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CLIFFORD BUNDLES OVER KAHLER-NORDEN MANIFOLDS

Şenay KARAPAZAR [1]

ABSTRACT

Kahler-Norden manifolds have been studied extensively. We constructed Kahler-Norden Spin manifold and spinor bundle S on Kahler-Norden manifold M (Karapazar, 2008, Değirmenci and Karapazar, 2010). In this work we define Clifford bundle $Cl(TM)$ of the tangent bundle TM over Kahler-Norden manifold M and give a theorem related to Clifford bundle $Cl(TM)$. Finally, by using this theorem we prove a product rule.

Keywords: Kahler-Norden manifolds, Spinor bundles, Clifford bundle.

KAHLER-NORDEN MANİFOLDLARI ÜZERİNDE CLIFFORD DEMETLERİ

ÖZ

Kahler-Norden manifoldları yaygın olarak çalışılmaktadır. M Kahler-Norden manifoldu üzerinde Kahler-Norden spin manifoldu ve S Spinor demedi inşa edilmiştir (Karapazar, 2008, Değirmenci and Karapazar, 2010). Bu çalışmada M Kahler-Norden manifoldu üzerinde TM tanjant demedinin $Cl(TM)$ Clifford demedini tanımladık ve $Cl(TM)$ Clifford demedi ile ilişkili bir teorem verdik. Son olarak da bu teoremi kullanarak çarpım kuralını ispatladık.

Anahtar Kelimeler: Kahler-Norden manifoldu, Spinor demedi, Clifford demedi.

[1], Anadolu University, Science Faculty, Department of Mathematics, 26470, Eskişehir, Turkey.

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1. INTRODUCTION

A Kahler-Norden manifold is defined by a triple (M^{2n}, J, g), where M is a smooth manifold, J an almost complex structure on M, g semi-Riemannian metric on M with anti-hermitian property $g(JX, JY) = -g(X,Y)$ for any $X, Y \in \chi(M)$ and J is parallel with respect to the Levi-Civita connection ∇^g, i.e., $\nabla^g J = 0$ [Salimov and İşcan, 2009]. The structure group of such a Kahler-Norden manifold is complex orthogonal group $O(n,\mathbb{C})$. The complex orthogonal group $O(n,\mathbb{C})$ is a subgroup of the general complex linear group $GL(n,\mathbb{C})$ and consists of all matrices A of the form $A^t A = I$. The elements of $O(n,\mathbb{C})$ preserve the symmetric bilinear form

$$b(z,w) = z_1 w_1 + z_2 w_2 + \cdots + z_n w_n$$

on $\mathbb{C}^n$ where $z = (z_1, \cdots, z_n), w = (w_1, \cdots, w_n) \in \mathbb{C}^n$. The complex orthogonal matrices with determinant 1 form a subgroup of $O(n,\mathbb{C})$ called as special complex orthogonal group, $SO(n,\mathbb{C})$. The complex quadratic form with respect to the symmetric bilinear form b is given by

$$Q(z) = b(z,z) = {z_1}^2 + {z_2}^2 + \cdots + {z_n}^2$$

for any $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$. In this paper we deal with Kahler-Norden manifolds having the structure group $SO(n,\mathbb{C})$. Hence, the manifold M can be equipped with the symmetric bilinear form b: Let $\mathcal{U}$ be an open covering of M. Then, for every point $x \in M$ there exists $U \in \mathcal{U}$ containing x and a homeomorphism $\phi_U : U \to \widetilde{U} \subset \mathbb{C}^n$. Considering $T_x M$ as a complex vector space for every $Z, W \in T_x M$, define

$$b_x(Z,W) = b((\phi_U)_{*x}(Z), (\phi_U)_{*x}(W))$$

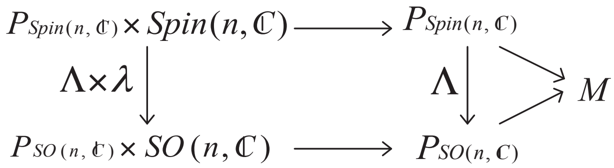
where $(\phi_U)_{*x}$ is the differential of ϕ_U at the point x. It can be seen that this definition is well-defined. The complex quadratic form $Q_x(Z) = b_x(Z,Z)$ is the quadratic form with respect to symmetric bilinear form b_x on $T_x M$.

The complex spin group $Spin(n,\mathbb{C})$ is a two-fold covering of special complex orthogonal group $SO(n,\mathbb{C})$. The representation of the complex Clifford algebra $\mathbb{C}l_n$ is well-known in literature [Lawson and Michelsohn, 1989, Friedrich, 2000]. If $n = 2k$ is even, then $\mathbb{C}l_{2k} \cong \mathbb{C}(2^k)$ and if $n = 2k+1$ is odd, then $\mathbb{C}l_{2k+1} \cong \mathbb{C}(2^k) \oplus \mathbb{C}(2^k)$. When $n = 2k$ or $n = 2k+1$ the vector space $\mathbb{C}(2^k)$ is called the vector space of complex n-spinors and denoted by Δ_n. So-called spinor representation of the complex Clifford algebra $\mathbb{C}l_n$ is denoted by κ. The restriction of κ to $Spin(n,\mathbb{C}) \subset \mathbb{C}l_n$ is a group homomorphism and $\kappa : Spin(n,\mathbb{C}) \to Aut(\Delta_n)$ is called spinor representation of $Spin(n,\mathbb{C})$.

2. COMPLEX SPIN STRUCTURE

A complex spin structure on a manifold is the existence of a $Spin(n,\mathbb{C})$ $-$principal bundle associated with a map extending the two-fold covering $\lambda : Spin(n,\mathbb{C}) \to SO(n,\mathbb{C})$ to a two-fold covering of $P_{SO(n,\mathbb{C})}$. More precisely, the following definitions have been given in (Karapazar, 2008, Değirmenci and Karapazar, 2010):

Definition 2.1 A complex spin structure on Kahler-Norden manifold (M^{2n}, J, g) is a $Spin(n,\mathbb{C})-$ principal bundle $P_{Spin(n,\mathbb{C})}$ together with a two-fold covering map $\Lambda : P_{Spin(n,\mathbb{C})} \to P_{SO(n,\mathbb{C})}$ such that the following diagram commutes:



where the rows stand for the action of the respective group on the corresponding principal bundle.

Definition 2.2 A Kahler-Norden spin manifold is a Kahler-Norden manifold (M^{2n}, J, g) whose frame bundle $P_{SO(n,\mathbb{C})}$ admits a complex spin structure.

Definition 2.3 The spinor bundle of a Kahler-Norden spin manifold is the following vector bundle associated to $P_{Spin(n,\mathbb{C})}$:

$$S = P_{Spin(n,\mathbb{C})} \times_{\kappa} \Delta_n$$

where κ is the spinor representation.

3. CLIFFORD BUNDLES

There are two equivalent ways of defining a Clifford bundle of a tangent bundle TM over a Kahler-Norden manifold M. One is the apparent generalization of $\mathbb{C}l_n$ and defined in the following way:

Each complex orthogonal transformation on $\mathbb{C}^n$ induces an orthogonal transformation on $\mathbb{C}l_n$ as $SO(n,\mathbb{C})$ induces an action on the tensor algebra and preserves the ideal. This induced map preserves the multiplication on $\mathbb{C}l_n$, so if ρ_n is the standart representation of $SO(n,\mathbb{C})$ on the complex vector space $\mathbb{C}^n$, then we get a map $cl(\rho_n): SO(n,\mathbb{C}) \to Aut(\mathbb{C}l_n)$ given by

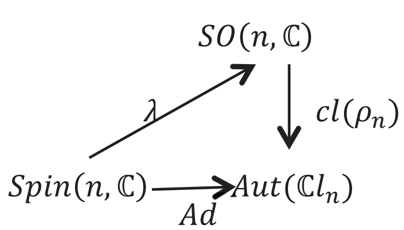
$$cl(\rho_n)(A)(v_1 \cdot v_2 \cdots v_k) = \rho_n(A)v_1 \cdot \rho_n(A)v_2 \cdots \rho_n(A)v_k$$

The map $cl(\rho_n)$ gives a representation of $SO(n,\mathbb{C})$ on complex clifford algebra $\mathbb{C}l_n$.

Definition 3.1 Let TM be a tangent bundle on Kahler-Norden manifold M. The Clifford bundle is defined as the associated bundle

$$Cl(TM) = P_{SO(n,\mathbb{C})} \times_{cl(\rho_n)} \mathbb{C}l_n$$

The adjoint representation of $Spin(n,\mathbb{C})$ on $\mathbb{C}l_n$ is given by $Ad: Spin(n,\mathbb{C}) \to Aut(\mathbb{C}l_n)$, $Ad_g(\psi) = g\psi g^{-1}$. The following diagram obtained with the help of the representation Ad commutes:



We can construct the following associated vector bundle by aid of the adjoint representation Ad:

$$P_{Spin(n,\mathbb{C})} \times_{Ad} \mathbb{C}l_n$$

There is an isomorphism between this associated vector bundle and Clifford bundle since the transition maps of these bundles are the same. So we can write Clifford bundle as associated bundle in the following way:

$$Cl(TM) = P_{Spin(n,\mathbb{C})} \times_{Ad} \mathbb{C}l_n = P_{SO(n,\mathbb{C})} \times_{cl(\rho_n)} \mathbb{C}l_n$$

Remark 3.2 As in the real case the Clifford bundle can be defined as

$$Cl(TM) = \coprod_{x \in M} \mathbb{C}l(T_x M, Q_x)$$

where $T_x M$ is an n-dimensional complex vector space, $(T_x M, Q_x)$ is quadratic space with respect to the complex quadratic form Q_x at the fiber over x and $\mathbb{C}l(T_x M, Q_x)$ is a complex Clifford algebra with respect to the quadratic space $(T_x M, Q_x)$. These two definitions of Clifford bundle are the same since the fiber over $x \in M$ of $P_{SO(n,\mathbb{C})} \times_{cl(\rho_n)} \mathbb{C}l_n$ is in $\mathbb{C}l_n$.

Because κ is defined on $\mathbb{C}l_n$, we obtain a Clifford multiplication map

$$\kappa : \mathbb{C}l_n \times \Delta_n \to \Delta_n$$
$$\kappa(x, \psi) = x \cdot \psi$$

Taking into account the Clifford bundle $Cl(TM)$ the Clifford multiplication carries over to a Clifford multiplication on Clifford bundle $Cl(TM)$ and the spinor bundle S. Then, the following theorem states that we can multiply the element of Clifford bundle with the element of the spinor bundle S.

Teorem 3.3 Let S be Kahler-Norden spinor bundle. Then, S is a bundle of modules over the bundle of algebras $Cl(TM)$.

Proof. The following diagram commutes:

$$\begin{array}{ccc} P_{Spin(n,\mathbb{C})} \times \mathbb{C}l_n \times \Delta_n & \xrightarrow{\kappa} & P_{Spin(n,\mathbb{C})} \times \Delta_n \\ \rho_g \downarrow & & \downarrow \rho_g \\ P_{Spin(n,\mathbb{C})} \times \mathbb{C}l_n \times \Delta_n & \xrightarrow[\kappa]{} & P_{Spin(n,\mathbb{C})} \times \Delta_n \end{array}$$

Explicitly, it can be written as follows:

$$\begin{array}{ccc} (p, \psi, m) & \xrightarrow{\kappa} & (p, \psi m) \\ \rho_g \downarrow & & \downarrow \rho_g \\ (pg^{-1}, g\psi g^{-1}, gm) & \xrightarrow[\kappa]{} & (pg^{-1}, g\psi m) \end{array}$$

Therefore, κ turn into a mapping

$$\mu : Cl(TM) \times S \longrightarrow S$$
$$([p,\psi],[p,m]) \longmapsto [p, \psi m]$$

It can be easily seen to have desired properties.

By using above theorem now we prove a useful property of spinor derivative, i.e., a product rule.

Proposition 3.4 Let X and Y be vector fields and ψ be a spinor. Then, the following relation holds:

$$\nabla_X(Y \cdot \psi) = \nabla_X Y \cdot \psi + Y \cdot \nabla_X \psi$$

Proof. Let $s_\alpha : U_\alpha \to S$ be a local frame. The vector field Y can be expressed as $Y = [\bar{s}_\alpha, Y_{s_\alpha}]$ where $Y_{s_\alpha} : U_\alpha \to \mathbb{R}^{2n}$ is a smooth map and $\bar{s}_\alpha : U_\alpha \to P_{Spin(n,\mathbb{C})}$ is a local frame. With this,

$$\nabla_X Y = [\bar{s}_\alpha, X(Y_{s_\alpha}) + {s_\alpha}^* \tilde{\omega}(X) Y_{s_\alpha}]$$

and

$$Y \cdot \psi = [\bar{s}_\alpha, Y_{s_\alpha} \cdot \psi_{s_\alpha}]$$

where $\tilde{\omega}$ is a $Spin(n,\mathbb{C})$ −valued connection 1-form on $P_{Spin(n,\mathbb{C})}$, ψ_{s_α} is a spinor on U_α and $\psi = [\bar{s}_\alpha, \psi_{s_\alpha}]$. Hence, we get

$$Y \cdot \nabla_X \psi = [\bar{s}_\alpha, Y_{s_\alpha} \cdot X(Y_{s_\alpha}) + Y_{s_\alpha} \cdot {\bar{s}_\alpha}^* \tilde{\omega}(X) \cdot \psi_{s_\alpha}]$$

$$\nabla_X(Y \cdot \psi) = [\bar{s}_\alpha, X(Y_{s_\alpha} \cdot \psi_{s_\alpha}) + {\bar{s}_\alpha}^* \tilde{\omega}(X) \cdot Y_{s_\alpha} \cdot \psi_{s_\alpha}]$$

Moreover, it holds the following identity from [Karapazar,2008]

$$\kappa(e) \circ \kappa(v) - \kappa(v) \circ \kappa(e) = \kappa(\lambda_*(e)v)$$

for any $e \in \mathfrak{spin}(n,\mathbb{C})$ and $v \in \mathbb{C}^n$. In this identity we can write as $e = {\bar{s}_\alpha}^* \tilde{\omega}(X)$ and $v = Y_{s_\alpha}$. Then, we get

$${\bar{s}_\alpha}^* \tilde{\omega}(X) \cdot Y_{s_\alpha} \cdot \psi_{s_\alpha} = {s_\alpha}^* \omega(X) Y_{s_\alpha} \cdot \psi_{s_\alpha} + Y_{s_\alpha} \cdot {\bar{s}_\alpha}^* \tilde{\omega}(X) \cdot \psi_{s_\alpha}$$

Altogether, this yields the following:

$$\begin{aligned} \nabla_X(Y \cdot \psi) &= [\bar{s}_\alpha, X(Y_{s_\alpha} \cdot \psi_{s_\alpha}) + {\bar{s}_\alpha}^* \tilde{\omega}(X) \cdot Y_{s_\alpha} \cdot \psi_{s_\alpha}] \\ &= [\bar{s}_\alpha, X(Y_{s_\alpha}) \cdot \psi_{s_\alpha} + Y_{s_\alpha} \cdot X(\psi_{s_\alpha}) + ({s_\alpha}^* \tilde{\omega}(X) Y_{s_\alpha}) \cdot \psi_{s_\alpha} + Y_{s_\alpha} \cdot {\bar{s}_\alpha}^* \tilde{\omega}(X) \\ &\qquad \cdot \psi_{s_\alpha}] \\ &= [\bar{s}_\alpha, X(Y_{s_\alpha}) \cdot \psi_{s_\alpha} + {s_\alpha}^* \tilde{\omega}(X) Y_{s_\alpha} \cdot \psi_{s_\alpha}] + [\bar{s}_\alpha, Y_{s_\alpha} \cdot X(\psi_{s_\alpha}) + Y_{s_\alpha} \cdot \\ &\quad {\bar{s}_\alpha}^* \tilde{\omega}(X) \cdot \psi_{s_\alpha}] \\ &= \nabla_X Y \cdot \psi + Y \cdot \nabla_X \psi \end{aligned}$$

4. CONCLUSION

$Spin(n,\mathbb{C})$ is a two-fold covering of $SO(n,\mathbb{C})$. With the aid of this two-fold covering we constructed spinors on a certain class of Kahler-Norden manifolds (Karapazar (2008), Değirmenci and Karapazar, 2010). We defined the spinor bundle on these manifolds. In this work we define the Clifford bundle in two ways and give a theorem related to Clifford bundle $Cl(TM)$. By means of this theorem the Clifford multiplication κ lead to the Clifford multiplication μ between Clifford bundle

$Cl(TM)$ and the spinor bundle S. By using the Clifford multiplication μ it is given a multiplication rule of a spinor covariant derivative.

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