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Comparison of certain value-at-risk estimation methods for the two-parameter Weibull loss distribution

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ABSTRACT

The Weibull distribution is one of the most important distributions that is utilized as a probability model for loss amounts in connection with actuarial and financial risk management problems. This paper considers the Weibull distribution and its quantiles in the context of estimation of a risk measure called Value-at-Risk (VaR). VaR is simply the maximum loss in a specified period with a pre-assigned probability level. We attempt to present certain estimation methods for VaR as a quantile of a distribution and compare these methods with respect to their deficiency (*Def*) values. Along this line, the results of some Monte Carlo simulations, that we have conducted for detailed investigations on the efficiency of the estimators as compared to MLE, are provided.

1. Introduction

The Weibull distribution belongs to a class of probability distributions that are known as heavy-tailed distributions; it is even a super-exponential distribution [1, p. 23–78]. With this feature of it, the Weibull distribution is a natural choice for the probability distribution of losses with potentially high outcomes (heavy tail) in finance and insurance [2,3], mostly in the context of the ruin risk modeling. Benckert and Lung [4], Beirlant and Teugels [5] and Mikosch [6] give stimulating ideas and examples that emphasize the heavy-tailed distributional modeling regarding the actuarial science and insurance applications.

By definition, a risk measure is a mapping from a set of random variables, that stand for risks, to the real numbers. The motivation behind the use of risk measures in finance and actuarial science is to determine some critical quantities like required capital, risk reserves and premiums.

Value-at-Risk (VaR) is one of the oldest risk measures that has been intensively used in finance and insurance business. It is basically defined as the maximum expected loss for a given probability. VaR, as a risk measure, is seen very practical to use in many real life risk management practices but it is also criticized for some of its inadequacies in the measurement of risk and risk oriented decision making [7]. Even so, it is such an essential measure that many comprehensive risk measures like Tail Value-at-Risk (TVaR), Conditional Tail Expectation (CTE), Conditional VaR (CVaR) and Expected Shortfall (ES) of a random variable (r.v.) can be expressed as a function of the VaRs of that r.v. [8–10].

Given a loss random variable X, say an aggregate claim amount for an insurance portfolio, with probability distribution function $F_X(x)$, VaR at level α $(0 \leq \alpha \leq 1)$, denoted by $\mathrm{VaR}_\alpha(X)$, is actually an α-quantile risk measure for X. Formally, the α-quantile risk measure for X is defined as $Q_\alpha[X] = \inf\{x : F_X(x) \geq \alpha\}$ which is a left-continuous and non-decreasing

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function of α. For all real valued X and α values, there is an equivalence relation between the quantiles and distribution functions; $Q_\alpha[X] \le x \leftrightarrow \alpha \ge F_X(x)$. Then, by definition,

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha), \tag{1.1}$$

where $F_X^{-1}(\alpha)$ is the inverse of function $F_X(x)$ at a chosen α value.

The subjects of VaRs and quantiles of r.v.'s have been tackled by many authors. Among the recent ones of them, Embrechts et al. [11], Denuit et al. [9] and Gebizlioglu and Yagcı [12] present some results about these quantities for the dependent risks.

Due to the overwhelming significance of quantiles both in statistical theory and risk theory and in their applications, as well, the estimation of $Q_\alpha[X]$ and thus $\mathrm{VaR}_\alpha(X)$ has been always a crucial matter of study. Thereupon, there have been various attempts to obtain efficient estimators of $\mathrm{VaR}_\alpha(X)$ which can be through the parametric approaches at portfolio or position level or through the nonparametric approaches [13, p. 1740–1747]. The nonparametric estimator and the maximum likelihood estimator (MLE) are the two most commonly used estimators of $\mathrm{VaR}_\alpha(X)$ [14]. The nonparametric method of estimating the α-quantile of F is known to be the method of sample quantiles. It is well known and widely used in statistical inference and applications [15]. The α-quantile of F in the sample quantile context is defined as $X_{n,[n\alpha]+1}$ where $X_{n,[n\alpha]+1}$ is the $([n\alpha]+1)$th order statistic of the sample and [.] is the greatest integer value function. On the other hand; for a given value of α, the MLE of $\mathrm{VaR}_\alpha(X)$ is calculated by inserting the MLE estimators of the parameters into the inverse function F^{-1}.

In this paper, we consider different estimation methods for $\mathrm{VaR}_\alpha(X)$ and study how they behave for different shape parameters and for different sample sizes that are drawn from a Weibull distribution as a loss distribution for a loss amount random variable. We compare the performances of nine parametric estimators including MLE with respect to their *Def* values through Monte Carlo simulation. *Def* is a mean square error (MSE) based measure of the joint efficiency of estimators of a set of parameters, say (θ, β), of a probability distribution. It is calculated as the sum of MSEs, $Def(\hat{\theta}, \hat{\beta}) = \mathrm{MSE}(\hat{\theta}) + \mathrm{MSE}(\hat{\beta})$, for the estimates $(\hat{\theta}, \hat{\beta})$, which are obtained by a chosen method of estimation. As the methods of estimation for the parameters at hand, we compute and compare moment estimator (ME), generalized spacing estimator (GSE), modified maximum likelihood estimator I (MMLE-I), modified maximum likelihood estimator II (MMLE-II), Tiku's modified maximum likelihood estimator (TMMLE), least squares estimator (LSE), weighted least squares estimator (WLSE) and percentile estimator (PE).

Recently, Kantar and Şenoğlu [16] have estimated the parameters of the Weibull distribution using the estimators mentioned above with known shape parameter and compared them with respect to their bias, MSE and *Def* values.

A new approach has been implemented in our work: a two-parameter Weibull loss distribution with a known location parameter is considered and some different estimators of its scale and the shape parameters are compared with respect to the *Def* values. For the first time in the research area, the best method that gives the highest efficiency for estimating $\mathrm{VaR}_\alpha(X)$ is shown here for several sample sizes and shape parameter values.

The paper is arranged as follows: Section 2 introduces the Weibull distribution. A concise description of the estimators mentioned above is given in Section 3. Section 4 presents the results of the simulation study. Section 5 presents a real life example with data from the insurance sector. The conclusions are given in Section 6.

2. The Weibull distribution

Cumulative distribution function (CDF) of the two-parameter Weibull distribution is given by

$$F(x;\theta,\beta) = 1 - \exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}, \quad x \ge 0\ \theta > 0,\ \beta > 0, \tag{2.1}$$

where θ is the scale parameter and β is the shape parameter. The corresponding pdf is expressed by

$$f(x) = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}}, \quad x \ge 0,\ \theta > 0,\ \beta > 0. \tag{2.2}$$

The Weibull distribution is reversed J shaped if $\beta < 1$, and bell shaped if $\beta > 1$. When $\beta = 1$, it reduces to the density function of the well-known exponential distribution.

Among the distributions with asymmetric features, the Weibull distribution is used in many applications in the areas of risk management, insurance, finance life testing, reliability engineering and biology, as well, because of its flexible properties mentioned above.

Under the two-parameter Weibull loss distribution, $\mathrm{VaR}_\alpha(X)$ is estimated by

$$\widehat{\mathrm{VaR}}_\alpha(X) = \{-\ln(1-\alpha)\}^{1/\hat{\beta}}\,\hat{\theta} \tag{2.3}$$

where α is the confidence level.

It is clear that the estimation and the efficiency of estimators for VaR rely on the estimation of the parameters of the Weibull distribution considered here.

Many methods have been proposed to estimate the unknown parameters of the Weibull distribution. MLE is the most widely used method among the others because it has the asymptotic efficiency property under a parametric model. However, it loses efficiency when the usual regularity conditions are not satisfied or the sample size is small. Now the question is if the estimated values of the unknown parameters of the two-parameter Weibull distribution were obtained by using the estimators given above, what would be the efficiencies of them as compared to MLE. In this paper, we answer this question by comparing all the nine estimators mentioned above via the Monte Carlo simulation study in terms of their means, MSEs and *Def* values.

3. Estimation of the parameters

The estimators that we consider for the scale parameter θ and the shape parameter β of the two-parameter Weibull distribution are described in the following subsections:

3.1. Maximum likelihood estimators (MLEs)

If $X_1, X_2, \ldots, X_n$ is a random sample from the two-parameter Weibull distribution with unknown scale and shape parameters, then the log-likelihood function of the sample is given by;

$$\ln L = n \ln \beta - n\beta \ln \theta + (\beta - 1) \sum_{i=1}^{n} \ln x_i - \theta^{-\beta} \sum_{i=1}^{n} x_i^{\beta}.$$

In order to maximize the log-likelihood function with respect to θ and β, we obtain the first derivatives of the log-likelihood function and equate them to zero as shown below;

$$\frac{\partial \ln L}{\partial \theta} = -n\beta\theta^{-1} + \beta\theta^{-(\beta+1)} \sum_{i=1}^{n} x_i^{\beta} = 0$$

and

$$\frac{\partial \ln L}{\partial \beta} = n\beta^{-1} - n \ln \theta + \sum_{i=1}^{n} \ln x_i - \theta^{-\beta} \sum_{i=1}^{n} x_i^{\beta} \ln x_i + \theta^{-\beta} \ln \theta \sum_{i=1}^{n} x_i^{\beta} = 0.$$

It is clear that the MLEs do not have explicit solutions. Therefore, we resort to iterative methods. Iterative solutions of these equations would yield the MLEs of the scale parameter θ and the shape parameter β.

3.2. Moment estimators (MEs)

MEs are obtained by equating theoretical moments to their corresponding sample moments as shown below;

$$\bar{X} = \theta\, \Gamma\left(1 + \frac{1}{\beta}\right), \tag{3.1}$$

$$s^2 = \left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right] \theta^2, \tag{3.2}$$

where $\bar{X}$ and s^2 are the sample mean and the sample variance, respectively. Dividing the square of (3.1) by (3.2) we obtain

$$\frac{\bar{X}^2}{s^2} = \frac{\left\{\Gamma\left(1 + \frac{1}{\beta}\right)\right\}^2}{\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)}. \tag{3.3}$$

It is clear that the only unknown parameter in this equation is β. ME of β is the iterative solution of this equation and is represented by $\widehat{\beta}_{\mathrm{ME}}$. Thus, $\widehat{\theta}_{\mathrm{ME}}$ can be obtained from the following equation:

$$\widehat{\theta}_{\mathrm{ME}} = \frac{\overline{X}}{\Gamma\left(1 + \frac{1}{\widehat{\beta}_{\mathrm{ME}}}\right)}.$$

3.3. Generalized spacing estimators (GSEs)

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution function F_ϕ with unknown parameters θ and β and $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the corresponding order statistics.

The GSE of $\phi = (\theta, \beta)$ is obtained by minimizing the following function:

$$T(\phi) \stackrel{\text{def}}{=} \sum_{i=1}^{n} h(nD_i(\phi)),$$

where $h : (0, \infty) \rightarrow R$ is a strictly convex function, $D_i(\phi) = F_\phi(X_{(i)}) - F_\phi(X_{(i-1)})$ $(i = 1, 2, \ldots, n)$, $F_\phi(X_{(0)}) \overset{\text{def}}{=} 0$ and $F_\phi(X_{(n)}) \overset{\text{def}}{=} 1$. The minimization procedure is conducted by using iterative methods. In this study, $h(x)$ is chosen to be $-\log x$ from the various other choices of convex functions since GSE with $h(x) = -\log x$ shows a better performance than the other choices of the function $h(x)$ in terms of MSE [17,18].

3.4. Modified maximum likelihood estimators (MMLEs)

Cohen and Whitten [19] proposed modifications of the MLEs for estimating the unknown parameters of the two-parameter Weibull distribution. MMLE-I and MMLE-II estimators of the shape parameter β, i.e., $\widehat{\beta}_{\text{MMLEI}}$ and $\widehat{\beta}_{\text{MMLEII}}$, are obtained by solving the following equations with respect to β;

$$\frac{n x_{(1)}^{\beta}}{-\ln[n/(n+1)]} = \sum_{i=1}^{n} x_i^{\beta}$$

and

$$n^2 \left(\frac{x_{(1)}}{\Gamma\left(1 + \frac{1}{\beta}\right)} \right)^{\beta} = \sum_{i=1}^{n} x_i^{\beta},$$

respectively. Thus, the MMLE-I and MMLE-II estimators of the scale parameter θ are obtained by inserting the estimates $\widehat{\beta}_{\text{MMLE-I}}$ and $\widehat{\beta}_{\text{MMLE-II}}$ in the following equation:

$$\hat{\theta} = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\hat{\beta}_{\text{MMLE}}} \right)^{1/\hat{\beta}_{\text{MMLE}}}.$$

3.5. Tiku's modified maximum likelihood estimators (TMMLEs)

Tiku [20,21] and Tiku and Akkaya [22] suggest a simple method for deriving explicit estimators of the parameters. The method is based on linearizing the likelihood equations around the first two terms of the Taylor series expansion. TMMLEs of θ and β of the two-parameter Weibull distribution are obtained via the Extreme Value distribution having a pdf given by;

$$f(w) = \frac{1}{\eta} \exp\left(\frac{w - \delta}{\eta} \right) \exp\left(-\exp^{\left(\frac{w-\delta}{\eta}\right)} \right), \quad w \in IR;\ \eta > 0,\ \delta \in IR,$$

where $\delta = \ln\theta$ is the location parameter and $\eta = 1/\beta$ is the scale parameter.

First, the estimators of the Extreme Value distribution parameters are obtained and then the parameters of the Weibull distribution are found by using the following inverse transformations:

$$\theta = \exp(\delta) \quad \text{and} \quad \beta = 1/\eta.$$

This is due to the fact that if the distribution of the random variable X is Weibull then $\ln X$ has an Extreme Value distribution. TMMLEs of θ and β are obtained as

$$\widehat{\theta}_{\text{TMMLE}} = \exp(\widehat{\delta}) \quad \text{and} \quad \widehat{\beta}_{\text{TMMLE}} = \frac{1}{\widehat{\eta}},$$

where

$$\widehat{\delta} = K + D\hat{\eta}, \qquad \widehat{\eta} = \{B + \sqrt{(B^2 + 4nC)}\}/2n,$$

$$K = \sum_{i=1}^{n} \beta_i x_{(i)}/m, \qquad D = \sum_{i=1}^{n} (\alpha_i - 1)/m, \qquad B = \sum_{i=1}^{n} (\alpha_i - 1)(x_{(i)} - K), \qquad C = \sum_{i=1}^{n} \beta_i (x_{(i)} - K)^2,$$

$$m = \sum_{i=1}^{n} \beta_i, \qquad \alpha_i = \exp(t_{(i)})(1 - t_{(i)}), \qquad \beta_i = \exp(t_{(i)}) \quad \text{and} \quad t_{(i)} = \ln(-\ln(1 - i/(n+1))), \quad 1 \le i \le n.$$

It should be noted that TMMLEs of η and δ are bias corrected by replacing D with $-(1/m)\sum_{i=1}^{n} \beta_i t_{(i)}$ and the divisor $2n$ in $\widehat{\eta}$ with $2m$ when $n \le 15$.

3.6. Least squares estimators (LSEs)

The LSE method suggested in [23] is based on the knowledge of the distribution function $F(X)$. LSEs are obtained by minimizing

$$\sum_{i=1}^{n} \left(F(X_{(i)}) - \frac{i}{n+1} \right)^2$$

with respect to the unknown parameters. Let $X_1, X_2, \ldots, X_{n-1}$ be independent and identically distributed (i.i.d.) observations from a distribution function $F(X)$, $X_{(1)} < X_{(2)} < \cdots < X_{(n-1)}$ are ordered random variables and $\frac{i}{n+1}$ is the expected value of $F(X_{(i)})$. Therefore, the LSEs of the scale and the shape parameters of the Weibull distribution are obtained by minimizing the following equation with respect to θ and β;

$$\sum_{i=1}^{n}\left(1-\exp\left\{-\left[(x_{(i)})/\theta\right]^{\beta}\right\}-\frac{i}{n+1}\right)^{2}.$$

3.7. Weighted least squares estimators (WLSEs)

The WLSEs of the parameters are found by minimizing the function

$$\sum_{i=1}^{n}\frac{1}{\operatorname{Var}(F(X_{(i)}))}\left(F(X_{(i)})-\frac{i}{n+1}\right)^{2}$$

with respect to the unknown parameters. It is known that $V(F(X_{(i)})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}$ (see [23]). Therefore, in the case of the Weibull distribution, the WLSEs of the parameters θ and β are obtained by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1-\exp\left\{-\left[(x_{(i)})/\theta\right]^{\beta}\right\}-\frac{i}{n+1}\right)^{2}$$

with respect to θ and β.

3.8. Percentile estimators (PEs)

The basic idea behind the PEs is to minimize the difference between the theoretical percentiles and the sample percentiles [24,25]. The PEs of the parameters are obtained by minimizing the function

$$\sum_{i=1}^{n}\left[x_{(i)}-F^{-1}(p_i)\right]^{2}$$

with respect to the unknown parameters. Although various different estimators of p_i have been used in the literature, here we take p_i as $\frac{i}{n+1}$. In the case of the Weibull distribution, the PEs of the scale and the shape parameters are obtained by minimizing

$$\sum_{i=1}^{n}\left[x_{(i)}-\theta\left[-\ln\left(1-\frac{i}{n+1}\right)\right]^{\frac{1}{\beta}}\right]^{2}$$

with respect to θ and β.

4. Estimation of VaR and simulation study for efficiency determinations

The $\mathrm{VaR}_\alpha(X)$ for random variable X with the two-parameter Weibull distribution is defined in Eq. (2.3) of Section 2. As said there too, in the attempt of estimating $\mathrm{VaR}_\alpha(X)$, the efficiency of its estimators is to be determined as the joint efficiency of the estimators of the parameters of the Weibull distribution.

In this regard, we have performed a large scale Monte Carlo simulation to evaluate the performances of the different estimators for θ, β and $\mathrm{VaR}_\alpha(X)$. The estimators are compared according to their means and MSEs and *Def* values for the sample sizes $n = 10, 20, 50, 100, 200$ and for the shape parameters $\beta = 0.5, 1.5, 3$ and 6. Throughout the study, the scale parameter θ is taken to be 1 since all the estimators are scale invariant. The simulation replication number is taken to be $1000 \times n$ for each different sample size n. All the computations were conducted in MATLAB 7.5. The simulated means and MSEs of the different estimators of the Weibull parameters and the $Def(\hat{\theta}, \hat{\beta}) = \mathrm{MSE}(\hat{\theta}) + \mathrm{MSE}(\hat{\beta})$ values are given in Tables 1–4.

The use of the concept of *Def* is essential for comparing the different estimators of $\mathrm{VaR}_\alpha(X)$, since the efficiency of $\widehat{\mathrm{VaR}}_\alpha(X) = \{-\ln(1-\alpha)\}^{1/\hat{\beta}}\,\hat{\theta}$ depends on the joint efficiency of the estimators $\hat{\theta}$ and $\hat{\beta}$.

4.1. Comparisons of the biases of the estimators

In the following subsections, the biases of the estimated parameters are compared with respect to the estimators that have been employed. The computed *Def* values of the estimators are also compared. We name $n = 10$ and 20 as the small samples, $n = 50$ as the moderate sample and $n = 100$ and 200 as the large samples in the context of the comparisons.

Table 1
Simulated means and MSE values for the parameters θ and β; $\theta = 1,\ \beta = 0.5$.

Estimator	$\widehat{\theta}$		$\widehat{\beta}$		*Def*
	Mean	MSE	Mean	MSE	
$n = 10$					
MLE	1.1457	0.5980	0.5853	0.0365	0.6345
TMMLE	1.2623	0.7583	0.5144	0.0228	0.7811
ME	1.5129	1.2291	0.7511	0.0997	1.3288
PE	1.3078	1.2872	0.5231	0.0592	1.3464
GSE	1.2490	0.7554	0.4673	0.0203	0.7758
LSE	1.2693	0.8306	0.5095	0.0442	0.8748
WLSE	1.2481	0.7716	0.5101	0.0381	0.8098
MMLE-I	1.0395	0.5243	0.4886	0.0354	0.5597
MMLE-II	0.8924	0.4617	0.4043	0.0532	0.5149
$n = 20$					
MLE	1.0767	0.2620	0.5377	0.0121	0.2740
TMMLE	1.0257	0.2366	0.5358	0.0119	0.2484
ME	1.3581	0.5371	0.6508	0.0409	0.5780
PE	1.1256	0.6342	0.4882	0.0352	0.6695
GSE	1.1303	0.3001	0.4681	0.0095	0.3096
LSE	1.1356	0.3282	0.4996	0.0147	0.3428
WLSE	1.1176	0.3000	0.5043	0.0130	0.3130
MMLE-I	0.9906	0.2605	0.4718	0.0215	0.2820
MMLE-II	0.8984	0.2331	0.4172	0.0270	0.2601
$n = 50$					
MLE	1.0318	0.0985	0.5120	0.0036	0.1021
TMMLE	1.0106	0.0945	0.5117	0.0036	0.0981
ME	1.2166	0.2085	0.5795	0.0149	0.2234
PE	0.9979	0.3191	0.4704	0.0201	0.3392
GSE	1.0551	0.1042	0.4774	0.0035	0.1078
LSE	1.0556	0.1179	0.4952	0.0045	0.1224
WLSE	1.0470	0.1083	0.4995	0.0039	0.1122
MMLE-I	0.9727	0.1119	0.4695	0.0136	0.1255
MMLE-II	0.8902	0.1166	0.4185	0.0220	0.1386
$n = 100$					
MLE	1.0115	0.0462	0.5065	0.0015	0.0477
TMMLE	0.9999	0.0449	0.5067	0.0015	0.0464
ME	1.1209	0.1000	0.5459	0.0075	0.1075
PE	0.9148	0.2128	0.4623	0.0144	0.2273
GSE	1.0237	0.0473	0.4859	0.0016	0.0489
LSE	1.0195	0.0515	0.5000	0.0022	0.0537
WLSE	1.0152	0.0479	0.5027	0.0018	0.0497
MMLE-I	0.9635	0.0686	0.4724	0.0107	0.0793
MMLE-II	0.8975	0.0752	0.4317	0.0158	0.0910
$n = 200$					
MLE	1.0124	0.0224	0.5030	0.0008	0.0232
TMMLE	1.0065	0.0221	0.5032	0.0008	0.0228
ME	1.0818	0.0553	0.5283	0.0046	0.0598
PE	0.9031	0.1556	0.4667	0.0106	0.1662
GSE	1.0189	0.0229	0.4910	0.0008	0.0237
LSE	1.0167	0.0244	0.4996	0.0010	0.0254
WLSE	1.0142	0.0229	0.5013	0.0009	0.0238
MMLE-I	0.9624	0.0396	0.4691	0.0083	0.0479
MMLE-II	0.9044	0.0461	0.4336	0.0125	0.0586

4.1.1. Comparisons of the biases of $\hat{\theta}$

For the shape parameter $\beta = 0.5$, it is seen that MMLE-I outperforms other estimators for small samples, PE performs best for moderate samples and TMMLE has the minimum biases for the large samples. ME shows the worst performance for the sample sizes $n \leq 100$ and the performance of PE is quite poor for $n = 200$. It can be seen that MMLE-II underestimates θ for all cases.

For the shape parameter $\beta = 1.5$, MLE and ME exhibit the lowest bias among the others for the sample sizes $n \leq 100$. TMMLE performs best for the sample size $n = 200$. MMLE-I and MMLE-II have the maximum biases for all the sample sizes.

For the shape parameter $\beta = 3.0$, WLSE, LSE, ME, MLE and GSE have the minimum biases for the sample sizes 10, 20, 50, 100 and 200, respectively. MMLE-I shows the worst performance for all cases considered here. MMLE-I and MMLE-II underestimate θ for all cases when $\beta = 1.5$ and 3.

Table 2
Simulated means and MSE values for the parameters θ and β; $\theta = 1$, $\beta = 1.5$.

Estimator	$\widehat{\theta}$		$\widehat{\beta}$		*Def*
	Mean	MSE	Mean	MSE	
$n = 10$					
MLE	0.9991	0.0476	1.7612	0.3220	0.3696
TMMLE	1.0328	0.0510	1.5456	0.1966	0.2476
ME	1.0005	0.0476	1.7748	0.3113	0.3589
PE	1.0301	0.0532	1.4625	0.2006	0.2538
GSE	1.0300	0.0507	1.3961	0.1696	0.2203
LSE	1.0328	0.0553	1.5238	0.3450	0.4002
WLSE	1.0281	0.0526	1.5261	0.3025	0.3552
MMLE-I	0.9575	0.0509	1.4745	0.3277	0.3786
MMLE-II	0.9625	0.0535	1.5343	0.4506	0.5041
$n = 20$					
MLE	1.0012	0.0245	1.6227	0.1135	0.1380
TMMLE	0.9850	0.0242	1.6163	0.1112	0.1355
ME	1.0018	0.0246	1.6335	0.1156	0.1402
PE	1.0173	0.0268	1.4496	0.1015	0.1283
GSE	1.0183	0.0255	1.4123	0.0822	0.1077
LSE	1.0170	0.0275	1.5082	0.1285	0.1561
WLSE	1.0135	0.0261	1.5208	0.1139	0.1401
MMLE-I	0.9703	0.0279	1.4244	0.1936	0.2216
MMLE-II	0.9775	0.0293	1.4757	0.2444	0.2737
$n = 50$					
MLE	1.0024	0.0099	1.5421	0.0339	0.0436
TMMLE	0.9954	0.0099	1.5416	0.0338	0.0437
ME	1.0024	0.0100	1.5471	0.0372	0.0472
PE	1.0080	0.0108	1.4521	0.0430	0.0538
GSE	1.0100	0.0101	1.4375	0.0319	0.0419
LSE	1.0085	0.0112	1.4975	0.0447	0.0559
WLSE	1.0061	0.0105	1.5092	0.0382	0.0487
MMLE-I	0.9782	0.0128	1.4021	0.1194	0.1323
MMLE-II	0.9856	0.0135	1.4446	0.1387	0.1522
$n = 100$					
MLE	1.0010	0.0047	1.5195	0.0141	0.0188
TMMLE	0.9975	0.0047	1.5204	0.0141	0.0189
ME	1.0012	0.0048	1.5224	0.0154	0.0202
PE	1.0035	0.0051	1.4638	0.0198	0.0249
GSE	1.0053	0.0048	1.4584	0.0144	0.0192
LSE	1.0042	0.0053	1.4976	0.0206	0.0259
WLSE	1.0027	0.0050	1.5059	0.0167	0.0217
MMLE-I	0.9817	0.0075	1.4138	0.0933	0.1008
MMLE-II	0.9869	0.0079	1.4549	0.1028	0.1107
$n = 200$					
MLE	1.0030	0.0026	1.5137	0.0065	0.0091
TMMLE	1.0011	0.0026	1.5139	0.0066	0.0091
ME	1.0034	0.0026	1.5177	0.0073	0.0099
PE	1.0048	0.0027	1.4839	0.0092	0.0119
GSE	1.0052	0.0026	1.4777	0.0066	0.0092
LSE	1.0057	0.0028	1.5017	0.0094	0.0122
WLSE	1.0045	0.0026	1.5066	0.0075	0.0101
MMLE-I	0.9874	0.0053	1.4317	0.0788	0.0841
MMLE-II	0.9940	0.0057	1.4678	0.0883	0.0940

For the shape parameter $\beta = 6$, WLSE outperforms the other estimators for the small samples. PE and LSE perform better for the moderate samples. ME and MMLE-II are preferred for the sample sizes 100 and 200, respectively. MMLE-I has the maximum biases and underestimates θ for all the sample sizes.

4.1.2. *Comparisons of the biases of* $\hat{\beta}$

For the shape parameter $\beta = 0.5$, LSE performs best for the small and large samples and WLSE is preferred for the moderate samples. ME shows the worst performance for the small samples. MMLE-II exhibits less bias for the moderate and large samples. MMLE-I and MMLE-II underestimate β for all cases considered here.

Table 3
Simulated means and MSE values for the parameters θ and β; $\theta = 1,\ \beta = 3$.

Estimator	$\widehat{\theta}$		$\widehat{\beta}$		*Def*
	Mean	MSE	Mean	MSE	
$n = 10$					
MLE	0.9923	0.0122	3.4982	1.3222	1.3344
TMMLE	1.0091	0.0124	3.0737	0.8346	0.8470
ME	0.9915	0.0122	3.4674	1.3078	1.3200
PE	1.0080	0.0123	2.8645	0.7194	0.7317
GSE	1.0079	0.0123	2.7755	0.7248	0.7371
LSE	1.0075	0.0134	3.0325	1.4249	1.4383
WLSE	1.0056	0.0129	3.0377	1.2442	1.2571
MMLE-I	0.9738	0.0132	2.9392	1.2630	1.2762
MMLE-II	0.9852	0.0133	3.2636	1.8056	1.8189
$n = 20$					
MLE	0.9959	0.0061	3.2313	0.4447	0.4508
TMMLE	0.9877	0.0062	3.2200	0.4366	0.4429
ME	0.9955	0.0061	3.2164	0.4425	0.4487
PE	1.0047	0.0061	2.8877	0.3222	0.3283
GSE	1.0043	0.0062	2.8131	0.3315	0.3377
LSE	1.0032	0.0067	3.0083	0.5435	0.5502
WLSE	1.0014	0.0064	3.0357	0.4843	0.4907
MMLE-I	0.9794	0.0083	2.8437	0.7694	0.7777
MMLE-II	0.9907	0.0080	3.1099	0.9813	0.9893
$n = 50$					
MLE	1.0001	0.0025	3.0795	0.1369	0.1394
TMMLE	0.9966	0.0025	3.0788	0.1371	0.1396
ME	1.0000	0.0025	3.0753	0.1399	0.1424
PE	1.0043	0.0025	2.9215	0.1259	0.1284
GSE	1.0040	0.0025	2.8700	0.1303	0.1328
LSE	1.0031	0.0027	2.9895	0.1777	0.1805
WLSE	1.0019	0.0026	3.0149	0.1548	0.1574
MMLE-I	0.9877	0.0034	2.8061	0.5169	0.5203
MMLE-II	0.9974	0.0035	3.0193	0.5966	0.6001
$n = 100$					
MLE	0.9998	0.0013	3.0503	0.0639	0.0652
TMMLE	0.9979	0.0013	3.0509	0.0642	0.0655
ME	0.9996	0.0013	3.0479	0.0659	0.0672
PE	1.0022	0.0013	2.9611	0.0600	0.0613
GSE	1.0018	0.0013	2.9266	0.0620	0.0633
LSE	1.0011	0.0014	3.0081	0.0889	0.0904
WLSE	1.0004	0.0013	3.0235	0.0738	0.0751
MMLE-I	0.9900	0.0021	2.8450	0.3786	0.3807
MMLE-II	0.9982	0.0021	3.0128	0.3984	0.4005
$n = 200$					
MLE	0.9991	0.0006	3.0157	0.0267	0.0273
TMMLE	0.9981	0.0006	3.0169	0.0267	0.0273
ME	0.9991	0.0006	3.0158	0.0272	0.0277
PE	1.0004	0.0006	2.9659	0.0272	0.0277
GSE	1.0002	0.0006	2.9436	0.0285	0.0291
LSE	0.9993	0.0006	2.9983	0.0409	0.0415
WLSE	0.9991	0.0006	3.0069	0.0320	0.0325
MMLE-I	0.9889	0.0013	2.8086	0.3229	0.3241
MMLE-II	0.9966	0.0013	2.9731	0.3408	0.3421

For the shape parameter $\beta = 1.5$, the biases of the LSE are lower than the other estimators for all the sample sizes. ME did not do well for the small samples and the performance of MMLE-I is quite poor for the moderate and large samples. MMLE-I underestimates β for all the sample sizes.

For the shape parameter $\beta = 3.0$, LSE outperforms others for the small and large samples. For the moderate sample, WLSE exhibits the highest performance. The worst performance is with MLE for the small samples. For the moderate and large samples, MMLE-I shows the worst performance. As in the case of $\beta = 1.5$, MMLE-I underestimates β for all the sample sizes.

For the shape parameter $\beta = 6.0$, LSE has the minimum biases for the small and moderate samples. WLSE and MMLE-II perform quite well for the sample sizes 100 and 200, respectively. ME has the maximum biases for the small samples while MMLE-I has the maximum biases for the moderate and large samples.

Table 4
Simulated means and MSE values for the parameters θ and β; $\theta = 1,\ \beta = 6$.

Estimator	$\widehat{\theta}$		$\widehat{\beta}$		*Def*
	Mean	MSE	Mean	MSE	
$n = 10$					
MLE	0.9940	0.0031	7.0248	5.5192	5.5223
TMMLE	1.0024	0.0031	6.1705	3.4578	3.4609
ME	0.9933	0.0031	7.0289	5.7836	5.7867
PE	1.0008	0.0031	5.7019	3.1214	3.1245
GSE	1.0018	0.0031	5.5733	2.9894	2.9925
LSE	1.0016	0.0034	6.0603	5.4726	5.4760
WLSE	1.0005	0.0033	6.0861	4.9090	4.9123
MMLE-I	0.9839	0.0035	6.1145	7.3810	7.3845
MMLE-II	0.9920	0.0034	6.7538	7.7471	7.7505
$n = 20$					
MLE	0.9977	0.0016	6.4626	1.6793	1.6809
TMMLE	0.9935	0.0016	6.4415	1.6509	1.6525
ME	0.9972	0.0016	6.4932	1.9595	1.9611
PE	1.0013	0.0016	5.7479	1.3167	1.3183
GSE	1.0019	0.0016	5.6290	1.2527	1.2543
LSE	1.0009	0.0017	6.0200	2.0754	2.0771
WLSE	1.0002	0.0017	6.0781	1.8216	1.8233
MMLE-I	0.9902	0.0019	5.7451	3.0686	3.0705
MMLE-II	0.9978	0.0018	6.4648	4.1293	4.1311
$n = 50$					
MLE	0.9986	0.0006	6.1776	0.5430	0.5435
TMMLE	0.9968	0.0006	6.1748	0.5433	0.5439
ME	0.9984	0.0006	6.1896	0.6507	0.6513
PE	1.0003	0.0006	5.8398	0.5323	0.5329
GSE	1.0004	0.0006	5.7594	0.5035	0.5041
LSE	0.9997	0.0006	6.0081	0.7555	0.7562
WLSE	0.9993	0.0006	6.0536	0.6397	0.6403
MMLEI	0.9922	0.0009	5.6322	2.0458	2.0467
MMLEII	0.9987	0.0008	6.2059	2.4220	2.4228
$n = 100$					
MLE	1.0004	0.0003	6.0622	0.2367	0.2370
TMMLE	0.9994	0.0003	6.0634	0.2370	0.2374
ME	1.0002	0.0003	6.0719	0.2874	0.2878
PE	1.0013	0.0003	5.8743	0.2604	0.2607
GSE	1.0014	0.0003	5.8157	0.2486	0.2489
LSE	1.0009	0.0004	5.9643	0.3463	0.3466
WLSE	1.0007	0.0003	6.0014	0.2839	0.2842
MMLE-I	0.9954	0.0005	5.6548	1.5742	1.5747
MMLE-II	1.0012	0.0005	6.1565	1.7900	1.7905
$n = 200$					
MLE	0.9987	0.0001	6.0304	0.1136	0.1137
TMMLE	0.9982	0.0002	6.0321	0.1142	0.1144
ME	1.0007	0.0002	6.0642	0.1348	0.1350
PE	0.9992	0.0001	5.9210	0.1292	0.1293
GSE	0.9992	0.0001	5.8876	0.1201	0.1202
LSE	1.0009	0.0002	6.0425	0.1710	0.1712
WLSE	1.0008	0.0002	6.0538	0.1375	0.1376
MMLE-I	0.9949	0.0003	5.5846	1.2637	1.2640
MMLE-II	1.0000	0.0003	6.0218	1.3003	1.3006

4.2. Comparisons of the Def values of the estimators

For the shape parameter $\beta = 0.5$, simulation results show that MMLE-I and MMLE-II outperform the other estimators with the smallest deficiency when the sample size is 10. For the sample size 20, TMMLE and MMLE-II work best for estimating the unknown parameters. TMMLE and MLE yield the smallest deficiency for the moderate and large samples.

For the shape parameter $\beta = 1.5$, GSE demonstrates the strongest performance with the lowest deficiency for the small and moderate sample sizes. For the large samples, MLE and TMMLE perform better than the others.

PE and GSE provide the smallest deficiency for the shape parameter $\beta = 3.0$, therefore they are the best estimators for estimating the unknown parameters of the two-parameter Weibull distribution for the sample sizes $n \leq 100$. For the large samples, MLE and TMMLE demonstrate the strongest performance with the lowest deficiency as expected.

Table 5
The average CPU times per sample for each of the different parameter estimators.

Estimator	$n = 10$	20	50	100	200
$\beta = 0.5$					
MLE	0.4688	0.7282	1.6468	3.4656	8.4124
TMMLE	[a]	[a]	[a]	[a]	[a]
ME	0.1938	0.1968	0.1968	0.1908	0.1968
PE	0.4188	0.6876	1.3468	2.9250	6.3970
GSE	0.5594	0.7936	1.9000	3.9470	9.1094
LSE	0.4312	0.6280	1.3562	2.7468	2.7468
WLSE	0.4562	0.7374	1.5500	1.5812	7.7656
MMLE-I	0.2406	0.2750	0.4562	0.8344	1.9094
MMLE-II	0.2532	0.2812	0.4592	0.8094	1.9062
$\beta = 1.5$					
MLE	0.4720	0.7438	1.5970	3.3532	8.5342
TMMLE	[a]	[a]	[a]	[a]	[a]
ME	0.1876	0.1844	0.1906	0.1970	0.1970
PE	0.4000	0.5718	1.2344	2.5406	6.0968
GSE	0.5000	0.8188	1.8658	3.9094	8.9688
LSE	0.4124	0.6374	1.3686	2.8344	6.4500
WLSE	0.4656	0.7126	1.5282	3.3032	7.8876
MMLE-I	0.8124	0.2814	0.4468	0.8064	1.9000
MMLE-II	0.2468	0.2812	0.4562	0.4468	1.8876
$\beta = 3$					
MLE	0.4626	0.7406	1.6688	3.5782	8.7344
TMMLE	[a]	[a]	[a]	[a]	[a]
ME	0.1906	0.1906	0.1844	0.1842	0.1938
PE	0.3782	0.5906	1.275	2.6436	6.1406
GSE	0.4906	0.8408	1.872	3.9282	9.2156
LSE	0.6374	0.6532	1.4062	2.8126	6.5156
WLSE	0.4562	0.4688	1.5874	3.4124	8.0374
MMLEI	0.2282	0.2688	0.4532	0.8186	1.9062
MMLEII	0.2376	0.2844	0.4468	0.8282	1.8686
$\beta = 6$					
MLE	0.5064	0.7624	1.75	3.6906	8.9032
TMMLE	[a]	[a]	[a]	[a]	[a]
ME	0.1938	0.1874	0.1938	0.1908	0.1906
PE	0.3906	0.6000	1.3250	2.7438	6.2968
GSE	0.6844	0.8376	1.9626	4.0844	9.6094
LSE	0.4312	0.6782	1.4280	2.9500	6.7000
WLSE	0.4686	0.7562	1.6688	3.5500	8.4906
MMLE-I	0.2218	0.2688	0.4438	0.7938	1.8970
MMLE-II	0.2470	0.2844	0.4564	0.8156	1.8750

[a] Less than 0.0001 CPU.

GSE and PE give the smallest deficiency values for the small and moderate samples when $\beta = 6.0$. MLE and TMMLE apparently show similar better performances than the other estimators for the large samples.

4.3. Comparisons of the computational complexities for the estimators

It should be noted that the iterative methods in parameter estimations are generally problematic due to (i) multiple roots, (ii) nonconvergence of iterations and (iii) convergence to wrong values.

Vaughan [26], Tiku and Akkaya [22] and Kantar and Şenoğlu [16] discuss these complications to an extent. We consider the central processing unit (CPU) times needed for the estimations for comparing the estimators since a CPU time indicates computational complexities. About the estimators that are considered in this paper, we have needed some iterative methods except the TMMLE method. It should be noted that the TMMLE are the explicit functions of the sample observations. If the efficiencies of two different estimators are close to each other, the choice of the better method can be made according to the CPU times [19]. In this respect, the average CPU times for the Monte Carlo runs are given in Table 5.

It is seen in Table 5 that the CPU times of the iterative methods per sample, except for ME, increase rapidly as the sample size increases.

5. An application

In the work for this paper, some car accident data of 500 incidents, recorded during the period January, 2009–December, 2009 by one of the leading insurance companies in Turkey, were used for the real life exemplary purposes. The data comprise the amount of claims in TL in the car accidents. The data set can be obtained from the third author upon request.

Upon the application of the Kolmogorov–Smirnov, Anderson–Darling and Chi-squared tests; the null hypothesis "H_0: loss distribution is Weibull" was not rejected. Therefore, the decision was that the distribution of the car accidents' claim size data is a Weibull distribution.

The estimates of the scale and the shape parameters for each estimation method are given below.

Estimator	$\widehat{\theta}$	$\widehat{\beta}$	$\widehat{\text{VaR}}_\alpha(X)$
MLE	1515.092	0.6727	7740.749
TMMLE	1417.325	0.6790	7132.489
ME	841.7330	0.4354	10 461.096
PE	372.4570	0.3411	9290.433
GSE	1523.863	0.6645	7943.848
LSE	1059.518	0.9710	3279.764
WLSE	1065.453	1.0818	2937.694
MMLEI	3941.552	1.4792	8275.664
MMLEII	4055.947	1.5056	8405.806

According to the simulation results given in Tables 1 and 2, TMMLE or GSE are chosen as the best estimators since the estimated value of β is between 0.5 and 1.5.

6. Conclusion

$\text{VaR}_\alpha(X)$ is estimated by $\{-\ln(1-\alpha)\}^{1/\hat{\beta}}\,\hat{\theta}$ for the two-parameter Weibull distribution. One of the most commonly used methods for estimating $\text{VaR}_\alpha(X)$ in the literature is the MLE method. Nonetheless, in our work we have sought the best method, including the MLE method, for estimating $\text{VaR}_\alpha(X)$ for several sample sizes and the known location parameter cases by comparing the methods of estimation through the *Def* values.

It is shown that MLE, GSE, LSE, WLSE, PE, ME, MMLE-I and MMLE-II methods of estimation involve solving nonlinear equations for the parameters, whereas TMMLEs have been proved to be explicit functions of the sample observations and do not need any iterative computational processes. For large samples, MLE or TMMLE are strongly recommended for estimating θ and β, and estimating VaR or quantiles thereof, in most of the cases. The performances of the MLEs and the TMMLEs are quite close to each other, since they are asymptotically equivalent.

When the computing time is important and the sample size is large, the TMMLE method proves to be the best recommendable estimator in general.

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