Interval Estimation of the System Reliability for Weibull Distribution based on Ranked Set Sampling Data

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Abstract

Inference for the system reliability R is one of the most popular problems in the areas of engineering, statistics, biostatistics and etc. Therefore, there exist considerable numbers of studies concerning this problem. Traditionally, simple random sampling (SRS) is used for estimating the system reliability. However, in recent years, ranked set sampling (RSS), cost effective and efficient alternative of SRS, is used to estimate the system reliability. In this study, we consider the interval estimation of R when both the stress and the strength are independent Weibull random variables based on RSS. We first obtain the asymptotic confidence interval (ACI) of R by using the maximum likelihood (ML) methodology. The bootstrap confidence interval (BCI) of R is also constructed as an alternative to ACI. An extensive Monte-Carlo simulation study is conducted to compare the performances of ACI and BCI of R for different settings. Finally, a real data set is analyzed to demonstrate the implementation of the proposed methods.

Keywords: Stress-strength model, ranked set sampling, asymptotic confidence interval, bootstrap confidence interval, Monte-Carlo simulation.

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1. Introduction

The estimation of the system reliability $R = P(X < Y)$ has been considered many times using both parametric and non-parametric methods [6, 10, 12, 13, 15, 19, 23, 27]. Here, X and Y represent the stress and the strength, respectively. It is clear that if the stress exceeds the strength, i.e., $X > Y$ the system would fail, otherwise it continues to work. The basic assumption of this topic is that both X and Y are independent random variables. For more detailed information, see Kotz et al. [14].

In the statistical literature, estimation of R has been examined by a quite number of authors under various distributions of X and Y based on simple random sampling (SRS) data. However, in recent years, the ranked set sampling (RSS)

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method, originated by McIntyre [16], was used for estimating R, since it is plausible and efficient alternative of SRS [9, 18, 20, 21, 22].

Akgul and Senoglu [3] obtained the point estimators of R when the stress X and the strength Y are both independent Weibull random variables with common shape and different scale parameters based on RSS by using the ML and the modified ML (MML) methodologies. In this paper, we extend their study to the interval estimation of R. We obtain the asymptotic confidence interval (ACI) of R by using the asymptotic properties of ML and MML estimators [2]. We also construct the bootstrap confidence interval (BCI) of R by using two different resampling methods proposed by Chen et al. [8] and Modarres et al. [17]. Different than the ACI, we just use the ML estimator in obtaining BCI of R similar to the common usage in the literature. The ACI performs well for the large sample sizes. However, for the small and moderate sample sizes, we prefer the BCI.

This paper is organized as follows: In Section 2, we give brief description of RSS method. In section 3, we mention the point estimation of R and construct ACI and BCI of R. In section 4, an extensive Monte-Carlo simulation study is performed. A real data set is analyzed to demonstrate the implementation of the proposed methods in Section 5. Final comments and conclusions are given in Section 6.

2. Ranked Set Sampling

The RSS is proposed to use when the variable of interest is more easily ranked than quantified [5, 11]. Steps of the RSS procedure are given below:

i. Select m random sets via SRS each of size m.
ii. Without doing any certain measurements, rank the units with respect to the variable of interest.
iii. Select the $i-$th smallest observations from the $i-$th set, $(i = 1, \dots, m)$ and obtain the certain measurements of these observations.
iv. This complete process is called as a cycle. The cycle is repeated r times, therefore the sample size is obtained as $n = mr$.

For better understanding this entire process, see the following table:

Cycle	Set	Ranked Units 1	2		m	Certain measurements
1	1	$\boldsymbol{X_{(1)11}}$	$X_{(2)11}$	⋯	$X_{(m)11}$	$\boldsymbol{X_{(1)11}}$
	2	$X_{(1)21}$	$\boldsymbol{X_{(2)21}}$	⋯	$X_{(m)21}$	$\boldsymbol{X_{(2)21}}$
	⋮	⋮	⋮	⋮	⋮	⋮
	m	$X_{(1)m1}$	$X_{(2)m1}$	⋯	$\boldsymbol{X_{(m)m1}}$	$\boldsymbol{X_{(m)m1}}$
⋮		⋮	⋮	⋮	⋮	
r	1	$\boldsymbol{X_{(1)1r}}$	$X_{(2)1r}$	⋯	$X_{(m)1r}$	$\boldsymbol{X_{(1)1r}}$
	2	$X_{(1)2r}$	$\boldsymbol{X_{(2)2r}}$	⋯	$X_{(m)2r}$	$\boldsymbol{X_{(2)2r}}$
	⋮	⋮	⋮	⋮	⋮	⋮
	m	$X_{(1)mr}$	$X_{(2)mr}$	⋯	$\boldsymbol{X_{(m)mr}}$	$\boldsymbol{X_{(m)mr}}$

Here, $X_{(i)ic}$, $i = 1, \dots, m$ and $c = 1, \dots, r$ is called RSS sample. If the ranking is perfect, i.e. $X_{(1)1c} \leq X_{(2)2c} \leq \cdots \leq X_{(m)mc}$, the probability density function (pdf) of $X_{(i)ic}$ is the pdf of the $i-$th order statistics. In this study, all the computations are performed under the assumption of the perfect ranking. It should be noted that for easy understanding and simplicity, we use the notations X_{ic} instead of $X_{(i)ic}$.

3. Estimation of R

In this section, we mention the point estimation of the system reliability R briefly since it is considered by Akgul and Senoglu [3] and we construct the ACI and the BCI of R.

Before starting the estimation procedure, we give some descriptions about the collection of the RSS samples used in estimation of R.

In the context of stress-strength model, let X_{ic}, $i = 1, \ldots, m_x$, $c = 1, \ldots, r_x$ denote the ranked set sample of size n and Y_{jl}, $j = 1, \ldots, m_y$, $l = 1, \ldots, r_y$ denote the ranked set sample of size m. Here, m_x and m_y are the set sizes and r_x and r_y are the number of cycles for X and Y, respectively. It is clear that the sample sizes for the stress and the strength are $n = m_x r_x$ and $m = m_y r_y$, respectively.

3.1. Point Estimation of R. Let $X \sim Weibull(p, \sigma_1)$ and $Y \sim Weibull(p, \sigma_2)$ be two independent random variables, then the system reliability R is obtained as given below

$$R = \int_0^\infty \left(1 - e^{-t^p/\sigma_1}\right) \frac{p}{\sigma_1} t^{p-1} e^{-t^p/\sigma_2} dt = \frac{\sigma_2}{\sigma_1 + \sigma_2}. \tag{3.1}$$

It is clear that the estimator of R is obtained by inserting the estimators of σ_1 and σ_2 into the equation given in (3.1). Similar to Akgul and Senoglu [3], we use the ML methodology to obtain the estimators of the parameters p, σ_1 and σ_2. The log-likelihood ($\ln L$) function is obtained as shown below

$$\begin{aligned} \ln L \propto\ & \ln C + (n+m)\ln p - n \ln \sigma_1 - m \ln \sigma_2 + \\ & (p-1)\sum_{c=1}^{r_x}\sum_{i=1}^{m_x} \ln x_{ic} + \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(i-1)\ln\left(1 - e^{-x_{ic}^p/\sigma_1}\right) - \\ & \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(m_x - i + 1)\left(\frac{x_{ic}^p}{\sigma_1}\right) + (p-1)\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\ln y_{jl} + \\ & \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(j-1)\ln\left(1 - e^{-y_{jl}^p/\sigma_2}\right) - \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(m_y - j + 1)\left(\frac{y_{jl}^p}{\sigma_2}\right). \end{aligned} \tag{3.2}$$

Then, we take the derivatives of $\ln L$ with respect to the parameters of interest and equate them to zero as given in the following equations

$$\begin{aligned}
\frac{\partial \ln L}{\partial p} &= \frac{n+m}{p}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\ln x_{ic} + \frac{1}{\sigma_1}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(i-1)\frac{x_{ic}^p \ln x_{ic}}{e^{x_{ic}^p/\sigma_1}-1} - \\
&\quad \frac{1}{\sigma_1}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(m_x-i+1)x_{ic}^p \ln x_{ic} + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\ln y_{jl} + \\
&\quad \frac{1}{\sigma_2}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(j-1)\frac{y_{jl}^p \ln y_{jl}}{e^{y_{jl}^p/\sigma_2}-1} - \frac{1}{\sigma_2}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(m_y-j+1)y_{jl}^p \ln y_{jl} = 0, \\
\frac{\partial \ln L}{\partial \sigma_1} &= -\frac{n}{\sigma_1} - \frac{1}{\sigma_1^2}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(i-1)\frac{x_{ic}^p}{e^{x_{ic}^p/\sigma_1}-1} + \frac{1}{\sigma_1^2}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(m_x-i+1)x_{ic}^p = 0, \\
\frac{\partial \ln L}{\partial \sigma_2} &= -\frac{m}{\sigma_2} - \frac{1}{\sigma_2^2}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(j-1)\frac{y_{jl}^p}{e^{y_{jl}^p/\sigma_2}-1} + \frac{1}{\sigma_2^2}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}(m_y-j+1)y_{jl}^p = 0.
\end{aligned}$$

Solutions of these equations are called as the ML estimators of the parameters. However, these equations cannot be solved explicitly. We therefore resort to iterative methods as in Akgul and Senoglu [3]. For an alternative to ML, they also used the MML methodology originated by Tiku [24, 25] which gives the explicit solutions for the unknown parameters rather than the numerical solutions, see also Akgul [1]. Besides providing the close form estimators, the MML estimators are also asymptotically equivalent to ML estimators [7, 26]. In this study, we use the ML and MML estimators of R obtained by Akgul and Senoglu [3] to construct the confidence interval of R.

3.2. Interval estimation for R**.** Now, we consider the interval estimation of the system reliability R. For this purpose, we construct the ACI and the BCI of R. We use asymptotic properties of the ML estimators to construct the ACI of R. Then, for small and moderate sample sizes we constitute the BCI of R.

3.2.1. *ACI of R.* In this subsection, we construct the ACI of R, by using the asymptotic distribution of $\hat{R} = \hat{\sigma}_1/(\hat{\sigma}_1+\hat{\sigma}_2)$. To do this, we first obtain the Fisher information matrix defined below

$$\mathbf{I}_{RSS}(\boldsymbol{\theta}) = -\begin{bmatrix} E\left(\frac{\partial^2 \ln L}{\partial p^2}\right) & E\left(\frac{\partial^2 \ln L}{\partial p \partial \sigma_1}\right) & E\left(\frac{\partial^2 \ln L}{\partial p \partial \sigma_2}\right) \\ E\left(\frac{\partial^2 \ln L}{\partial \sigma_1 \partial p}\right) & E\left(\frac{\partial^2 \ln L}{\partial \sigma_1^2}\right) & E\left(\frac{\partial^2 \ln L}{\partial \sigma_1 \partial \sigma_2}\right) \\ E\left(\frac{\partial^2 \ln L}{\partial \sigma_2 \partial p}\right) & E\left(\frac{\partial^2 \ln L}{\partial \sigma_2 \partial \sigma_2}\right) & E\left(\frac{\partial^2 \ln L}{\partial \sigma_2^2}\right) \end{bmatrix} \tag{3.3}$$

Here, $\boldsymbol{\theta}$ represents (p, σ_1, σ_2). The elements of Fisher information matrix are denoted by I_{ij}, $i, j = 1, 2, 3$ and given below

$$\begin{aligned}
I_{11} &= -E\left(\frac{\partial^2 \ln L}{\partial p^2}\right) \\
&= \frac{1}{p^2}\Bigg\{ \left(n\left(m_x-1\right)+m\left(m_y-1\right)\right)E_1 + \\
&\quad \left(2n\left(m_x-1\right)\ln\sigma_1 + 2m\left(m_y-1\right)\ln\sigma_2\right)E_2 + \\
&\quad 2\zeta\left(3\right)\left(n\left(m_x-1\right)\left(\ln\sigma_1\right)^2 + m\left(m_y-1\right)\left(\ln\sigma_2\right)^2\right)\Bigg\} + \\
&\quad \frac{1}{p^2}\left\{\left(n+m\right)\left(1+\Gamma''\left(2\right)\right) + 2\Gamma'\left(2\right)\left(n\ln\sigma_1 + m\ln\sigma_2\right) + n(\ln\sigma_1)^2 + m(\ln\sigma_2)^2\right\}, \\
I_{22} &= -E\left(\frac{\partial^2 \ln L}{\partial \sigma_1^2}\right) = n\left(m_x-1\right)\frac{2\zeta\left(3\right)-2}{\sigma_1^2} + \frac{n}{\sigma_1^2}, \\
I_{33} &= -E\left(\frac{\partial^2 \ln L}{\partial \sigma_2^2}\right) = m\left(m_y-1\right)\frac{2\zeta\left(3\right)-2}{\sigma_2^2} + \frac{m}{\sigma_2^2}, \\
I_{12} &= I_{21} = -E\left(\frac{\partial^2 \ln L}{\partial p \partial \sigma_1}\right) \\
&= -\frac{n\left(m_x-1\right)}{p\sigma_1}\left(\left(2\zeta\left(3\right)-2\right)\ln\sigma_1 + E_3\right) - \frac{n}{p\sigma_1}\left(\ln\sigma_1 + \Gamma'\left(2\right)\right), \\
I_{13} &= I_{31} = -E\left(\frac{\partial^2 \ln L}{\partial p \partial \sigma_2}\right) \\
&= -\frac{m\left(m_y-1\right)}{p\sigma_2}\left(\left(2\zeta\left(3\right)-2\right)\ln\sigma_2 + E_3\right) - \frac{m}{p\sigma_2}\left(\ln\sigma_2 + \Gamma'\left(2\right)\right), \\
I_{23} &= I_{32} = -E\left(\frac{\partial^2 \ln L}{\partial \sigma_1 \partial \sigma_2}\right) = 0,
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= 2\zeta\left(3\right) + \left(1/3\right)\zeta\left(3\right)\pi^2 - 6\zeta\left(3\right)\gamma + 2\zeta\left(3\right)\gamma^2 + 6\zeta\left(1,3\right) - 4\zeta\left(1,3\right)\gamma + 2\zeta\left(2,3\right), \\
E_2 &= -2\zeta\left(3\right)\gamma + 3\zeta\left(3\right) + 2\zeta\left(1,3\right), \\
E_3 &= -2\zeta\left(3\right)\gamma + 2\gamma + 3\zeta\left(3\right) + 2\zeta\left(1,3\right) - 3
\end{aligned}$$

and $\zeta(\cdot)$ and $\zeta(\cdot,\cdot)$ are the Riemann zeta function, γ is the Euler constant. For more detailed information, one may refer to Akgul [1] and Chen et al. [8].

We use the following theorems to compute the asymptotic distribution of $\hat{R}$.

3.1. Theorem. *As $n \to \infty$ and $m \to \infty$, then*

$$\left(\sqrt{m}\left(\hat{p}-p\right), \sqrt{n}\left(\widehat{\sigma}_1-\sigma_1\right), \sqrt{n}\left(\widehat{\sigma}_2-\sigma_2\right)\right) \xrightarrow{d} N_3\left(0, A^{-1}\left(p, \sigma_1, \sigma_2\right)\right), \tag{3.4}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}.$$

Here,

$$a_{11}=\frac{I_{11}}{m}, a_{22}=\frac{I_{22}}{n}, a_{33}=\frac{I_{33}}{n}, a_{12}=a_{21}=\frac{I_{12}}{\sqrt{nm}}, a_{13}=a_{31}=\frac{I_{13}}{\sqrt{nm}}.$$

Proof. The proof follows from the asymptotic distribution of the ML estimators under regularity conditions and the central limit theorem. □

3.2. Theorem. *As $n\to\infty$ and $m\to\infty$, then*

$$\sqrt{n}\left(\hat{R}-R\right)\xrightarrow{d}N\left(0,B\right). \tag{3.5}$$

Proof. The proof follows from Theorem 3.1 and delta method. Here,

$$B=\left(\frac{\partial R}{\partial p},\frac{\partial R}{\partial \sigma_1},\frac{\partial R}{\partial \sigma_2}\right)\ A^{-1}\left(\begin{array}{c}\frac{\partial R}{\partial p}\\ \frac{\partial R}{\partial \sigma_1}\\ \frac{\partial R}{\partial \sigma_2}\end{array}\right),$$

where

$$\left(\frac{\partial R}{\partial p},\frac{\partial R}{\partial \sigma_1},\frac{\partial R}{\partial \sigma_2}\right)=\frac{1}{\left(\sigma_1+\sigma_2\right)^2}\left(0,-\sigma_2,\sigma_1\right)$$

and

$$A^{-1}=\frac{1}{u}\left[\begin{array}{ccc} a_{22}a_{33} & -a_{21}a_{33} & -a_{22}a_{31}\\ -a_{21}a_{33} & a_{11}a_{33}-a_{13}^2 & a_{12}a_{31}\\ -a_{22}a_{31} & a_{12}a_{31} & a_{11}a_{22}-a_{12}^2\end{array}\right].$$

In the definition of A^{-1}, u is defined as $u=\det A\ =a_{11}a_{22}a_{33}-a_{12}^2a_{33}-a_{13}^2a_{22}$. □

The estimate of $Var\left(\hat{R}\right)$ is $\widehat{Var}\left(\hat{R}\right)=B|_{p=\hat{p},\sigma_1=\widehat{\sigma}_1,\sigma_2=\widehat{\sigma}_2}$. Thus,

$$\sqrt{n}\left(\hat{R}-R\right)/\sqrt{\widehat{Var}\left(\hat{R}\right)}\sim N\left(0,1\right).$$

This result yields the asymptotic $100\left(1-\alpha\right)\%$ confidence interval for R as

$$\left(\hat{R}-z_{\alpha/2}\sqrt{\frac{\widehat{Var}\left(\hat{R}\right)}{n}},\hat{R}+z_{\alpha/2}\sqrt{\frac{\widehat{Var}\left(\hat{R}\right)}{n}}\right), \tag{3.6}$$

where, $z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standard normal distribution, i.e., $N\left(0,1\right)$.

It should be noted that ACI of R can alternatively be constructed by inserting the MML estimator of R into the equation (3.6), because of the reason given in subsection 3.1.

3.2.2. *BCI of R.* In this subsection, we construct BCIs for the system reliability of R by using two different resampling methods. The first method is introduced by Chen et al. [8] and the second method is proposed by Modarres et al. [17]. It should be noted that the BCI of R based on Resampling Method I and Resampling Method II are represented by BCI-I and BCI-II, respectively. These methods are defined below.

Resampling Method I: BCI-I

Step 1: Divide the RSS samples x_{ic} $(i = 1, \ldots, m_x, c = 1, \ldots, r_x)$ and y_{jl} $(j = 1, \ldots, m_y, l = 1, \ldots, r_y)$ into m_x and m_y subgroups each contains r_x and r_y observations, respectively.

Step 2: Resample from each subgroups with replacement.

Step 3: Combine all m_x and m_y subgroups each of sizes r_x and r_y and obtain the RSS resamples x^*_{ic} $(i = 1, \ldots, m_x, c = 1, \ldots, r_x)$ and y^*_{jl}, $(j = 1, \ldots, m_y, l = 1, \ldots, r_y)$, respectively. Here, $*$ notation represents the sample drawn with replacement.

Step 4: By using x^*_{ic} and y^*_{jl}, compute the bootstrap estimates of R, say $\hat{R}^*$.

Step 5: Repeat step 1-4, B times to get the bootstrap estimates $\hat{R}^*_1, \ldots, \hat{R}^*_B$ of R.

Step 6: Rank them from the smallest the largest $(\hat{R}^*_{(1)}, \ldots, \hat{R}^*_{(B)})$.

Step 7: The approximate $100(1-\alpha)\%$ BCI of R is constructed as below

$$\left(\hat{R}^*_{((\alpha/2)B)}, \hat{R}^*_{(1-(\alpha/2)B)}\right) \tag{3.7}$$

It should be stated that we adopt the procedure described by Chen et al. [8] to obtain the BCI of R, see also Akgul et al. [4].

Resampling Method II: BCI-II

Step 1: Combine all observations for each RSS sample x_{ic} $(i = 1, \ldots, m_x, c = 1, \ldots, r_x)$ and y_{jl} $(j = 1, \ldots, m_y, l = 1, \ldots, r_y)$, say $x^\diamond_{ic}$ and $y^\diamond_{jl}$, respectively.

Step 2: Randomly draw m_x elements from $x^\diamond_{ic}$, say $x^\diamond_1, \ldots, x^\diamond_{m_x}$ and m_y elements from $y^\diamond_{jl}$, say $y^\diamond_1, \ldots, y^\diamond_{m_y}$, order them from the smallest to largest as $x^\diamond_{(1)} \leq \cdots \leq x^\diamond_{(m_x)}$ and $y^\diamond_{(1)} \leq \cdots \leq y^\diamond_{(m_y)}$, and retain $x^*_{i1} = x^\diamond_{(i)}$ and $y^*_{j1} = y^\diamond_{(j)}$, respectively.

Step 3: Perform Step 2 for $i = 1, \ldots, m_x$ and $j = 1, \ldots, m_y$, respectively.

Step 4: Repeat Step 2 and 3 r_x and r_y times to obtain x^*_{ic} $(i = 1, \ldots, m_x, c = 1, \ldots, r_x)$ and y^*_{jl} $(j = 1, \ldots, m_y, l = 1, \ldots, r_y)$, respectively, and compute the bootstrap estimates of R, say $\hat{R}^*$.

For BCI-II of R follow Steps 5-7 given in resampling method I. For more detailed information, see Modarres et al. [17] and Akgul et al. [4].

4. Simulation Study

In this section, we perform an extensive Monte-Carlo simulation study to compare the average confidence lengths (ACL) and the coverage probabilities (CP) of the confidence intervals constructed in this study. In our simulation setup, we take the set sizes and the number of cycles as $(m_x, m_y) = (3,3), (3,4), (4,4), (4,5)$ and (5,5) and $r_x = r_y = 5$ and 10, respectively. Therefore, in the context of RSS, the sample sizes for X and Y are obtained as $n = m_x r_x$ and $m = m_y r_y$. It should be noted that the sample sizes of SRS observation are also denoted as n and m.

The SRS and the RSS observation are generated under the assumption of both densities have Weibull distribution with the common shape and the different scale

Table 1. Average confidence lengths and CPs based on the ACI of the ML and the MML estimators of R and the BCI of R under RSS when p =1.5.

		SRS					
		ACI_{ML}		ACI_{MML}		BCI	
$r_x = r_y$	(m_x, m_y)	ACL	CP	ACL	CP	ACL	CP
		$\sigma_1 = 1,\ \sigma_2 = 1$					
5	(3,3)	0.3502	0.91	0.3502	0.91	0.3675	0.92
	(3,4)	0.3289	0.94	0.3288	0.93	0.3409	0.93
	(4,4)	0.3051	0.92	0.3051	0.92	0.3158	0.92
	(4,5)	0.2901	0.92	0.2900	0.93	0.2989	0.92
	(5,5)	0.2739	0.93	0.2739	0.93	0.2811	0.93
10	(3,3)	0.2505	0.93	0.2504	0.93	0.2552	0.93
	(3,4)	0.2346	0.92	0.2346	0.92	0.2380	0.93
	(4,4)	0.2176	0.95	0.2176	0.95	0.2206	0.94
	(4,5)	0.2065	0.93	0.2064	0.93	0.2089	0.93
	(5,5)	0.1948	0.93	0.1948	0.93	0.1969	0.93
		$\sigma_1 = 1, \sigma_2 = 2$					
5	(3,3)	0.3264	0.92	0.3256	0.92	0.3290	0.91
	(3,4)	0.3037	0.90	0.3023	0.91	0.3012	0.9
	(4,4)	0.2852	0.92	0.2847	0.92	0.2864	0.91
	(4,5)	0.2703	0.93	0.2696	0.93	0.2686	0.93
	(5,5)	0.2556	0.93	0.2552	0.93	0.2555	0.92
10	(3,3)	0.2338	0.94	0.2335	0.94	0.2317	0.93
	(3,4)	0.2190	0.93	0.2185	0.93	0.2162	0.93
	(4,4)	0.2034	0.93	0.2033	0.94	0.2011	0.93
	(4,5)	0.1928	0.93	0.1926	0.93	0.1894	0.92
	(5,5)	0.1823	0.93	0.1821	0.93	0.1799	0.93
		$\sigma_1 = 2, \sigma_2 = 1$					
5	(3,3)	0.3260	0.90	0.3250	0.90	0.3288	0.91
	(3,4)	0.3055	0.91	0.3053	0.92	0.3079	0.91
	(4,4)	0.2848	0.92	0.2843	0.93	0.2845	0.92
	(4,5)	0.2710	0.94	0.2708	0.94	0.2713	0.93
	(5,5)	0.2563	0.93	0.2558	0.93	0.2560	0.93
10	(3,3)	0.2340	0.93	0.2338	0.94	0.2317	0.93
	(3,4)	0.2191	0.94	0.2191	0.94	0.2185	0.93
	(4,4)	0.2035	0.94	0.2033	0.93	0.2013	0.94
	(4,5)	0.1931	0.93	0.1931	0.94	0.1907	0.93
	(5,5)	0.1824	0.95	0.1823	0.94	0.1793	0.94

parameters. The parameter settings are taken as $p = 1.5$, σ_1, σ_2 =(1,1), (1,2) and (2,1). All the computations are performed in Matlab R2013a based on 1000 Monte-Carlo runs.

The 95% ACIs are constructed by using the asymptotic distributions of the ML estimators of p, σ_1 and σ_2, and replacing them with the corresponding MML estimators based on SRS and RSS. For the 95% BCIs, we use B=1000 bootstrap resamples. They are computed based on the ML estimators of R under SRS and RSS. Results are reported in Table 1.

From Table 1, the ACLs for both ACIs and BCIs based on RSS are shorter than the corresponding confidence lengths based on SRS. It clear that the lengths of the confidence intervals decrease when the sample sizes (n, m) increase, as expected.

Table 1. (Continued)

		RSS							
		ACI_{ML}		ACI_{MML}		$BCI-I$		$BCI-II$	
$r_x = r_y$	(m_x, m_y)	ACL	CP	ACL	CP	ACL	CP	ACL	CP
		$\sigma_1 = 1,\ \sigma_2 = 1$							
5	(3,3)	0.2611	0.91	0.2612	0.91	0.2504	0.90	0.2648	0.91
	(3,4)	0.2356	0.93	0.2353	0.92	0.2230	0.90	0.2347	0.91
	(4,4)	0.2061	0.93	0.2061	0.93	0.1933	0.90	0.2044	0.91
	(4,5)	0.1891	0.94	0.1889	0.92	0.1751	0.91	0.1858	0.92
	(5,5)	0.1702	0.94	0.1702	0.94	0.1571	0.92	0.1663	0.92
10	(3,3)	0.1867	0.95	0.1866	0.94	0.1828	0.93	0.1879	0.94
	(3,4)	0.1678	0.95	0.1675	0.91	0.1637	0.93	0.1675	0.93
	(4,4)	0.1466	0.95	0.1465	0.94	0.1423	0.94	0.1460	0.94
	(4,5)	0.1343	0.95	0.1342	0.93	0.1297	0.94	0.1333	0.93
	(5,5)	0.1208	0.94	0.1207	0.94	0.1163	0.93	0.1197	0.93
		$\sigma_1 = 1, \sigma_2 = 2$							
5	(3,3)	0.2338	0.91	0.2380	0.91	0.2300	0.90	0.2416	0.91
	(3,4)	0.2090	0.91	0.2081	0.90	0.2014	0.90	0.2114	0.90
	(4,4)	0.1838	0.92	0.1862	0.91	0.1770	0.91	0.1877	0.92
	(4,5)	0.1682	0.93	0.1675	0.92	0.1604	0.92	0.1684	0.92
	(5,5)	0.1521	0.93	0.1535	0.92	0.1462	0.92	0.1538	0.93
10	(3,3)	0.1671	0.92	0.1691	0.92	0.1683	0.92	0.1728	0.93
	(3,4)	0.1498	0.92	0.1485	0.90	0.1493	0.92	0.1533	0.92
	(4,4)	0.1314	0.94	0.1322	0.92	0.1330	0.94	0.1357	0.94
	(4,5)	0.1198	0.94	0.1193	0.92	0.1193	0.93	0.1229	0.93
	(5,5)	0.1081	0.94	0.1088	0.93	0.1082	0.93	0.1113	0.94
		$\sigma_1 = 2, \sigma_2 = 1$							
5	(3,3)	0.2330	0.92	0.2367	0.92	0.2302	0.90	0.241	0.91
	(3,4)	0.2106	0.92	0.217	0.90	0.2064	0.90	0.2165	0.91
	(4,4)	0.1842	0.92	0.1861	0.91	0.1775	0.91	0.1877	0.92
	(4,5)	0.1689	0.93	0.1725	0.93	0.1623	0.92	0.1717	0.92
	(5,5)	0.1522	0.93	0.1536	0.93	0.1457	0.92	0.1534	0.93
10	(3,3)	0.1671	0.93	0.1696	0.91	0.1685	0.92	0.1725	0.93
	(3,4)	0.1506	0.93	0.1548	0.90	0.1523	0.93	0.1559	0.93
	(4,4)	0.1312	0.93	0.1325	0.92	0.1325	0.93	0.1357	0.93
	(4,5)	0.1201	0.93	0.1225	0.92	0.1211	0.94	0.1239	0.94
	(5,5)	0.1079	0.94	0.1086	0.93	0.1085	0.93	0.1112	0.93

In the context of RSS, when we compare the ACIs of R we realized that the ACLs of ACI_{ML} and the ACLs of ACI_{MML} are more or less the same and they close to each other as the set sizes increase. On the other hand, the ACLs of BCI-I is shorter than the ACLs of BCI-II. Also, the BCI-I is provides the shortest ACLs among the others.

In terms of CPs, when $\sigma_1 = \sigma_2 = 1$, the CPs of ACI_{ML} and ACI_{MML} of R based on RSS are closer to nominal value than their SRS counterparts. However, the CPs of BCI of R based on SRS is better than the CPs of BCIs of R based on RSS when $r_x = r_y = 5$. As the number of cycles increase, the performances of BCIs are shown similarity for both SRS and RSS with respect to the CPs.

When $\sigma_1 = 1, \sigma_2 = 2$ and $\sigma_2 = 2, \sigma_1 = 1$, the CPs of ACI_{ML} based on RSS is better than the corresponding CPs based on SRS. However, the CPs of ACI_{MML} based on RSS is lower approximately 1% and 2% than its SRS counterpart overall. In view of BCIs, the CPs of BCI-I and BCI-II based on RSS are more or less the same with the CPs of BCI based on SRS in most of the cases especially when $r_x = r_y = 10$.

5. Data Analysis

In this section, we analyze the strength data set to illustrate the implementation of the interval estimation procedure, proposed in this paper. This data set is about the strength measured in GPA for single carbon fibers of lengths 20 mm (Data Set I) and 50 mm (Data Set II) with sample sizes 69 and 65, respectively, see Ghitany et al. [10]. Besides the single carbon fibers of lengths 20 mm and 50 mm, the single carbon fibers of lengths 20 mm and 10 mm are also considered in the context of the estimation of the system reliability R [6, 15]. However, different than these studies, we consider the strength data (Data Set I and Data Set II) as population of interest, see Akgul et al. [4]. Then, we select samples randomly from these populations via SRS and RSS. Therefore, by selecting 21 observations from Data Set I and Data Set II, we obtain the random samples based on SRS, namely X and Y, respectively. By taking the set sizes $m_x = m_y = 3$ and the number of cycles $r_x = r_y = 7$, then applying RSS procedure given in Section 2, we obtain the corresponding samples based on RSS, called as X and Y, respectively.

Akgul and Senoglu [3] obtain the ML and the MML estimates of R based on SRS and RSS. Now, we construct the 95% ACIs for R. Also, 95% BCIs of R are constructed based on 5000 bootstrap replications. The results are reported in Table 2.

Table 2. 95% ACIs and BCIs of R for the strength data based on SRS and RSS.

Sampling Methods	*CIs*	*Lower*	*Upper*	*Length*
SRS	ACI_{ML}	0.2228	0.5135	0.2906
	ACI_{MML}	0.2206	0.5108	0.2902
	BCI	0.2362	0.5096	0.2734
RSS	ACI_{ML}	0.2734	0.4853	0.2118
	ACI_{MML}	0.2701	0.4811	0.2110
	$BCI-I$	0.2767	0.4916	0.2149
	$BCI-II$	0.2693	0.4682	0.1989

It is clear from Table 2 that the lengths of confidence intervals based on RSS are shorter than the length of confidence intervals based on SRS. Also, ACIs of R under the ML and MML estimates based on SRS and RSS are more or less the same in its own right. To illustrate this situation, we draw the histograms for the replications of BCI (based on SRS), BCI-I (based on RSS) and BCI-II (based on RSS) of R under ML and MML estimates based on 5000 replication in Figure 1.

Obviously, all histograms match with the normal distribution. Furthermore, the means and the standard deviations of bootstrap replications of R are calculated and it is seen that they are in agreement with the ML and the MML estimates of R based on SRS and RSS given in Akgul and Senoglu [3]. Since, the distribution

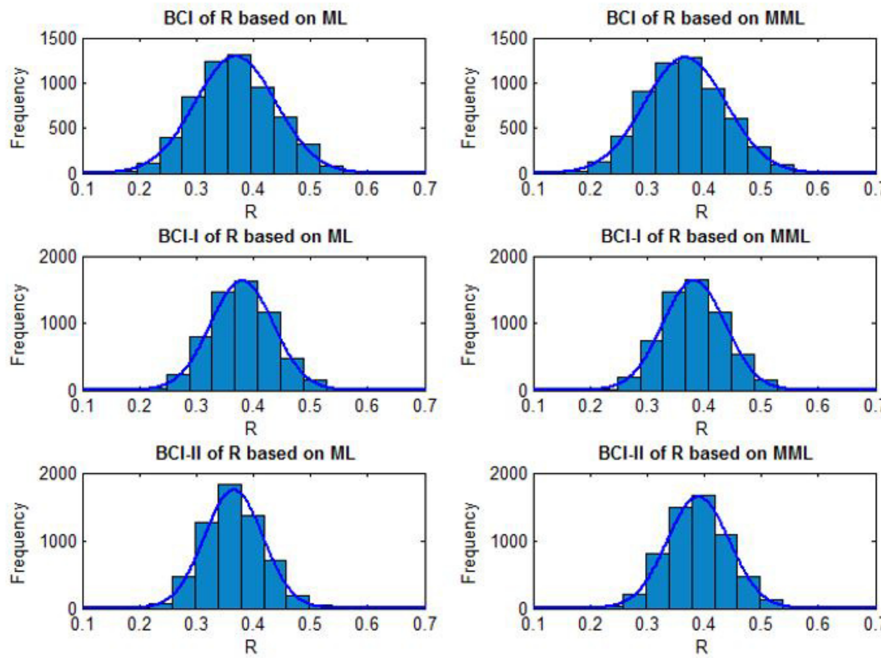

Figure 1. Histogram of bootstrap replications of R with normal curves.

of the bootstrap replications of R is normal, we can use the mean for estimate of R [23].

6. Conclusions

In this paper, we consider the interval estimation of $R = P(X < Y)$ when the stress X and the strength Y are both independent Weibull random variables with common shape and different scale parameters based on RSS data. We provide the asymptotic distributions of the ML estimators which are used to construct ACI of R. By substituting the ML estimators with the MML estimators, we also derive the ACI of R based on MML estimators. For the small and the moderate sample sizes we construct the BCI of R. The ACIs and the BCI of R are compared with their SRS counterparts.

In the context of RSS data, it is observed that the ACI based on ML estimator works well even for small sample sizes in terms of CP. From the real data example, it is seen that length of the confidence intervals based on RSS are smaller than their SRS counterparts.

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