Commun Nonlinear Sci Numer Simulat 14 (2009) 1810–1815

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Short communication

Travelling wave solutions of nonlinear evolution equations by using the first integral method

Filiz Tascan [a,*], Ahmet Bekir [b], Murat Koparan [c]

[a] Eskisehir Osmangazi University, Art-Science Faculty, Department of Mathematics, Eskisehir, Turkey
[b] Dumlupinar University, Art-Science Faculty, Department of Mathematics, Kütahya, Turkey
[c] Anadolu University, Education Faculty, Department of Elementary Education, Eskisehir, Turkey

A R T I C L E I N F O

Article history:
Received 8 July 2008
Received in revised form 9 July 2008
Accepted 9 July 2008
Available online 19 July 2008

PACS:
02.30.Jr
02.70.Wz
05.45.Yv
94.05.Fg

Keywords:
Travelling wave solutions
First integral method
Modified Zakharov–Kuznetsov (MZK) equation
ZK-MEW equation

A B S T R A C T

In this paper, we established travelling wave solutions for some (2 + 1)-dimensional nonlinear evolution equations. The first integral method was used to construct travelling wave solutions of nonlinear evolution equations. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. The first integral method presents a wider applicability for handling nonlinear wave equations.

1. Introduction

The investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In the past several decades, new exact solutions may help to find new phenomena. A variety of powerful methods, such as inverse scattering method [1,2], bilinear transformation [3], the tanh–sech method [4–6], extended tanh method [7–9], sine–cosine method [10,11], homogeneous balance method [12], Exp-function method [13–15] and Riccati equation method [16] were used to develop nonlinear dispersive and dissipative problems.

The pioneer work Feng [17] introduced the first integral method for a reliable treatment of the nonlinear PDEs. The useful first integral method is widely used by many such as in [18,19] and by the reference therein.

Our first interest in the present work is in implementing the first integral method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. In Section 2, we describe this method for

* Corresponding author. Tel.: +90 222 2393750; fax: +90 222 2393578.
E-mail addresses: ftascan@ogu.edu.tr (F. Tascan), abekir@dumlupinar.edu.tr (A. Bekir), mkoparan@anadolu.edu.tr (M. Koparan).

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doi:10.1016/j.cnsns.2008.07.009

finding exact travelling wave solutions of nonlinear evolution equations. In Section 3 to Section 4, we illustrate this method in detail with the celebrated the modified Zakharov–Kuznetsov (MZK) equation and ZK-MEW equation. In Section 5, some conclusions are given.

2. The first integral method

Raslan has summarized for using first integral method [20].

Step 1. Consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots) = 0. \tag{2.1}$$

Using a wave variable $\xi = x - ct$. We can rewrite Eq. (2.1) in the following nonlinear ODE

$$Q(U, U', U'', U''', \ldots) = 0. \tag{2.2}$$

where the prime denotes the derivation with respect to ξ. Eq. (2.2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

Step 2. Suppose that the solution of ODE (2.2) can be written as follows:

$$u(x,t) = f(\xi). \tag{2.3}$$

Step 3. We introduce a new independent variable

$$X(\xi) = f(\xi), \qquad Y = f_\xi(\xi), \tag{2.4}$$

which leads a system of

$$\begin{aligned} X_\xi(\xi) &= Y(\xi), \\ Y_\xi(\xi) &= F(X(\xi), Y(\xi)). \end{aligned} \tag{2.5}$$

Step 4. By the qualitative theory of ordinary differential equations [21], if we can find the integrals to (2.5) under the same conditions, then the general solutions to (2.5) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to (2.5) which reduces (2.2) to a first order integrable ordinary differential equation. An exact solution to (2.1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division theorem: Suppose that $P(w,z), Q(w,z)$ are polynomials in $C(w,z)$ and $P(w,z)$ is irreducible in $C(w,z)$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C(w,z)$ such that

$$Q[w,z] = P[w,z]G[w,z]. \tag{2.6}$$

3. The modified Zakharov–Kuznetsov equation

Consider the modified Zakharov–Kuznetsov equation

$$u_t + u^2 u_x + u_{xxx} + u_{xyy} = 0. \tag{3.1}$$

Using the wave variable $\xi = x + y - ct$, the Eq. (3.1) is carried to a ODE

$$-cu' + u^2 u' + 2u''' = 0. \tag{3.2}$$

where the prime denotes the derivation with respect to ξ. Integrating (3.2) with respect to ξ and considering the zero constants for intergation we obtain

$$-cu + \frac{u^3}{3} + 2u'' = 0. \tag{3.3}$$

Using (2.4) we get

$$\dot{X}(\xi) = Y(\xi), \tag{3.4}$$

$$\dot{Y}(\xi) = \frac{cX(\xi)}{2} - \frac{X^3(\xi)}{2}. \tag{3.5}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (3.4), (3.5), and $q(X,Y) = \sum_{i=0}^{m} a_i(X) Y^i$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X) Y^i = 0, \tag{3.6}$$

where $a_i(X), (i = 0, 1, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$. Eq. (3.6) is called the first integral to (3.4), (3.5), due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\begin{aligned} \frac{dq}{d\xi} &= \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi}, \\ &= [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i \end{aligned} \tag{3.7}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in Eq. (3.6).

CaseI : Suppose that $m = 1$, by equating the coefficients of $Y^i (i = 0, 1, 2)$ on both sides of Eq. (3.7), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{3.8}$$

$$\dot{a}_0(X) = g(X) + h(X)a_0(X), \tag{3.9}$$

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{cX}{2} - \frac{X^3}{2}\right). \tag{3.10}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (3.8) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find $a_0(X)$

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0. \tag{3.11}$$

Substituting $a_0(X), a_1(X)$ and $g(X)$ in Eq. (3.10) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1 = \pm\frac{\sqrt{3}}{3}\mathrm{i}, \quad B_0 = 0, \quad c = \pm\frac{2\sqrt{3}}{3}\mathrm{i}A_0. \tag{3.12}$$

where A_0 is free parameter. Using Eq. (3.12) into Eq. (3.6), we obtain

$$Y(\xi) = \mp\frac{\sqrt{3}}{3}\mathrm{i}X^2(\xi) - A_0, \tag{3.13}$$

Combining (3.13) with (3.4), we obtain the exact solution to (3.5) and then the exact solution to the modified Zakharov–Kuznetsov equation (3.1) can be written as

$$X(\xi) = -\left(\frac{2\sqrt{3}}{\mathrm{i}}A_0\right)^{1/2}\tan\left[\left(\frac{\sqrt{3}\mathrm{i}}{6}A_0\right)^{1/2}(\xi + c_0)\right], \tag{3.14}$$

where c_0 is integration constant. If we take

$$A = -\left(\frac{2\sqrt{3}}{\mathrm{i}}A_0\right)^{1/2} \quad \text{and} \quad B = \left(\frac{\sqrt{3}\mathrm{i}}{6}A_0\right)^{1/2},$$

then the travelling wave solution to the modified Zakharov–Kuznetsov equation (3.1) can be written as

$$u(x, y, t) = A\tan\left[B\left(x + y \mp \frac{2\sqrt{3}}{3}\mathrm{i}A_0t + c_0\right)\right]. \tag{3.15}$$

Case II : Suppose that $m = 2$, by equating the coefficients of $Y^i (i = 0, 1, 2, 3)$ on both sides of Eq. (3.7), we have

$$\dot{a}_2(X) = h(X)a_2(X), \tag{3.16}$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{3.17}$$

$$\dot{a}_0(X) = -2a_2(X)\dot{Y} + g(X)a_1(X) + h(X)a_0(X), \tag{3.18}$$

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{cX}{2} - \frac{X^3}{2}\right). \tag{3.19}$$

Since $a_2(X)$ is a polynomial of X, then from (3.16) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = \frac{A_1}{2}X^2 + B_0X + A_0. \tag{3.20}$$

$$a_0(X) = \left(\frac{A_1^2}{8} + \frac{1}{12}\right)X^4 + \frac{A_1B_0}{2}X^3 + \left(\frac{A_0A_1}{2} + \frac{B_0^2}{2} - \frac{c}{2}\right)X^2 + A_0B_0X + d. \tag{3.21}$$

Substituting $a_0(X), a_1(X), a_2(X)$ and $g(X)$ in Eq. (3.19) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1 = \pm\frac{2\sqrt{3}}{3}\mathrm{i}, \quad B_0 = 0, \quad c = \frac{\sqrt{3}}{3}\mathrm{i}A_0, \quad d = \frac{A_0^2}{4}, \tag{3.22}$$

where A_0 is free parameter. Using Eq. (3.22) into Eq. (3.6), we obtain

$$Y(\xi) = -\frac{\sqrt{3}}{6}\mathrm{i}X^2(\xi) - \frac{A_0}{2}. \tag{3.23}$$

Combining (3.23) with (3.4), we obtain the exact solution to (3.5) and then the exact solution to the modified Zakharov–Kuznetsov equation (3.1) can be written as

$$X(\xi) = -\frac{1}{\mathrm{i}}\left(\sqrt{3}A_0\mathrm{i}\right)^{1/2}\tan\left[\frac{\left(3\sqrt{3}A_0\mathrm{i}\right)^{1/2}}{6}(\xi + c_0)\right], \tag{3.24}$$

where c_0 is integration constant. If we take

$$C = \left(\sqrt{3}A_0 i\right)^{1/2},$$

then the travelling wave solution to the modified Zakharov–Kuznetsov equation (3.1) can be written as

$$u(x,y,t) = \mathrm{i}C\tan\left[\frac{\sqrt{3}}{6}C\left(x + y - \frac{C^2}{3}t + c_0\right)\right]. \tag{3.25}$$

Comparing our results and Wazwaz's results [22] with Bekir's results [23] then it can be seen that the results are same.

4. The ZK-MEW equation

Let us consider the ZK-MEW equation [24]:

$$u_t + a(u^3)_x + (bu_{xt} + ru_{yy})_x = 0, \tag{4.1}$$

where a, b and r are known constants.

Using the wave variable $\xi = x + y - ct$ and proceeding as before we find

$$-cu' + a(u^3)' - bcu''' + ru''' = 0. \tag{4.2}$$

Integrating (4.2) with respect to ξ and neglecting constants of integration we find

$$-cu + au^3 + (r - bc)u'' = 0. \tag{4.3}$$

The constant of integration equals zero since the solitary wave solution and its derivatives equal zero as $\xi \to \pm\infty$. Using (2.4) we get

$$\dot{X}(\xi) = Y(\xi), \tag{4.4}$$

$$\dot{Y}(\xi) = \frac{c}{r - bc}X(\xi) - \frac{a}{r - bc}X^3(\xi), \quad r - bc \neq 0. \tag{4.5}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (4.4), (4.5), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{4.6}$$

where $a_i(X), (i = 0, 1, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$. Eq. (4.6) is called the first integral to (4.4), (4.5), due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}\xi} &= \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi}, \\ &= [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i \end{aligned} \tag{4.7}$$

In this example, we assume that $m = 1$ in Eq. (4.6). Suppose that $m = 1$, by equating the coefficients of $Y^i (i = 0, 1, 2)$ on both sides of Eq. (4.7), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{4.8}$$

$$\dot{a}_0(X) = g(X) + h(X)a_0(X), \tag{4.9}$$

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{c}{r-bc}X - \frac{a}{r-bc}X^3\right). \tag{4.10}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (4.8) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find $a_0(X)$

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0. \tag{4.11}$$

Substituting $a_0(X), a_1(X)$ and $g(X)$ in Eq. (4.10) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{rA_1^3}{2ab} - \frac{A_1}{b}, \quad B_0 = 0, \quad c = \frac{r}{b} + \frac{2a}{A_1^2 b}, \tag{4.12}$$

where A_1 is free parameter.

Similarly, as for the case of (4.12), the exact solution is

$$u(x, y, t) = \sqrt{\frac{rA_1^2 + 2a}{ab}} \tanh\left[\frac{A_1}{2}\sqrt{\frac{rA_1^2 + 2a}{ab}}(\xi + c_0)\right], \tag{4.13}$$

where c_0 is integration constant. If we take

$$D = \sqrt{\frac{rA_1^2 + 2a}{ab}},$$

then the travelling wave solution to the ZK-MEW Eq. (4.1) can be written as

$$u(x, y, t) = D \tanh\left[\frac{DA_1}{2}\left(x + y - \left(\frac{r}{b} + \frac{2a}{A_1^2 b}\right)t + c_0\right)\right]. \tag{4.15}$$

Comparing our results and Inc's results [24] with Wazwaz's results [25] then it can be seen that the results are same.

5. Conclusion

The first integral method was successfully used to establish travelling wave solutions. Many well known nonlinear wave equations were handled by this method. The performance of this method is reliable and effective and gives more solutions. This method has more advantages: it is direct and concise. The availability of computer systems like *Maple* facilitates the tedious algebraic calculations. The method which we have proposed in this letter is also a standard, direct and computerizable method, which allows us to solve complicated and tedious algebraic calculation.

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