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Matrix realization of dual quaternionic electromagnetism

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Abstract: In this paper, a new representational model based on dual quaternionic matrices is proposed for classical electromagnetism. After demonstrating the isomorphic matrix representations of dual quaternions, Maxwell's equations and the constitutive relations for electromagnetism are expressed in terms of dual quaternionic matrices. For this purpose, new 8×8 matrices connected with quaternion basis elements have been introduced.

Keywords: Dual Quaternion, Electromagnetism, Maxwell's equations, Matrices
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1 Introduction

Quaternions as a generalization of complex numbers to three dimensions were discovered by Sir William Rowan Hamilton [1] in 1843. Today three types of quaternions are available: real, complex and dual. A real quaternion is a basic special mathematical entity and contains four real components. Similarly, a complex quaternion and a dual quaternion are composed of two real quaternions and also have four complex and dual components, respectively.

Studies about quaternions in classical electromagnetism can be traced back to J. C. Maxwell's works. Although in his famous book *Treatise on Electricity and Magnetism* [2] Maxwell used three dimensional vector representation to formulate electromagnetism, he also gave their quaternionic forms in a number of places. Since quaternions include vectors and scalars, Maxwell used the VQ and SQ symbols to refer to the vector part

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and scalar part of quaternion Q, respectively [3]. Complex quaternions (also named biquaternions) were used to reformulate classical electrodynamics and Maxwell's equations were reduced to their complex quaternion form by Imaeda [4]. Using a new operator, Colombo *et al.* [5] have then investigated Imaeda's approach in a larger framework to deduce Maxwell's equations and to study their behavior in two Minkowski space-times, one with electric charges and the other with magnetic monopoles. Lambek [6], Ward [7], Gürlebeck and Sprössig [8] have also discussed complex quaternions in physics. Since quaternion algebra has a more expressive nature than vectors, various equations such as Maxwell's four equations, the equation of continuity, and the four dimensional potential of electromagnetism were given in complex quaternion notation [9]–[20]. Similar work about electromagnetism related with complex quaternions has been done by Negi *et al.* [21]. In their paper, after demonstrating the isomorphic matrix representation of complex quaternions, Maxwell's equations and the equation of motion were derived in compact and expanded forms, respectively, for quaternions and matrix representations.

In the paper by Tolan *et al.* [22] the field equations for electromagnetism, the potential and Maxwell equations, have been investigated with new octonionic equations and these equations have been compared with their vectorial representations. Dirac's operator and Maxwell's equations in vacuum have been derived in the algebra of split octonions by Gogberashvili [23]. Similarly, Bisht *et al.* [24, 25] have expressed electromagnetic and dyonic field equations in terms of split octonions as well.

Except for the well-known practical applications of rigid body movements in three dimensional space and differential algebra, dual quaternion formalism has not been used as frequently in other areas of physics as it deserves. However, dual quaternions play important roles in field theory, supersymmetry [26] and in expressing the Galilean space-time transformation [27] as well. The formulation of classical electromagnetism by dual quaternions is quite new. In recent work [28], classical electromagnetism has been reformulated by using this type of quaternion. Maxwell's equations have been rewritten in terms of dual quaternions and these four equations have been combined in a single equation. Some dual quaternionic equations related to electromagnetism have also been developed.

Although the equations of classical electromagnetism can be written in a number of different forms, dual quaternionic version of these equations has not been studied adequately. One of the purposes of this paper is also to represent these equations in dual quaternionic matrix form and to give some arguments confirming that the dual quaternionic approach facilitates investigation of electromagnetism. The matrix equations obtained have been compared with their dual quaternionic representations. We also aim to promote a better understanding between dual quaternions and their matrices. As mentioned before, dual quaternions are composed of two real quaternions. For this reason, we produced new 4 and 8-dimensional matrices. The formulation presented in this paper establishes a relationship between dual quaternionic electromagnetism and the equivalent matrix algebra.

2 Dual quaternions

A dual number invented by Clifford [29] in 1873 is

$$Q = q + \epsilon q'. \tag{1}$$

Here ϵ is known as the dual unit and has the following properties:

$$\epsilon \neq 0, \quad 0\,\epsilon = \epsilon\,0 = 0, \quad 1\epsilon = \epsilon 1 = \epsilon, \quad \epsilon^2 = 0. \tag{2}$$

The real numbers q and q' are called the real and dual parts of Q, respectively. It should be emphasized that dual numbers are extension of real numbers.

A real quaternion with four components can be written as

$$\mathbf{q} = q_0\boldsymbol{e}_0 + q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3. \tag{3}$$

Here q_0, q_1, q_2, and q_3 are real numbers. $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are quaternion basis elements that obey the following multiplication rules

$$\boldsymbol{e}_0^2 = 1, \;\; \boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = \boldsymbol{e}_3^2 = -1, \quad \boldsymbol{e}_0\boldsymbol{e}_j = \boldsymbol{e}_j\boldsymbol{e}_0 = \boldsymbol{e}_j \;\; (j = 1, 2, 3), \tag{4a}$$

$$\boldsymbol{e}_1\boldsymbol{e}_2 = -\boldsymbol{e}_2\boldsymbol{e}_1 = \boldsymbol{e}_3, \;\; \boldsymbol{e}_3\boldsymbol{e}_1 = -\boldsymbol{e}_1\boldsymbol{e}_3 = \boldsymbol{e}_2, \;\; \boldsymbol{e}_2\boldsymbol{e}_3 = -\boldsymbol{e}_3\boldsymbol{e}_2 = \boldsymbol{e}_1. \tag{4b}$$

A dual quaternion $\mathbf{Q}$ is defined in a similar way to dual numbers:

$$\begin{aligned}\mathbf{Q} = \mathbf{q} + \epsilon\mathbf{q}' &= (q_0 + \epsilon q_0')\boldsymbol{e}_0 + (q_1 + \epsilon q_1')\boldsymbol{e}_1 + (q_2 + \epsilon q_2')\boldsymbol{e}_2 + (q_3 + \epsilon q_3')\boldsymbol{e}_3 \\ &= Q_0\boldsymbol{e}_0 + Q_1\boldsymbol{e}_1 + Q_2\boldsymbol{e}_2 + Q_3\boldsymbol{e}_3.\end{aligned} \tag{5}$$

Here Q_0, Q_1, Q_2 and Q_3 are dual numbers. A dual quaternion $\mathbf{Q}$ consists of a scalar part $S_{\mathrm{Q}} = Q_0$ and vector part $V_{\mathrm{Q}} = \boldsymbol{Q} = Q_1\boldsymbol{e}_1 + Q_2\boldsymbol{e}_2 + Q_3\boldsymbol{e}_3$.

The product of two dual quaternions $\mathbf{P}$ and $\mathbf{Q}$ is

$$\mathbf{PQ} = (P_0 + \boldsymbol{P})(Q_0 + \boldsymbol{Q}) = P_0Q_0 + P_0\boldsymbol{Q} + Q_0\boldsymbol{P} - \boldsymbol{P}\circ\boldsymbol{Q} + \boldsymbol{P}\times\boldsymbol{Q}, \tag{6}$$

where the dot and cross product indicate, respectively, the usual three-dimensional scalar and vector products.

For any dual quaternion $\mathbf{Q}$ there exists a complex conjugate,

$$\mathbf{Q}^* = S_{\mathrm{Q}} - V_{\mathrm{Q}} = Q_0 - Q_1\boldsymbol{e}_1 - Q_2\boldsymbol{e}_2 - Q_3\boldsymbol{e}_3, \tag{7}$$

while the dual conjugate $\mathbf{Q}^{\mathrm{c}}$ is given by

$$\begin{aligned}\mathbf{Q}^{\mathrm{c}} &= (q_0 - \epsilon q_0')\boldsymbol{e}_0 + (q_1 - \epsilon q_1')\boldsymbol{e}_1 + (q_2 - \epsilon q_2')\boldsymbol{e}_2 + (q_3 - \epsilon q_3')\boldsymbol{e}_3 \\ &= Q_0^{\mathrm{c}}\boldsymbol{e}_0 + Q_1^{\mathrm{c}}\boldsymbol{e}_1 + Q_2^{\mathrm{c}}\boldsymbol{e}_2 + Q_3^{\mathrm{c}}\boldsymbol{e}_3.\end{aligned} \tag{8}$$

Here c denotes the dual conjugate. The complex conjugate and dual conjugate are also dual quaternions.

The norm of a dual quaternion in general is a dual scalar,

$$N_Q = \mathbf{Q}\mathbf{Q}^* = \mathbf{Q}^*\mathbf{Q} = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2. \tag{9}$$

The inverse of a dual quaternion $\mathbf{Q}$ (non-zero norm) is also a dual quaternion and can be defined as

$$\mathbf{Q}^{-1} = \frac{\mathbf{Q}^*}{N_Q}. \tag{10}$$

Dual quaternions of norm unity are called unit dual quaternions.

3 Matrix representations of dual quaternions

In this section, we aim to provide the necessary background for the mathematical formulations to be developed in this paper. In the first section, the skew-symmetric matrix form of a dual quaternion $\mathbf{Q}$ is obtained by using 4×4 matrix representations of the real quaternion basis elements $\boldsymbol{e}_{0,1,2,3}$. In the second section, new dual quaternionic 8×8 matrices are produced and their properties are also investigated.

3.1 4 × 4 Matrix representations

Dual quaternionic matrices can be produced in a similar way to the definition of a dual number Q in Eq. (1). The dual unit ϵ is represented by a special 2×2 matrix as

$$\mathcal{E} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{11}$$

with the property

$$\mathcal{E}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{12}$$

Thus, in terms of matrices, a dual number $Q = q + \epsilon q'$ may be expressed as

$$\mathcal{Q} = q(I_2) + q'(\mathcal{E} \times I_2) = \begin{bmatrix} q & q' \\ 0 & q \end{bmatrix}, \tag{13}$$

where I_2 is the unit matrix of order two.

On the other hand, for a real quaternion $\mathbf{q} = q_0\boldsymbol{e}_0 + q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3$ we can use the following 4×4 real representation of the matrices ψ_0, ψ_1, ψ_2 and ψ_3 [21]:

$$\psi_0 = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

$$\psi_1 = \begin{bmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{15}$$

$$\psi_2 = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{16}$$

$$\psi_3 = \begin{bmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

Here σ_2 is defined as

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{18}$$

where i is the imaginary unit ($i = \sqrt{-1}$).

Thus, a real quaternion $\mathbf{q}$ is expressed by the following skew-symmetric 4×4 matrix

$$\mathsf{q} = q_0\psi_0 + q_1\psi_1 + q_2\psi_2 + q_3\psi_3 = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix}. \tag{19}$$

The ψ_0, ψ_1, ψ_2 and ψ_3 matrices satisfy the same algebra as that of Eq. (4):

$$\psi_0^2 = I_4, \quad \psi_1^2 = \psi_2^2 = \psi_3^2 = -I_4, \;\; \psi_0\psi_j = \psi_j\psi_0 \;\; (j = 1, 2, 3), \tag{20a}$$

$$\psi_1\psi_2 = -\psi_2\psi_1 = \psi_3, \;\; \psi_3\psi_1 = -\psi_1\psi_3 = \psi_2, \;\; \psi_2\psi_3 = -\psi_3\psi_2 = \psi_1, \tag{20b}$$

where I_4 is the unit matrix of order 4.

In terms of 4×4 matrices, it is possible to represent a dual quaternion $\mathbf{Q} = \mathbf{q} + \epsilon\mathbf{q}'$ as the summation of skew-symmetric matrices q and q' that correspond to the real quaternions $\mathbf{q}$ and $\mathbf{q}'$, respectively:

$$\mathsf{Q} = \mathsf{q} + \epsilon\mathsf{q}' = \begin{bmatrix} q_0 + \epsilon q_0' & q_1 + \epsilon q_1' & q_2 + \epsilon q_2' & q_3 + \epsilon q_3' \\ -q_1 - \epsilon q_1' & q_0 + \epsilon q_0' & -q_3 - \epsilon q_3' & q_2 + \epsilon q_2' \\ -q_2 - \epsilon q_2' & q_3 + \epsilon q_3' & q_0 + \epsilon q_0' & -q_1 - \epsilon q_1' \\ -q_3 - \epsilon q_3' & -q_2 - \epsilon q_2' & q_1 + \epsilon q_1' & q_0 + \epsilon q_0' \end{bmatrix}. \tag{21a}$$

Therefore in view of Eqs. (14)–(17) and (21a), the dual quaternion matrix Q is written by

$$\mathsf{Q} = Q_0\psi_0 + Q_1\psi_1 + Q_2\psi_2 + Q_3\psi_3 = \begin{bmatrix} Q_0 & Q_1 & Q_2 & Q_3 \\ -Q_1 & Q_0 & -Q_3 & Q_2 \\ -Q_2 & Q_3 & Q_0 & -Q_1 \\ -Q_3 & -Q_2 & Q_1 & Q_0 \end{bmatrix}. \tag{21b}$$

where Q_0, Q_1, Q_2, Q_3 are dual numbers. The column matrix representation of an arbitrary dual quaternion $\mathbf{Q}$ with respect to the basis $\boldsymbol{e}_{0,1,2,3}$ is merely the collection of its parameters:

$$\mathsf{Q} = \begin{bmatrix} Q_0, \ Q_1, \ Q_2, \ Q_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} Q_0, \ \boldsymbol{Q}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{22}$$

where the superscript T indicates the transpose of a matrix. By using the representational resemblance between real and dual quaternions in 4×4 matrix form many useful identities can be derived. The product of dual quaternions $\mathbf{P}$ and $\mathbf{Q}$ can be expressed as

$$\mathsf{PQ} = \begin{bmatrix} P_0 & -\boldsymbol{P}^{\mathrm{T}} \\ \boldsymbol{P} & P_0 I_3 + \tilde{\boldsymbol{P}} \end{bmatrix} \begin{bmatrix} Q_0 \\ \boldsymbol{Q} \end{bmatrix} = \begin{bmatrix} P_0 & -P_1 & -P_2 & -P_3 \\ P_1 & P_0 & -P_3 & P_2 \\ P_2 & P_3 & P_0 & -P_1 \\ P_3 & -P_2 & P_1 & P_0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}, \tag{23a}$$

or

$$\mathsf{PQ} = \begin{bmatrix} Q_0 & -\boldsymbol{Q}^{\mathrm{T}} \\ \boldsymbol{Q} & Q_0 I_3 - \tilde{\boldsymbol{Q}} \end{bmatrix} \begin{bmatrix} P_0 \\ \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} Q_0 & -Q_1 & -Q_2 & -Q_3 \\ Q_1 & Q_0 & Q_3 & -Q_2 \\ Q_2 & -Q_3 & Q_0 & Q_1 \\ Q_3 & Q_2 & -Q_1 & Q_0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}, \tag{23b}$$

where I_3 is the 3×3 unit matrix,

$$\tilde{\boldsymbol{P}} = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}, \text{and } \tilde{\boldsymbol{Q}} = \begin{bmatrix} 0 & -Q_3 & Q_2 \\ Q_3 & 0 & -Q_1 \\ -Q_2 & Q_1 & 0 \end{bmatrix}. \tag{24}$$

This property is very useful. Although dual quaternion multiplication is associative and distributive with respect to addition and substraction, it isn't commutative. But matrix Eq. (23) shows that P and Q can commute simply with a sign change. Defining the following matrices:

$$\breve{\mathsf{P}} = \begin{bmatrix} P_0 & -\boldsymbol{P}^{\mathrm{T}} \\ \boldsymbol{P} & P_0 I_3 + \tilde{\boldsymbol{P}} \end{bmatrix} = \begin{bmatrix} P_0 & -P_1 & -P_2 & -P_3 \\ P_1 & P_0 & -P_3 & P_2 \\ P_2 & P_3 & P_0 & -P_1 \\ P_3 & -P_2 & P_1 & P_0 \end{bmatrix} \tag{25}$$

and

$$\breve{\mathsf{Q}} = \begin{bmatrix} Q_0 & -\boldsymbol{Q}^{\mathrm{T}} \\ \boldsymbol{Q} & Q_0 I_3 - \tilde{\boldsymbol{Q}} \end{bmatrix} = \begin{bmatrix} Q_0 & -Q_1 & -Q_2 & -Q_3 \\ Q_1 & Q_0 & Q_3 & -Q_2 \\ Q_2 & -Q_3 & Q_0 & Q_1 \\ Q_3 & Q_2 & -Q_1 & Q_0 \end{bmatrix}, \tag{26}$$

then the commutative property of multiplication can be expressed in a compact form as

$$\breve{\mathsf{P}}\mathsf{Q} = \breve{\mathsf{Q}}\mathsf{P}. \tag{27}$$

3.2 $\mathbf{8 \times 8}$ Matrix representations

Since dual quaternions have eight real components, we can also obtain their 8×8 matrix representations. For this purpose, the 8×8 matrix forms of quaternion basis elements $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and the dual unit ϵ must first be produced.

The 8×8 matrix corresponding to dual unit ϵ can be realized as the following operation

$$\varepsilon = \mathcal{E} \otimes I_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & I_4 \\ 0 & 0 \end{bmatrix}. \tag{28}$$

Here $\otimes$ denotes the Kronecker product for matrices of arbitrary sizes.

By the same token, now it is possible to obtain 8×8 matrices that correspond to the quaternion basis elements $\boldsymbol{e}_{0,1,2,3}$ in the following manner:

$$\alpha_0 = I_2 \otimes I_4 = \begin{bmatrix} I_4 & 0 \\ 0 & I_4 \end{bmatrix}, \tag{29}$$

$$\alpha_1 = I_2 \otimes \psi_1 = \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_1 \end{bmatrix}, \tag{30}$$

$$\alpha_2 = I_2 \otimes \psi_2 = \begin{bmatrix} \psi_2 & 0 \\ 0 & \psi_2 \end{bmatrix}, \tag{31}$$

and finally,

$$\alpha_3 = I_2 \otimes \psi_3 = \begin{bmatrix} \psi_3 & 0 \\ 0 & \psi_3 \end{bmatrix}. \tag{32}$$

As can be seen, these matrices can be generalized by the following formulas

$$\alpha_j = I_2 \otimes \psi_j = \begin{bmatrix} \psi_j & 0 \\ 0 & \psi_j \end{bmatrix} \quad (j = 1, 2, 3), \tag{33}$$

where ψ_j are the 4×4 matrices defined by the Eqs. (14–17). Similar to Eq. (4), α_0, α_1, α_2, and α_3, the basis elements in matrix form, satisfy the following multiplication rules [21]:

$$\alpha_0^2 = -\alpha_1^2 = -\alpha_2^2 = -\alpha_3^2 = I_8, \;\; \alpha_0\alpha_j = \alpha_j\alpha_0 = \alpha_j \;\; (j = 1, 2, 3), \tag{34a}$$

$$\alpha_1\alpha_2 = -\alpha_2\alpha_1 = \alpha_3, \;\; \alpha_3\alpha_1 = -\alpha_1\alpha_3 = \alpha_2, \;\; \alpha_2\alpha_3 = -\alpha_3\alpha_2 = \alpha_1. \tag{34b}$$

Now, an 8×8 matrix corresponding to the dual quaternion **Q** must be formulated. In view of Eq. (5) a dual quaternionic matrix Q can be constituted by the following formula:

$$\mathsf{Q} = (q_0 + \varepsilon q_0')\alpha_0 + (q_1 + \varepsilon q_1')\alpha_1 + (q_2 + \varepsilon q_2')\alpha_2 + (q_3 + \varepsilon q_3')\alpha_3. \tag{35a}$$

As can be seen, ε performs the role of the dual unit ϵ while the matrices $\alpha_{0,j}$ correspond to the quaternion basis elements $\boldsymbol{e}_{0,1,2,3}$. So it is possible to express the dual quaternion

Q as the following 8 × 8 matrix:

$$\mathsf{Q}=\begin{bmatrix} q_0 & q_1 & q_2 & q_3 & q'_0 & q'_1 & q'_2 & q'_3 \\ -q_1 & q_0 & -q_3 & q_2 & -q'_1 & q'_0 & -q'_3 & q'_2 \\ -q_2 & q_3 & q_0 & -q_1 & -q'_2 & q'_3 & q'_0 & -q'_1 \\ -q_3 & -q_2 & q_1 & q_0 & -q'_3 & -q'_2 & q'_1 & q'_0 \\ 0 & 0 & 0 & 0 & q_0 & q_1 & q_2 & q_3 \\ 0 & 0 & 0 & 0 & -q_1 & q_0 & -q_3 & q_2 \\ 0 & 0 & 0 & 0 & -q_2 & q_3 & q_0 & -q_1 \\ 0 & 0 & 0 & 0 & -q_3 & -q_2 & q_1 & q_0 \end{bmatrix}=\begin{bmatrix} \mathsf{q} & \mathsf{q}' \\ 0 & \mathsf{q} \end{bmatrix}. \tag{35b}$$

The sum of the elements along the principal diagonal yields the trace of Q,

$$Tr\mathsf{Q}=8q_0. \tag{36}$$

The 8 × 8 matrix representation of the complex conjugate $\mathbf{Q}^*$ can be defined in a similar way to Eq. (35):

$$\mathsf{Q}^\dagger=(q_0+\varepsilon q'_0)\alpha_0-(q_1+\varepsilon q'_1)\alpha_1-(q_2+\varepsilon q'_2)\alpha_2-(q_3+\varepsilon q'_3)\alpha_3 \tag{37a}$$

or, in expanded form

$$\mathsf{Q}^\dagger=\begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 & q'_0 & -q'_1 & -q'_2 & -q'_3 \\ q_1 & q_0 & q_3 & -q_2 & q'_1 & q'_0 & q'_3 & -q'_2 \\ q_2 & -q_3 & q_0 & q_1 & q'_2 & -q'_3 & q'_0 & q'_1 \\ q_3 & q_2 & -q_1 & q_0 & q'_3 & q'_2 & -q'_1 & -q'_0 \\ 0 & 0 & 0 & 0 & q_0 & -q_1 & -q_2 & -q_3 \\ 0 & 0 & 0 & 0 & q_1 & q_0 & q_3 & -q_2 \\ 0 & 0 & 0 & 0 & q_2 & -q_3 & q_0 & q_1 \\ 0 & 0 & 0 & 0 & q_3 & q_2 & -q_1 & q_0 \end{bmatrix}=\begin{bmatrix} \mathsf{q}^\dagger & \mathsf{q}'^\dagger \\ 0 & \mathsf{q}^\dagger \end{bmatrix}. \tag{37b}$$

Quaternion conjugation is used in the formulation of electromagnetism with dual quaternions.

On the other hand, the dual conjugate of **Q** can also be represented as

$$\bar{\mathsf{Q}}=(q_0-\varepsilon q'_0)\alpha_0+(q_1-\varepsilon q'_1)\alpha_1+(q_2-\varepsilon q'_2)\alpha_2+(q_3-\varepsilon q'_3)\alpha_3, \tag{38a}$$

where the expanded matrix form is as the following

$$\bar{\mathsf{Q}} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 & -q_0' & -q_1' & -q_2' & -q_3' \\ -q_1 & q_0 & -q_3 & q_2 & q_1' & -q_0' & q_3' & -q_2' \\ -q_2 & q_3 & q_0 & -q_1 & q_2' & -q_3' & -q_0' & q_1' \\ -q_3 & -q_2 & q_1 & q_0 & q_3' & q_2' & -q_1' & -q_0' \\ 0 & 0 & 0 & 0 & q_0 & q_1 & q_2 & q_3 \\ 0 & 0 & 0 & 0 & -q_1 & q_0 & -q_3 & q_2 \\ 0 & 0 & 0 & 0 & -q_2 & q_3 & q_0 & -q_1 \\ 0 & 0 & 0 & 0 & -q_3 & -q_2 & q_1 & q_0 \end{bmatrix} = \begin{bmatrix} \mathsf{q} & -\mathsf{q} \\ 0 & \mathsf{q} \end{bmatrix}. \tag{38b}$$

4 Matrix representations of dual quaternionic electromagnetism

In general, two basic dual quaternions are used to represent Maxwell's equations in dual quaternionic form [28]. One of them is the dual quaternionic differential operator that is given by

$$\mathbf{D} = \boldsymbol{\nabla} + \epsilon\frac{\partial}{\partial t} = \left[\frac{\partial}{\partial x}\boldsymbol{e}_1 + \frac{\partial}{\partial y}\boldsymbol{e}_2 + \frac{\partial}{\partial z}\boldsymbol{e}_3\right] + \epsilon\frac{\partial}{\partial t}. \tag{39}$$

In the light of Eq. (35), the dual quaternionic differential operator $\mathbf{D}$ can be expressed as the following 8×8 real matrix formulation

$$\mathsf{D} = (\varepsilon\partial_t)\alpha_0 + \partial_x\alpha_1 + \partial_y\alpha_2 + \partial_z\alpha_3, \tag{40a}$$

which maps to

$$\mathsf{D} = \begin{bmatrix} 0 & \partial_x & \partial_y & \partial_z & \partial_t & 0 & 0 & 0 \\ -\partial_x & 0 & -\partial_z & \partial_y & 0 & \partial_t & 0 & 0 \\ -\partial_y & \partial_z & 0 & -\partial_x & 0 & 0 & \partial_t & 0 \\ -\partial_z & -\partial_y & \partial_x & 0 & 0 & 0 & 0 & \partial_t \\ 0 & 0 & 0 & 0 & 0 & \partial_x & \partial_y & \partial_z \\ 0 & 0 & 0 & 0 & -\partial_x & 0 & -\partial_z & \partial_y \\ 0 & 0 & 0 & 0 & -\partial_y & \partial_z & 0 & -\partial_x \\ 0 & 0 & 0 & 0 & -\partial_z & -\partial_y & \partial_x & 0 \end{bmatrix} = \begin{bmatrix} \nabla & \partial_t I_4 \\ 0 & \nabla \end{bmatrix}. \tag{40b}$$

Here, ∇ is a skew-symmetric 4×4 matrix that represents the vector part of the differential operator $\mathbf{D}$:

$$\nabla = \partial_x\,\psi_1 + \partial_y\,\psi_2 + \partial_z\,\psi_3 = \begin{bmatrix} 0 & \partial_x & \partial_y & \partial_z \\ -\partial_x & 0 & -\partial_z & \partial_y \\ -\partial_y & \partial_z & 0 & -\partial_x \\ -\partial_z & -\partial_x & \partial_x & 0 \end{bmatrix}. \tag{41}$$

Thus, D is called the operator matrix.

The other physical quantity is also a dual quaternion, $\mathbf{M}$, that includes both the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$,

$$\mathbf{M} = -\boldsymbol{E} + \epsilon\boldsymbol{H} = -[E_1\boldsymbol{e}_1 + E_2\boldsymbol{e}_2 + E_3\boldsymbol{e}_3] + \epsilon[H_1\boldsymbol{e}_1 + H_2\boldsymbol{e}_2 + H_3\boldsymbol{e}_3]. \tag{42}$$

In analogy to Eq. (35) with $q_0 = q_0' = 0,\ q_1 = -E_1,\ q_2 = -E_2,\ q_3 = -E_3$ and $q_1' = H_1,\ q_2' = H_2,\ q_3' = H_3$, its matrix form can be formulated as

$$\mathsf{M} = (-E_1 + \varepsilon H_1)\alpha_1 + (-E_2 + \varepsilon H_2)\alpha_2 + (-E_3 + \varepsilon H_3)\alpha_3. \tag{43a}$$

By using this formula, the isomorphic matrix representation of $\mathbf{M}$ describes the 8×8 matrix as

$$\mathsf{M} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 & 0 & H_1 & H_2 & H_3 \\ E_1 & 0 & E_3 & -E_2 & -H_1 & 0 & -H_3 & H_2 \\ E_2 & -E_3 & 0 & E_1 & -H_2 & H_3 & 0 & -H_1 \\ E_3 & E_2 & -E_1 & 0 & -H_3 & -H_2 & H_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -E_1 & -E_2 & -E_3 \\ 0 & 0 & 0 & 0 & E_1 & 0 & E_3 & -E_2 \\ 0 & 0 & 0 & 0 & E_2 & -E_3 & 0 & E_1 \\ 0 & 0 & 0 & 0 & E_3 & E_2 & -E_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathsf{E}^\dagger & \mathsf{H} \\ 0 & \mathsf{E}^\dagger \end{bmatrix}. \tag{43b}$$

Here $\mathsf{E}^\dagger$ and H are, respectively, 4×4 real quaternionic matrices that correspond to the conjugate of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$,

$$\mathsf{E}^\dagger = \mathsf{E}^{\mathrm{T}} = [E_1\psi_1 + E_2\psi_1 + E_3\psi_3]^{\mathrm{T}} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -E_3 & E_2 \\ -E_2 & E_3 & 0 & -E_1 \\ -E_3 & -E_2 & E_1 & 0 \end{bmatrix}^{\mathrm{T}} \tag{44}$$

and

$$\mathsf{H} = H_1\psi_1 + H_2\psi_2 + H_3\psi_3 = \begin{bmatrix} 0 & H_1 & H_2 & H_3 \\ -H_1 & 0 & -H_3 & H_2 \\ -H_2 & H_3 & 0 & -H_1 \\ -H_3 & -H_2 & H_1 & 0 \end{bmatrix}. \quad (45)$$

Henceforth we will call M the electromagnetic field matrix.

Using Gaussian and natural units $\varepsilon = \mu = c = 1$, operation of the dual quaternionic differential operator $\mathbf{D}$ on the dual quaternion $\mathbf{M}$ gives [28]

$$\mathbf{DM} = \frac{\partial \boldsymbol{H}}{\partial t} + \mathbf{S} \quad (46)$$

where $\mathbf{S}$ is the dual quaternionic current density

$$\mathbf{S} = \rho_v + \epsilon \boldsymbol{J}. \quad (47)$$

Eq. (46) is the expression of Maxwell's equations of classical electrodynamics; these four equations are combined into a single equation. Here ρ_v is the electric charge density while $\boldsymbol{J}$ is the electric current density. Like Eq. (46), by operation of matrix D on the electromagnetic field matrix M Maxwell's equations are now expressed in the following 8×8 matrix form:

$$\mathsf{DM} = \mathsf{J}, \quad (48)$$

where the matrix J is

$$\mathsf{J} = \begin{bmatrix} \rho_v & \frac{\partial H_1}{\partial t} & \frac{\partial H_2}{\partial t} & \frac{\partial H_3}{\partial t} & 0 & J_1 & J_2 & J_3 \\ -\frac{\partial H_1}{\partial t} & \rho_v & -\frac{\partial H_3}{\partial t} & \frac{\partial H_2}{\partial t} & -J_1 & 0 & -J_3 & J_2 \\ -\frac{\partial H_2}{\partial t} & \frac{\partial H_3}{\partial t} & \rho_v & -\frac{\partial H_1}{\partial t} & -J_2 & J_3 & 0 & -J_1 \\ -\frac{\partial H_3}{\partial t} & -\frac{\partial H_2}{\partial t} & \frac{\partial H_1}{\partial t} & \rho_v & -J_3 & -J_2 & J_1 & 0 \\ 0 & 0 & 0 & 0 & \rho_v & \frac{\partial H_1}{\partial t} & \frac{\partial H_2}{\partial t} & \frac{\partial H_3}{\partial t} \\ 0 & 0 & 0 & 0 & -\frac{\partial H_1}{\partial t} & \rho_v & -\frac{\partial H_3}{\partial t} & \frac{\partial H_2}{\partial t} \\ 0 & 0 & 0 & 0 & -\frac{\partial H_2}{\partial t} & \frac{\partial H_3}{\partial t} & \rho_v & -\frac{\partial H_1}{\partial t} \\ 0 & 0 & 0 & 0 & -\frac{\partial H_3}{\partial t} & -\frac{\partial H_2}{\partial t} & \frac{\partial H_1}{\partial t} & \rho_v \end{bmatrix}. \quad (49)$$

As can be seen, the trace of J gives

$$Tr\mathsf{J} = 8\rho_v \quad (50)$$

and provides easy determination of the electric charge density ρ_v.

Electromagnetic fields are generally functions of space and time. In the special case in which they are time-independent, by setting the time derivative equals to zero, then dual the quaternionic Eq. (46) gets a new form as

$$\mathbf{DM} = \left[\boldsymbol{\nabla} + \epsilon\frac{\partial}{\partial t}\right][-\boldsymbol{E} + \epsilon\boldsymbol{H}] = \rho_v + \epsilon\boldsymbol{J}. \tag{51}$$

Thus, the dual quaternionic matrix reformulation can be expressed as

$$\begin{bmatrix} \nabla & \partial_t I_4 \\ 0 & \nabla \end{bmatrix} \times \begin{bmatrix} \mathsf{E}^\dagger & \mathsf{H} \\ 0 & \mathsf{E}^\dagger \end{bmatrix} = \begin{bmatrix} \rho_v I_4 & \mathsf{j} \\ 0 & \rho_v I_4 \end{bmatrix}. \tag{52}$$

Here j is a 4×4 real quaternionic matrix that corresponds to the electric current density vector $\boldsymbol{J}$,

$$\mathsf{j} = j_1\psi_1 + j_2\psi_2 + j_3\psi_3 = \begin{bmatrix} 0 & j_1 & j_2 & j_3 \\ -j_1 & 0 & -j_3 & j_2 \\ -j_2 & j_3 & 0 & -j_1 \\ -j_3 & -j_2 & j_1 & 0 \end{bmatrix}. \tag{53}$$

Another consequence of Maxwell's equations can be obtained by the existence of a electromagnetic potential. Similarly to Eq. (39), it is possible to define the electromagnetic potential as the following:

$$\mathbf{P} = \boldsymbol{A} - \epsilon\varphi = [A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3] - \epsilon\varphi. \tag{54}$$

This is called the dual quaternionic electromagnetic potential. Here $\boldsymbol{A}$ and φ are the magnetic vector potential and the electric field potential, respectively. Now, we can formulate the 8×8 matrix form of the dual quaternionic electromagnetic potential as

$$\mathsf{P} = (-\varphi\varepsilon)\alpha_0 + A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3. \tag{55a}$$

This formula maps to following matrix:

$$\mathsf{P} = \begin{bmatrix} 0 & A_1 & A_2 & A_3 & -\varphi & 0 & 0 & 0 \\ -A_1 & 0 & -A_3 & A_2 & 0 & -\varphi & 0 & 0 \\ -A_2 & A_3 & 0 & -A_1 & 0 & 0 & -\varphi & 0 \\ -A_3 & -A_2 & A_1 & 0 & 0 & 0 & 0 & -\varphi \\ 0 & 0 & 0 & 0 & 0 & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & 0 & -A_1 & 0 & -A_3 & A_2 \\ 0 & 0 & 0 & 0 & -A_2 & A_3 & 0 & -A_1 \\ 0 & 0 & 0 & 0 & -A_3 & -A_2 & A_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathsf{A} & -\varphi I_4 \\ 0 & \mathsf{A} \end{bmatrix}, \tag{55b}$$

where A is

$$\mathsf{A} = A_1\psi_1 + A_2\psi_2 + A_3\psi_3 = \begin{bmatrix} 0 & A_1 & A_2 & A_3 \\ -A_1 & 0 & -A_3 & A_2 \\ -A_2 & A_3 & 0 & -A_1 \\ -A_3 & -A_2 & A_1 & 0 \end{bmatrix}. \tag{56}$$

By remembering the usual definitions of electric and magnetic fields;

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{H} = \nabla \times \boldsymbol{A}, \tag{57}$$

the dual quaternion product between the dual conjugate form of $\mathbf{D}$ and $\mathbf{P}$ leads to

$$\mathbf{D}^c\mathbf{P} = \left[\boldsymbol{\nabla} - \epsilon\frac{\partial}{\partial t}\right][\boldsymbol{A} - \epsilon\varphi] = \boldsymbol{H} + \epsilon\boldsymbol{E}. \tag{58}$$

As shown, this operation allows determination of fields $\boldsymbol{H}$ and $\boldsymbol{E}$ [28]. The isomorphic matrix mapping of the this equation can be derived easily as

$$\bar{\mathsf{D}}\mathsf{P} = \begin{bmatrix} \nabla & -\partial_t I_4 \\ 0 & \nabla \end{bmatrix} \times \begin{bmatrix} \mathsf{A} & -\varphi I_4 \\ 0 & \mathsf{A} \end{bmatrix} = \begin{bmatrix} \mathsf{H} & \mathsf{E} \\ 0 & \mathsf{H} \end{bmatrix}. \tag{59}$$

In the light of definitions (44) and (45), the expanded form of the field matrix can be written easily as following

$$\begin{bmatrix} \mathsf{H} & \mathsf{E} \\ 0 & \mathsf{H} \end{bmatrix} = \begin{bmatrix} 0 & H_1 & H_2 & H_3 & 0 & E_1 & E_2 & E_3 \\ -H_1 & 0 & -H_3 & H_2 & -E_1 & 0 & -E_3 & E_2 \\ -H_2 & H_3 & 0 & -H_1 & -E_2 & E_3 & 0 & -E_1 \\ -H_3 & -H_2 & H_1 & 0 & -E_3 & -E_2 & E_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & H_1 & H_2 & H_3 \\ 0 & 0 & 0 & 0 & -H_1 & 0 & -H_3 & H_2 \\ 0 & 0 & 0 & 0 & -H_2 & H_3 & 0 & -H_1 \\ 0 & 0 & 0 & 0 & -H_3 & -H_2 & H_1 & 0 \end{bmatrix} \tag{60}$$

and represents the components of the fields $\boldsymbol{H}$ and $\boldsymbol{E}$. Thus Maxwell's equations can be described in an isomorphic 8-dimensional real matrix representation.

As pointed out before, in order to reformulate classical electromagnetism, 4×4 matrices with dual coefficients can also be used instead of 8×8 matrices with real components.

Using definition (21b), the dual quaternionic differential operator **D** is expressed as

$$\mathsf{D} = \begin{bmatrix} \partial_0 & \nabla^{\mathrm{T}} \\ -\nabla & \partial_0 I_3 + \tilde{\nabla} \end{bmatrix} = \begin{bmatrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ -\partial_1 & \partial_0 & -\partial_3 & \partial_2 \\ -\partial_2 & \partial_3 & \partial_0 & -\partial_1 \\ -\partial_3 & -\partial_2 & \partial_1 & \partial_0 \end{bmatrix}, \tag{61}$$

where ∂_0, ∂_1, ∂_2, and ∂_3 are $\epsilon\partial_t$, ∂_x, ∂_y, and ∂_z, respectively. Here, matrix $\tilde{\nabla}$ is defined as

$$\tilde{\nabla} = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}. \tag{62}$$

The dual conjugate of the operator matrix D must also be

$$\bar{\mathsf{D}} = \begin{bmatrix} -\partial_0 & \nabla^{\mathrm{T}} \\ -\nabla & -\partial_0 I_3 + \tilde{\nabla} \end{bmatrix} = \begin{bmatrix} -\partial_0 & \partial_1 & \partial_2 & \partial_3 \\ -\partial_1 & -\partial_0 & -\partial_3 & \partial_2 \\ -\partial_2 & \partial_3 & -\partial_0 & -\partial_1 \\ -\partial_3 & -\partial_2 & \partial_1 & -\partial_0 \end{bmatrix}. \tag{63}$$

Similarly, the dual quaternion **M** that represents the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ can be rewritten as

$$\begin{aligned} \mathbf{M} = -\boldsymbol{E} + \epsilon \boldsymbol{H} &= (-E_1 + \epsilon H_1)\boldsymbol{e}_1 + (-E_2 + \epsilon H_2)\boldsymbol{e}_2 + (-E_3 + \epsilon H_3)\boldsymbol{e}_3 \\ &= M_1\boldsymbol{e}_1 + M_2\boldsymbol{e}_1 + M_3\boldsymbol{e}_3. \end{aligned} \tag{64}$$

Then, the column matrix form of **M** becomes

$$\mathsf{M} = \begin{bmatrix} 0, \; M_1, \; M_2, \; M_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0, \; \boldsymbol{M}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{65}$$

Using Eq. (23), Maxwell's equations of classical electrodynamics are now expressed in the following matrix form:

$$\begin{gathered} \begin{bmatrix} \partial_0 & -\nabla^{\mathrm{T}} \\ \nabla & \partial_0 I_3 + \tilde{\nabla} \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \rho_v \\ \boldsymbol{S} \end{bmatrix} \\ \begin{bmatrix} \partial_0 & -\partial_1 & -\partial_2 & -\partial_3 \\ \partial_1 & \partial_0 & -\partial_3 & \partial_2 \\ \partial_2 & \partial_3 & \partial_0 & -\partial_1 \\ \partial_3 & -\partial_2 & \partial_1 & \partial_0 \end{bmatrix} \begin{bmatrix} 0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} \rho_v \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}. \end{gathered} \tag{66}$$

Let us define the electromagnetic potential of an electrically charged particle $\mathbf{P} = \boldsymbol{A} - \epsilon\varphi = P_0\boldsymbol{e}_0 + P_1\boldsymbol{e}_1 + P_2\boldsymbol{e}_2 + P_3\boldsymbol{e}_3$ by the following dual matrix representation:

$$\mathsf{P} = \begin{bmatrix} P_0 & \boldsymbol{P}^{\mathrm{T}} \\ -\boldsymbol{P} & P_0 I_3 + \tilde{\boldsymbol{P}} \end{bmatrix} = \begin{bmatrix} P_0 & P_1 & P_2 & P_3 \\ -P_1 & P_0 & -P_3 & P_2 \\ -P_2 & P_3 & P_0 & -P_1 \\ -P_3 & -P_2 & P_1 & P_0 \end{bmatrix}, \tag{67}$$

where $\tilde{\boldsymbol{P}}$ is

$$\tilde{\boldsymbol{P}} = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}, \tag{68}$$

or

$$\mathsf{P} = \begin{bmatrix} P_0, \ P_1, \ P_2, \ P_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} P_0, \ \boldsymbol{P}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{69}$$

Similarly to Eq. (59), operating $\bar{\mathsf{D}}$ on P gives the elements of the electromagnetic field matrix F:

$$\begin{bmatrix} -\partial_0 & -\nabla^{\mathrm{T}} \\ \nabla & -\partial_0 I_3 + \tilde{\nabla} \end{bmatrix} \begin{bmatrix} P_0 \\ \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{F} \end{bmatrix}$$

$$\begin{bmatrix} -\partial_0 & -\partial_1 & -\partial_2 & -\partial_3 \\ \partial_1 & -\partial_0 & -\partial_3 & \partial_2 \\ \partial_2 & \partial_3 & -\partial_0 & -\partial_1 \\ \partial_3 & -\partial_2 & \partial_1 & -\partial_0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix}, \tag{70}$$

where F is

$$\mathsf{F} = \begin{bmatrix} 0, \ F_1, \ F_2, \ F_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0, \ H_1 + \epsilon E_1, \ H_2 + \epsilon E_2, \ H_3 + \epsilon E_3 \end{bmatrix}^{\mathrm{T}}. \tag{71}$$

Thus, the components of the electric field $\boldsymbol{E}$ and $\boldsymbol{B}$ can be obtained by evaluating matrix elements:

$$\begin{aligned} F_1 &= \partial_1 P_0 - \partial_0 P_1 - \partial_3 P_2 + \partial_2 P_3 \\ H_1 + \epsilon E_1 &= \epsilon\frac{\partial\varphi}{\partial x} - \epsilon\frac{\partial A_1}{\partial t} - \frac{\partial A_2}{\partial z} + \frac{\partial A_3}{\partial y} = \left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right] + \epsilon\left[\frac{\partial\varphi}{\partial x} - \frac{\partial A_1}{\partial t}\right] \end{aligned} \tag{72a}$$

$$\begin{aligned} F_2 &= \partial_2 P_0 + \partial_3 P_1 - \partial_0 P_2 - \partial_1 P_3 \\ H_2 + \epsilon E_2 &= \epsilon\frac{\partial\varphi}{\partial y} + \frac{\partial A_1}{\partial z} - \epsilon\frac{\partial A_2}{\partial t} - \frac{\partial A_3}{\partial x} = \left[\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right] + \epsilon\left[\frac{\partial\varphi}{\partial y} - \frac{\partial A_2}{\partial t}\right] \end{aligned} \tag{72b}$$

and

$$F_3 = \partial_3 P_0 - \partial_2 P_1 + \partial_1 P_2 - \partial_0 P_3$$
$$H_3 + \epsilon E_3 = \epsilon \frac{\partial \varphi}{\partial z} - \frac{\partial A_1}{\partial y} + \frac{\partial A_2}{\partial x} - \epsilon \frac{\partial A_3}{\partial t} = \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right] + \epsilon \left[\frac{\partial \varphi}{\partial z} - \frac{\partial A_3}{\partial t} \right]. \quad (72c)$$

5 Conclusions

As mentioned before, except for the well-known practical applications in rigid body movements in three dimensional space, especially in robotics, dual quaternionic formalism has not been used frequently in other areas of physics as it deserves. This work has been contributed to increase usage of this algebra by introducing special matrices for dual quaternions. The dual quaternionic matrices in electromagnetism that we derived not only provide a simple and elegant representation of Maxwell's and relavant field equations, but also simplify their manipulations. The use of matrix techniques also has an advantage in that it enables us to deal with the eight real components of the quaternionic equations simultaneously while maintaining the simplicity of the mathematical derivation.

In this paper, starting with definition of a dual number we have developed the isomorphism between dual quaternions and those of 4×4 and 8×8 matrices with dual and real components, respectively. Although the dual quaternion product doesn't satisfy the commutation property, by using the representational resemblance between real and dual quaternions in matrix form, dual quaternionic matrices can commute simply with a sign change (Eq. (23)). Since dual quaternions are composed of eight real components, new 8×8 matrices have been produced. For this purpose, first we derived a special matrix that corresponds to the dual unit ϵ. Eq. (28) shows that ε is the equivalent 8×8 matrix representation of ϵ. By combining the 8×8 matrices that correspond to the quaternion basis elements $\boldsymbol{e}_{0,1,2,3}$ and the dual unit ϵ, we obtained an 8×8 real matrix of the dual quaternion $\mathbf{Q}$. Then we described how a dual quaternion maps into its isomorphic matrix representation. Formulation of the matrix $\mathsf{Q} = (q_0 + \varepsilon q_0')\alpha_0 + (q_1 + \varepsilon q_1')\alpha_1 + (q_2 + \varepsilon q_2')\alpha_2 + (q_3 + \varepsilon q_3')\alpha_3$ is very similar to the definition of a dual quaternion $\mathbf{Q} = (q_0 + \epsilon q_0')\boldsymbol{e}_0 + (q_1 + \epsilon q_1')\boldsymbol{e}_1 + (q_2 + \epsilon q_2')\boldsymbol{e}_2 + (q_3 + \epsilon q_3')\boldsymbol{e}_3$. As seen, ε performs the role of the dual unit ϵ while matrices $\alpha_{0,1,2,3}$ correspond to $\boldsymbol{e}_{0,1,2,3}$. Thus, we have developed the isomorphism between dual quaternions and their matrices.

Transformation matrices are widely used for kinematic analysis and trajectory planning in robotics. Whenever a sensor is mounted on a robot hand, exposure of the relationship between the sensor and the hand is needed; this is called as the hand-eye calibration problem. Almost all efforts related to this problem aim to solve a homogeneous matrix equation of the form $\mathsf{AX} = \mathsf{XB}$ [30, 31]. Therefore, the matrix equations obtained in this paper can also contribute to solve this problem.

Indeed, the matrix representations of dual quaternions help us to simplify manipulations of equations. We have reformulated the equations of classical electromagnetism by using dual quaternionic matrices as Negi *et al.* [21] did with complex quaternionic matrices. Isomorphic matrix representations of Maxwell's equations have also been given. We have offered two different matrix formulations that are 8-dimensional with real components and 4-dimensional with dual components. The expressions with real matrices are quite explicit and simple. Because of separate locations in matrix, components of physical quantities such as the electric and magnetic fields or the electric charge density ρ_v and components of the electric current density $\boldsymbol{J}$ *etc.* can easily be shown and manipulated (Eqs. (43b) and (49)). On the other hand, a 8×8 matrix has 64 real quantities in it while a 4×4 matrix has only 16 dual quantities. One can think that 4×4 dual number matrix elements seem not so clear as real 8×8 matrices and this leads to an obstacle for the evaluation of results. But this is not true. Because of having a very useful symbol ϵ, dual numbers provide separation of quantities of different physical nature explicitly. Thus, it can be seen easily how to relate independent results. For example, the real parts of Eq. (72) are associated with components of the magnetic field $\boldsymbol{H}$ while the dual parts with symbol ϵ are related to components of the electric field $\boldsymbol{E}$.

Since dual quaternions are used generally to investigate spatial screw motion of a rigid body, the formulation of classical electromagnetism by this type of quaternions is quite new [28]. Therefore, this paper fills a gap and contains useful results. The dual quaternionic equations that are derived are similar to complex quaternionic [4]–[21] and octonionic formulations [22], [23]. Moreover, the mathematics of the paper is easy to follow and comprehend. The dual quaternionic formulation is preferable to the octonionic representation while it is as useful as the complex quaternionic version of electromagnetism. If the octonion formalism is used, in this case 8 basis elements must be used instead of 4. Because of having the very useful symbol ϵ, dual quaternions with only 4 basis elements can easily express physical quantities up to 8-dimensional space. Furthermore, this special structure helps us to separate quantities in different physical phenomena much more explicitly than octonions. Besides, there are many definitions for the products of octonion units [22]. These diversities in the octonion product lead to differences in the expressions of the differential operator, field equations, potential and source equations *etc.* and finally will cause many versions of the octonionic electromagnetism to arise as well. Therefore, a dual quaternionic formulation of the classical electromagnetism appears to be a fruitful choice.

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