{(2)}) = (1, \sqrt{x}, lnx)$, $g^{(2)} \in K_{0,2}$ gives to greatest value to $U(g)$, consequently corresponding distribution is $(\text{MaxMax(F)Ent})_2$.

Calculated $(\text{MinMax(F)Ent})_\text{m}$ and $(\text{MaxMax(F)Ent})_\text{m}$, $m = 1,2$ distributions are given in Table 4.

Table 4. Distributions of $(\text{MinMax(F)Ent})_\text{m}$ and $(\text{MaxMax(F)Ent})_\text{m}$, $\text{m} = 1{,}2$

$(\text{MinMax(F)Ent})_1$	$(\text{MinMax(F)Ent})_2$	$(\text{MaxMax(F)Ent})_1$	$(\text{MaxMax(F)Ent})_2$
0.9997	0.9998	0.9963	0.9982
0.9998	0.9999	0.9991	0.9996
0.9998	0.9999	0.9995	0.9998
0.9998	0.9999	0.9996	0.9999
0.9998	0.9999	0.9997	0.9999
0.9998	0.9999	0.9997	0.9999
0.9998	0.9999	0.9998	0.9999
0.9999	0.9999	0.9998	1.0000
0.9999	0.9999	0.9998	1.0000
0.9999	0.9999	0.9998	1.0000
0.9999	1.0000	0.9998	1.0000
0.9999	1.0000	0.9999	1.0000
0.9999	1.0000	0.9999	1.0000
0.9999	1.0000	0.9999	1.0000
0.9999	1.0000	0.9999	1.0000
0.9999	1.0000	0.9999	1.0000
0.9999	1.0000	0.9999	1.0000
0.9999	1.0000	0.9999	1.0000

In order to obtain the performance of $(\text{MinMax(F)Ent})_\text{m}$ and $(\text{MaxMax(F)Ent})_\text{m}$ distributions, we use various criterias as Root Mean Square Error (RMSE), Chi-Square (χ^2) and Max(F)Ent measure (H). The acquired results are demonstrated in Table 5-6.

Table 5. The obtained results for $(\text{MinMax(F)Ent})_\text{m}$, $\text{m} = 1{,}2$ distributions

Distributions of $(\text{MinMax(F)Ent})_\text{m}$	Moment Constraints	H	RMSE	χ^2
$(\text{MinMax(F)Ent})_1$	$(1, \sqrt{x})$	0.0375	0.6367	0.5212
$(\text{MinMax(F)Ent})_2$	$(1, \sqrt{x}, ln(1+x))$	0.0182	0.6368	0.5213

Table 6. The obtained results for $(\text{MaxMax(F)Ent})_\text{m}$, $\text{m} = 1{,}2$ distributions

Distributions of $(\text{MaxMax(F)Ent})_\text{m}$	Moment Constraints	H	RMSE	χ^2
$(\text{MaxMax(F)Ent})_1$	$(1, lnx)$	0.0896	0.6363	0.5205
$(\text{MaxMax(F)Ent})_2$	$(1, \sqrt{x}, lnx)$	0.0351	0.6366	0.5211

Tables 5-6 show that in the sense of RMSE and χ^2 criteria each of $(\text{MaxMax(F)Ent})_\text{m}$, $(\text{m} = 1{,}2)$ distribution is better than each of $(\text{MinMax(F)Ent})_\text{m}$, $(\text{m} = 1{,}2)$ distribution. Moreover, $(\text{MaxMax(F)Ent})_1$ distribution shows better performance along $(\text{MaxMax(F)Ent})_\text{m}, \text{m} = 1{,}2$ distributions.

8. CONCLUSION

In the present study the following results are achieved.

- It is proved the convexity property of Max(F)Ent measure, then it is formulated a Maximum Fuzzy Entropy Problem and proposed sufficient conditions for existence of its solution.
- A special functional $U(g)$ depended on moment vector functions g is defined by applying Lagrange multipliers method.

- Maximum Fuzzy Entropy Distribution (Max(F)EntD) is distribution of fuzzy values $\left(\mu_A(x_0), \mu_A(x_1), \dots, \mu_A(x_n)\right)$ maximizing Max(F)Ent measure subject to constraints generated by fixed moment vector function. Maximum Fuzzy Entropy Distribution $\left(\mu_A(x_0), \mu_A(x_1), \dots, \mu_A(x_n)\right)$ can be considered geometrically as points $\left(x_i, \mu_A(x_i)\right), i = 0,1, \dots, n$ of membership function $\mu_A(x)$. Consequently, interpreting these points as experimental data it is possible to select formula, in other words membership function, in accordance on mentioned data by known methods.
- Generalized Maximum Fuzzy Entropy Methods (GMax(F)EntM) in the form of MinMax(F)Ent and MaxMax(F)Ent methods are suggested on the basis of primary maximizing Max(F)Ent measure H_A for fixed moment vector function in order to obtain the special functional $U(g)$ with maximum entropy values of Max(F)Ent measure and secondary optimization for mentioned functional with respect to moment vector functions. Distributions obtained by these methods are defined as $(\mathrm{MinMax(F)Ent})_\mathrm{m}$ and $(\mathrm{MaxMax(F)Ent})_\mathrm{m}$ distributions.

According to obtained results, for this fuzzy data, $(\mathrm{MinMax(F)Ent})_\mathrm{m}$ and $(\mathrm{MaxMax(F)Ent})_\mathrm{m}$, $\mathrm{m} = 1,2$ distributions are compared by using different criterias in terms of modellig data. It is shown that the each of $(\mathrm{MaxMax(F)Ent})_\mathrm{m}, \mathrm{m} = 1,2$ distribution is more suitable in modelling fuzzy data than each of $(\mathrm{MinMax(F)Ent})_\mathrm{m}, \mathrm{m} = 1,2$ distributions in the sense of RMSE and χ^2 criterias. It is found that $(\mathrm{MaxMax(F)Ent})_1$ distribution can provide better results for the fuzzy data in terms of RMSE and χ^2 criterias. Consequently, obtained results are shown that developed methods can be applied succesfully in fuzzy data analysis.

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