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Prospective middle school mathematics teachers' preconceptions of geometric translations

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Abstract This article reports an analysis of 44 prospective middle school mathematics teachers' pre-existing knowledge of rigid geometric transformations, specifically the geometric translations. The main data source for this study was the participants' responses to the tasks that were presented during semi-structured clinical interviews. The findings of the study revealed that prospective teachers had difficulties recognizing, describing, executing, and representing geometric translations. The results indicated that teacher candidates held various views about the geometric translations: (1) translation as rotational motion, (2) translation as translational motion, and (3) translation as mapping. The results further revealed various interpretations of the vector that defines translations: (1) vector as a force, (2) vector as a line of symmetry, (3) vector as a direction indicator, and (4) vector as a displacement. Although many of the teacher candidates interviewed knew that a vector has a magnitude and a direction, this knowledge did not generally lead them to conclude that vectors define translations.

Keywords Preconceptions · Geometric translations · Prospective middle school mathematics teachers

1 Introduction

Geometry is an important part of the elementary and secondary school mathematics curriculum. "The study of geometry contributes to helping students develop the skills of visualization, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argument and proof" (Jones, 2002, p.125). Transformational geometry is one of the areas of geometry that has a potential to contribute to the development of students' reasoning and justification skills. Studying geometric transformations provides opportunities for learners to describe patterns, discover basic features of isometries, make generalizations, and develop spatial competencies (Clements, Battista,

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Sarama & Swaminathan, 1997; Portnoy, Grundmeier & Graham, 2006). "Mathematical properties from the various branches of geometry (topology, projective geometry, affine geometry, Euclidean geometry) can be described in terms of transformations which may be represented through several types of manipulative activities" (Williford, 1972, p. 260). According to NCTM's *Principles and Standards for School Mathematics* (2000), "Instructional programs from pre-kindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyze mathematical situations" (p. 41). Therefore, it is crucial for teachers to have good grasp of transformations in order to create learning environments which promote the development of robust mathematical ideas.

The present study examines the pre-existing knowledge of geometric translations that prospective middle school mathematics teachers possess before they are formally introduced to this topic. Prior knowledge has been defined in various ways (Dochy, 1994; Jonassen & Grabowski, 1993; Simons, 1999). According to Simons (1999), "prior knowledge can be described as all the knowledge learners have available when entering a learning environment, that is potentially relevant for learning new knowledge" (p. 579). Prior knowledge can be formal or informal, implicit or explicit, and it may be partially correct or wrong as compared with scientific standards (Dochy, 1994; Simons, 1999). It may consist of "an individual's personal stock of information, skills, experiences, beliefs and memories" (Alexander, Shallert & Hare, 1991, p. 317) and can also refer to intuitive knowledge or preconceptions, which include ideas held before formal instruction (Aguirre, 1988; Clement, Brown & Zietsman, 1989; Dochy, Segers & Buehl, 1999). Intuitive knowledge is "composed of a learner's intuitions, ideas and beliefs about mathematical entities" (Sirotic & Zaskis, 2007, p. 51). In general, prior knowledge can be considered as a combination of knowledge, skills, and beliefs about how mathematics works.

Several studies have emphasized the importance of formal and informal prior knowledge (Aguirre, 1988; Dochy, 1992; Howe & Jones, 1998; Sherin, 2006; Simons, 1999). Prior knowledge can be resistant to change (Fischbein, 1987; Simons, 1999) and may hinder the learning of new ideas (Aguirre, 1988; Dochy et al., 1999; Howe & Jones, 1998). According to Roschelle (1995), "learning proceeds primarily from prior knowledge, and only secondarily from the presented materials" (p. 37). Therefore, it is crucial to know what preconceptions learners have before proceeding to formal concepts.

Moreover, it is essential to investigate prospective teachers' preconceptions since they shape future instructional practices and student learning of geometric transformations. Preconceptions not only shape how teachers reason about a domain, but also shape their pedagogical reasoning (Ball, 1988). According to Feiman-Nemser and Remillard (1996), many teachers bring their beliefs from past school experiences into classrooms, which influences their methods of teaching. Therefore, it is important to consider prospective teachers not only as learners, but also as future educators and understand their learning of new ideas in teacher education programs.

Despite the importance of prior knowledge as emphasized in the literature, only a few studies (e.g., Ada & Kurtulus, 2010; Desmond, 1997; Harper, 2003; Jung, 2002; Portnoy et al., 2006; Thaqi, Gimenez & Rosich, 2011; Yanik & Flores, 2009) involved teacher candidates' understanding of geometric transformations. These studies generally focused on teacher candidates' difficulties with geometric transformations. More research is needed, not only to determine prospective teachers' difficulties with geometric transformations, but also to explore their nature of pre-existing knowledge and pre-service teachers' conceptualizations of transformations.

This study was a part of a larger study that examined the development of prospective middle school mathematics teachers' understandings of rigid geometric transformations (i.e., translations, reflections, and rotations) in the context of a technology-mediated environment. This article focuses on the first phase of the larger study and attempts to describe prospective teachers' prior knowledge of geometric translations in a non-technological environment before they are formally introduced to this topic. For the purpose of this study, in particular, the following research questions were investigated:

- What prior knowledge of geometric translations do prospective middle school mathematics teachers hold?
- What are the underlying characteristics of their preconceptions?
- What are the possible sources of their naïve ideas about geometric translations?

2 Theoretical framework

Preconceptions play an important role in learners' acquisitions of mathematical concepts. Learning can be supported by "activating prior knowledge from an earlier class or with a familiar context for new material to provide a linkage to a learner's previous experiences. Conversely, learning can be impeded by misconceptions that originate from personal experience, previous classes, or misapplication of prior knowledge to new content" (Krause, Kelly, Corkins & Tasooji, 2009, p. 2). In order to develop successful learning environments and instructional strategies, one needs to know not only the learners' preconceptions but also to understand the nature of their naïve knowledge structures (Özdemir, 2006). However, determining learners' intuitive structure of knowledge has been a controversial issue among researchers. Two fundamental theoretical perspectives regarding knowledge structures have been discussed among researchers: (1) knowledge-as-theory perspective (e.g., Carey, 1999; Chi, 2005; Ioannides & Vosniadou, 2002; Wellman & Gelman, 1992) and (2) knowledge-as-elements perspective (e.g., Harrison, Grayson & Treagust, 1999; Linn, Eylon & Davis, 2004; Özdemir & Clark, 2009). While the former perspective proposes that student ideas are coherent, the later perspective asserts that learners' intuitive structure of knowledge may consist of many quasi-independent elements (diSessa, 1993; diSessa, Gillespie & Esterly, 2004).

The study described here was guided by diSessa's (1993) epistemological framework for the learning of concepts regarding science and mathematics. From diSessa's perspective, the acquisition of concepts of science and mathematics "depends on the organization and systematization of prior knowledge that is often highly sensitive to context" (Wagner, 2003, p. 364). diSessa (1993) proposes that learners have partial and fragmented understanding of concepts that he calls "knowledge in pieces". In diDessa's framework, novices' knowledge structures are poorly integrated, contextually sensitive, and lacking the consistency of expert knowledge (Wagner, 2006). According to diSessa (1993), learners try to make sense out of the physical world by constructing pieces of knowledge that are based on their direct experiences. diSessa calls these pieces of knowledge *phenomenological primitives*, or *p-prims* which are fairly small knowledge structures and play significant roles in explaining physical phenomena. "P-prims are developed through a sense-of-mechanism that reflects our interactions with the physical world such as pushing, pulling, throwing, and holding" (Özdemir & Clark, 2007, p. 355).

A common example of a p-prim provided by diSessa (1993) is called "Ohm's p-prim". According to this p-prim, some agent produces an energizing force or impetus in order to overcome some resistance to achieve a result such as motion or some form of action (diSessa, 1993, 1996). This p-prim is "an abstraction from experiences involving an agency, an effect, and an impediment" (Hammer, 1996, p. 104). Another p-prim identified by diSessa is called "force as a mover." "Force as mover is directed (vector) form of Ohm's p-prim that prescribes that the motion of an object follows the directed effort making it move" (diSessa, 1996, p. 716). In this p-prim, pushing an object would cause it to move in the direction of the force. Although this is "not technically correct (force produces *acceleration* in the direction of the force—not necessarily motion—so an object already moving may not travel in the direction of the force)" (Nordine, 2007, p. 16), according to diSessa (1993), such an experience may lead naïve individuals to make overgeneralizations and conclude that force causes motion. Another p-prim in diSessa's framework includes *dying away*. This p-prim is generally used when someone wants to explain why moving objects slow down and stop.

A learner holds a rich and hierarchical collection of p-prims and people activate particular p-prims when they reason about a particular phenomena (diSessa, 2000). diSessa (2000) asserts that these p-prims, which include the intuitive knowledge of the world, shape people's expectations regarding events they observe and provide explanations for them. Therefore, one needs to examine not only how teacher candidates think about specific concepts like geometric transformations, but also how prior experiences (e.g., personal experiences and previous instruction) influence their conceptions. The purpose of this study is also to explore the sources of teacher candidates' pre-instructional beliefs and reasoning strategies about geometric translations.

3 Understanding of geometric transformations

This study focused on rigid geometric transformations which preserve the relative distances and angles of all points in the plane (Yanik & Flores, 2009). One example of such transformations is called translation which can be interpreted in two distinct ways: motion and mapping (Edwards, 2003; Hollebrands, 2003; Rosenbloom, 1969; Yaglom, 1962; Yanik & Flores, 2009). In the "motion" understanding of translation, all points in the plane can be carried onto the other points on the plane based on a specific distance and direction. In this understanding one may consider the plane as a background and manipulate geometric figures on top of the plane (Edwards, 2003).

In the "mapping" understanding, on the other hand, motion is not an essential part of translation. In this conception, translation can be considered as a special function that maps all points in the plane to other points at a certain distance and direction (Hollebrands, 2003). The plane consists of an infinite number of points and geometric figures are considered as sets of points which are a subset of the plane rather than as separate entities sitting on the plane (e.g., Edwards, 2003; Hollebrands, 2003). In this interpretation of geometric translations (or transformations), all points in the plane are considered as the domain of the function (Hollebrands, 2003). Therefore, one needs to apply translations to all points in the plane rather than a single object. Edwards (2003) considers the mapping conception of geometric transformations as representative of contemporary mathematicians' conceptualization of transformation geometry.

3.1 Student ideas about geometric transformations

Studies (e.g., Edwards, 2003; Glass, 2001; Hollebrands, 2003) revealed that elementary and high school students held a predominantly motion conception of transformations. Edwards (2003) found that when learning initial concepts within the domain of transformation geometry, students of various ages (e.g., middle school students, high school students, and college students) "had the same initial expectations of how transformations would work, and they made the same kinds of errors" (p. 4). For instance, in her several studies (e.g., Edwards, 1991, 1992, 1997; Edwards & Zazkis, 1993) Edwards revealed that students of different ages conceived transformations as motion. According to Edwards (2003), students who hold motion understanding of translations consider the plane as a background and manipulate geometric figures on top of the plane. Edwards further discusses that learners' general understandings of transformations arise from their embodied experience in the physical world. She said: "the embodied, natural understanding of motion that the learners brought to the experience is the source of their misconceptions. These 'misconceptions' are in actuality, conceptions that are adaptive and functional outside of the context of formal mathematics" (p. 9).

Hollebrands (2003) studied high school students' understandings of geometric transformations (e.g., translations, rotations, reflections, and dilations) in a technological environment in which the *Geometer's Sketchpad* (Jackiw, 2001) and the TI-92 calculator were used. Hollebrands proposed that in order to have a mapping understanding of transformations, one needs to understand four fundamental concepts: (1) the domain of transformations as all points in the plane, (2) parameters (e.g., vector, reflection line, center of rotation, and angle of rotation) that define transformations, (3) the relations and properties of transformations, and (4) transformations as being one-to-one mappings of points in the plane onto points in the plane. She found that the development of these understandings by students was important in their thinking about transformations as mappings rather than thinking about transformations as motions.

Glass (2001) worked with five eighth-grade students and explored their understandings of transformations (e.g., translations, reflections, and rotations) using the *Geometer's Sketchpad* (Jackiw, 1995). Analysis revealed that students used motion-, end result- and property-based reasons in identifying the transformation types being represented. The first category *motion* is based on the movement of the image from the location of the pre-image. According to Glass, students using this category showed an operational understanding of transformations. The second category *end result of motion* focused on comparisons between the pre-image and image characteristics that are direct results of the image's movement. Glass stated that the use of end result-based reasons demonstrates a slightly more structural understanding of transformation than reasons from the first category, "because the emphasis has shifted slightly from the movement of the image to characteristics of the image in comparison to the pre-image" (p. 173). The third category was reasoning related to properties of transformations. According to Glass, students most often used the first category *motion* for transformation types followed by the second and third category.

3.2 Prospective teachers' understanding of transformations

As students move through school years, instruction also emphasizes more abstract thinking about transformations such as seeing transformations as mappings of the plane onto itself. However, studies (e.g., Harper, 2003; Jung, 2002; Portnoy et al., 2006; Yanik & Flores, 2009) indicate that students' thinking does not change significantly over the elementary and secondary years. Past research revealed that teacher candidates' motion conception of transformations persists even after taking college level courses. For instance, Yanik (2006) investigated four teacher candidates' knowledge of rigid geometric transformations (i.e., translations, reflections, and rotations). Two subjects who were majoring in elementary education were in their last year of an undergraduate program. The other two were enrolled in a master's program in mathematics education. The study revealed that all the prospective teachers had incomplete understanding of geometric transformations. Specifically, teacher candidates had difficulty describing and executing transformations. They conceived translations as undefined motion and geometric figures as separate from the plane. Results of the study showed that everyday experiences (e.g., walking and moving) influenced prospective teachers' reasoning about transformations.

In her study of pre-service elementary teachers, Desmond (1997) found that prospective teachers had difficulty identifying correct translations and the translation vectors. For instance, only 17% of the 83 prospective elementary teachers studied could match the translation with the correct distance and direction. Results indicated that motion was the main conceptual model of translations held by the participants in her study.

Jung (2002) investigated two prospective secondary school mathematics teachers' understanding of geometric transformations (e.g., translation, rotation, reflection, and glide reflection) in a technological environment. The results indicated that initially prospective teachers had difficulty using mathematical symbols to express geometric transformations. Their responses depended upon and were guided by the pictorial representations (e.g., pictures), and they had a motion understanding of transformations.

Harper (2003) investigated four prospective elementary teachers' understanding of geometric transformations (i.e., translations, reflections, and rotations). Results of the study showed that teacher candidates had difficulty identifying and executing translations correctly. She found that pre-service teachers had difficulty using vectors to translate figures as well as determining the translation vector for a given pre-image and its corresponding image points. Harper further found that teacher candidates used informal vocabulary for using translations (e.g., slide and move). All participants had motion conceptions of translations.

Portnoy et al. (2006) explored 19 prospective middle and high school teachers' views of geometric transformations. Participants of the study conceived transformations primarily as procedures that can be applied to geometric objects (e.g., sliding, flipping, and rotating). Translation was considered as sliding a figure on the plane or "taking something and putting it in another place in the plane" (p. 201).

Ada and Kurtulus (2010) studied 126 third-year prospective elementary teachers' misconceptions in transformation geometry. The results of the study showed that although the teacher candidates seemed to know the algebraic representation of geometric translations and rotations, "they did not seem to understand the geometric meaning of them" (p. 901). For instance, only 16% of the 126 teacher candidates could explain the geometric meaning of translation and only 10% of the teacher candidates provided a correct explanation for the meaning of rotation.

Thaqi et al. (2011) conducted research to reveal 28 prospective teachers' attributed meanings for geometric transformations in Kossovo and Spain. The findings of the study revealed that prospective teachers in both countries had incomplete understanding of geometric transformations. In general, the teacher candidates conceived transformations as motion and their comments about transformations were based on their intuitions.

Many of the prior studies have focused on the identification of teacher candidates' difficulties with transformation geometry. Little is known, however, about what pre-existing knowledge of geometric transformations prospective teachers hold and the underlying characteristics of their preconceptions. This study attempted to provide insights into prospective teachers' prior knowledge of geometric transformations and possible sources of naïve ideas.

Past research underlines the need for "teacher educators to learn more about pre-service teachers' experiences, beliefs, and views that influence what they learn" (McCall, 1995, p. 340). Investigating prospective teachers' pre-existing knowledge of geometric transformations is crucial since it provides opportunity to improve the quality of mathematics teaching and learning. Knowing about these preconceptions can guide teacher educators in making pedagogical decisions such as how to introduce new ideas, what learning experiences to utilize, and in what order. According to Tobin (1993), choosing appropriate learning experiences is critical to address learners' preconceptions. If pre-service teachers' preconceptions are not addressed properly during the teacher education programs, they may keep these conceptions and transfer them to students when they actually start their profession.

Exploring the sources of prospective teachers' preconceptions provides opportunity to understand to what extent their preconceptions might be the result of knowledge that they formed in previous classes or personal experiences. These sources of preconceptions may provide ideas for curriculum changes and designing learning environments that enhance the understanding of transformational geometry.

4 Methodology

4.1 Setting and participants

This study was conducted in a teacher education program at a large urban public university in the central part of Turkey. All participants were chosen from the mathematics teacher education program. This is a 4-year bachelor's degree program in the Department of Elementary Education in the Faculty of Education. In the Mathematics Teacher Education program, prospective teachers are trained specifically to become middle school mathematics teachers who will be teaching pupils in the age range of 12 to 14 years old. In order to graduate from the program, teacher candidates need to complete at least 240 ECTS credits (approximately 144 US college credits hours) with 88 credits in undergraduate Mathematics, 23 credits in Mathematics Education, and 129 credits across Educational Psychology, Instructional Technology, Research and Measurement, Practical Training, and Social Foundations. Students take at least 30 credits per semester, and one school year consists of Fall and Spring semesters. One ECTS requires approximately 25 to 30 student learning hours which include all activities needed to accomplish the intended course learning outcomes.

A total of 44 second-year prospective middle school mathematics teachers (30 females and 14 males) in the age range of 19–22 voluntarily participated in the study. All participants had taken three elementary Mathematics courses and one introductory Geometry course at the college level before the study began. Transformational geometry is part of an Analytic Geometry course which is introduced to the prospective teachers in the third year of their teacher education program. Although the geometric transformations have been part of the college curriculum for a while, it has been included in the elementary school mathematics curriculum for just a few years.

Geometric translations are introduced to students in 6th grade as motions of geometric shapes. Sample activities require students to construct geometric figures on dot papers based on a given direction and magnitude (see Fig. 1). Although the vector concept is not formally introduced to students during geometric translations, "directed line segments" are used to specify the direction and the magnitude of the translation (see Fig. 1).Some characteristics of geometric translations (e.g., translations preserve the shape and size of a figure) are mentioned at this grade level. Starting with 7th grade, the coordinate system is introduced and students focus on the locations of geometric shapes on the plane. In 8th grade, students are introduced to compositions of transformations (e.g., translations, reflections, and rotations).

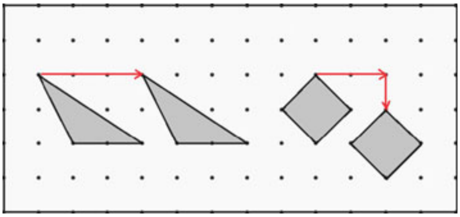


Fig. 1 Sample activities in 6th grade curriculum

The majority of participants stated that they did not remember whether or not they encountered geometric transformations in their school years though it did not seem totally new to them. It was assumed that teacher candidates still might have some preconceptions regarding geometric transformations based on their common sense, real-life experiences, and what they had been taught in other settings.

4.2 Data analysis

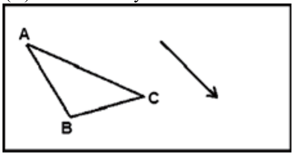
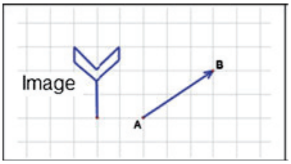
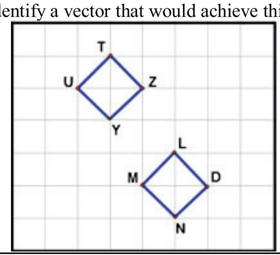
A qualitative design was chosen for this study to describe and analyze prospective teachers' understanding of geometric translations. According to Miles and Huberman (1994), qualitative designs are suitable for exploring rich descriptions and explanations of complex phenomena. Data analysis included four components of Miles and Huberman's (1994) interactive model: data collection, data reduction, data display, and conclusion drawing and verification.

4.2.1 Data collection

The main data source for this study was the participants' responses to the tasks that were presented during clinical interviews. At the beginning of the study, the researcher conducted a one-on-one semi-structured task-based clinical interview with each of the participants to assess their initial understanding of geometric translations. Semi-structured task-based clinical interviews allowed the researcher to discover prospective teachers' conceptions and ideas about translations. The interview length varied from 50 min to 1.5 h. The duration of the interview was adapted for each participant to explore emerging issues in depth. The researcher tried to obtain as much information as possible about participants' reasoning and justifications for the tasks to gain deeper insights about their understanding of geometric translations.

For the purpose of the study, a total of 13 tasks (see Table 1) were designed to probe prospective teachers' initial understandings of geometric translations. The researcher used various sources to design the interview tasks (e.g., Edwards, 2003; Flanagan, 2001; Molina, 1990; Yanik & Flores, 2009). These sources helped the researcher specify the types of tasks (e.g., description tasks and performance tasks), determine the order of interview question, and interpret the participant' responses (e.g., motion versus mapping conception of transformations). While some interview tasks were adopted and adapted from the previous research (e.g., Flanagan, 2001; Molina, 1990), others were prepared by the researcher in light of previous studies (e.g., Edwards, 2003; Flanagan, 2001; Yanik & Flores, 2009).The interview questions (see Table 1) asked the pre-service teachers to perform the following actions: provide descriptions for geometric translations (type 1, adopted from Flanagan, 2001, p. 390), recognize translations (type 2, adapted from Molina, 1990), perform translations using vectors (type 3, adapted from Flanagan, 2001), identify vectors for given translations (type 4), describe the role of vector in translations (type 5), represent translations (type 6, adapted from Flanagan, 2001), describe the plane and the

Table 1 Interview tasks

Task Type	The task is about	Relevant tasks
Type 1	Describing translations • Providing examples and non-examples of translations • Defining translations	1) Can you provide me an example of a translation? a) Why do you think this would be a translation? 2) Can you provide me a non-example of a translation? a. Why do you think this would be a non-example of a translation? 3) How would you define a geometric translation?
Type 2	Recognizing translations	4) Can this be a translation? (Consider student responses for tasks 1-3) a. (If the student considered translation as rotational motion ask him/her whether translational motion would be an example of translation) b. (If the student considered translational motion as translation, ask him/her whether rotational motion would be an example of translation)
Type 3	Performing translations using a vector	5) Suppose you are given the following triangle to translate using the vector (V). How would you execute the translation?  a. Could you explain how you determined where to place the image triangle? b. What can you say about pre-image and image triangles? 6)  Suppose you are shown the image above after it has been translated by vector AB. a. Can you draw the pre-image and show its location? b. Could you explain how you determined where to place the pre-image?
Type 4	Identifying the vector	7) Can you identify a vector that would achieve this translation? 
Type 5	Describing the role of vector in translations	8) What do you think about the role of vector in translation?
Type 6	Representing translations	9) Can you represent a translation with a function notation?
Type 7	Describing the plane and the relationships between geometric shapes and the plane	10) Can you describe the plane? 11) Can you describe the relationship between plane and the geometric figures?
Type 8	Understanding the domain for the translations	12) Suppose you are asked to execute a translation. a. Would you consider all points in the plane? 13) Would you consider only one point or figure? Why? Or Why not?

relationships between geometric shapes and the plane (type 7), and explain the domain for geometric transformations (type 8). The tasks that were adopted or adapted from these resources were appropriate for clinical interviews to help the researcher make in-depth analysis of prospective teachers' understandings of transformational geometry.

The researcher presented each type of task on a separate sheet of paper and provided paper, pencil, protractor, compass, colored pencils, ruler, scratch paper and hands-on manipulatives in case participants might want to use them. Each participant was asked to read the question and think aloud to share his/her reasoning. During the clinical interviews, the researcher had no intention of advancing the participants' understanding of geometric translations and vector concepts. All interviews were audio- and video-taped and later transcribed for data analysis purposes.

4.2.2 Data reduction

This phase began before the data were collected and continued throughout the study. The researcher began the data analysis with certain predetermined categories based on the past research (e.g., Edwards, 2003; Glass, 2001; Hollebrands, 2003; Yanik & Flores, 2009). Two strategies were used to organize the data: organizing data by the types of tasks (see Table 1) and organizing data by emerging themes (see Table 2). Initially, the types of tasks were categorized and then the answers given to these questions were coded based on the initial framework of the study which included the motion and mapping understandings of geometric translations (e.g., Edwards, 2003; Hollebrands, 2003; Rosenbloom, 1969; Yaglom, 1962; Yanik & Flores, 2009). The researcher prepared initial codes based on previous research (e.g., Edwards, 2003; Glass, 2001; Hollebrands, 2003; Yanik & Flores, 2009). Some of the initial codes were refined as the clinical interviews proceeded. Additional codes were also added if the initial codes did not fit the responses given by the participants. The main purposes of coding and analysis were to find patterns and create descriptions of each participant's prior knowledge of geometric translations.

Once each participant's ways of thinking and pre-existing knowledge of geometric translations were identified, the researcher then looked for the similarities and differences among the participants. For the purpose of phased analysis of the interviews, data were read repeatedly and examined to see whether there were emerging themes among the participants.

4.2.3 Data display

Data matrices were used to organize the data generated from the clinical interviews. In this way, the researcher was able to summarize the data and see how the participants were thinking about the various aspects of geometric translations. For instance, initially all participants' preconceptions of geometric translations and vector concepts were examined individually and three major conceptions of translations and four different interpretation of vectors were identified. The researcher then created a new data matrix to show the relationships among several preconceptions (see Table 3). These displays helped the researcher move on to the next step of analysis.

4.2.4 Conclusion drawing and verification

The last phase of the data analysis was aimed at finding patterns, regularities and possible explanations for the participants' pre-existing knowledge of geometric translations. The conclusions were drawn from the data collected throughout the study. To

Table 2 Codes generated from the data

Codes for prospective teachers' preconceptions for translations and vectors	
Codes for preconceptions for translations	Sample participant responses
Translation as rotational motion (this code is used when the participants considered the translation as rotational motion)	"Translation occurs when an object moves while rotating". "An example of a translation would be a wheel rolling on a flat surface". "A disc rolling down on an inclined surface"
Translation as translational motion (participants' responses were coded as "translation as translational motion" if they considered the translation as motion, which included translational motion only)	"Like moving an object from one place to another place without rotating it". "Sliding an object (in any direction)". "Moving an object without rotating it". "Translational motion"
Translation as mapping (This code is used when the participant considered the translation as mapping of a geometric figure rather than as motion)	"Translation is like a mapping of one point (or a geometric figure) to another point (or a geometric figure) on the plane. Translation does not involve motion"
Codes for preconceptions for vectors	Sample participant responses
Vector as a force (this code is used when the participants consider the vector as a force that applies on a figure)	"It is a force to move an object". "You use the vector in order to move an object from one point to another". "You use the vector to push or pull the figure"
Vector as a symmetry line (this code is used when the participants consider the role of vector as a symmetry line in geometric translations)	"I took the vector as a symmetry line". "The image triangle needs to be equal distance from the vector. I mean the distances between pre-image triangle and the vector, and the distances between image triangle and the vector should be the same". "You locate the vector right in the middle of pre-image and image figures, like symmetry line"
Vector as a direction indicator (this code is used when the participants consider that the vector determines the direction of the translation)	"The vector shows in which direction the figure will move". "The figure will be relocated somewhere on the plane in the direction of the vector". "You use the vector to determine which way the figure would go"
Vector as a displacement (this code is used when the participants consider the characteristics of the vector (i.e., length and the direction) to perform a translation)	"The length and the direction of the vector specify how the translation would occur". "The vector specifies where I exactly locate the figure". "I used the vector's length and direction when translating the figure"

ensure the validity of the results, three external colleagues also independently critiqued the researcher's decisions. These three reviewers provided the researcher detailed feedback on the participants' understanding of translations and vector concepts which assisted the researcher in making more accurate coding decisions and categorization of participants' responses (see Table 2).

According to Miles and Huberman (1994), all these stages (data collection, data reduction, data display, and conclusion drawing and verification) are not separate from analysis; they are part of data analysis. Therefore, the data collected through clinical interviews were analyzed through ongoing and retrospective analyses. Ongoing analysis took place during clinical interviews and a retrospective analysis focused on the interviews conducted with all participants. The purpose of the ongoing analysis was to understand each participant's ways of thinking. During each clinical interview, the researcher probed

Table 3 Prospective teachers' preconceptions about translations and vectors

	Vector as a force	Vector as a symmetry line	Vector as a direction indicator	Vector as a displacement	No idea	Total (*f*, %)
Translation as rotational motion	5	7	3	2	3	20 (46%)
Translation as translational motion	1	9	1	9	2	22 (50%)
Translation as mapping				1		1 (2%)
No Idea		1				1 (2%)
Total (*f*, %)	6 (14%)	17 (39%)	4 (9%)	12 (27%)	5 (11%)	N=44

with several questions to make the participant's thinking explicit and some of the questions were reused at different points in the interview. With this strategy, the researcher was able to record additional information regarding the tasks that were used in the interview. Retrospective analyses aimed at providing explanations for all participants' pre-existing knowledge of geometric translations.

5 Analyses

This section presents an interpretation of excerpts of the clinical interviews dedicated to presenting major conceptions of prospective teachers' initial understanding of rigid geometric translations and vector concepts (see Table 4). First, the prospective teachers' understandings of translations are examined. Next, their conceptions of vector are presented. Some possible interpretations of teacher candidates' understanding of translations and vector concepts are also mentioned. Lastly, overall findings are discussed.

Clinical interviews revealed that the majority of participants had difficulty describing translations. Almost all of the prospective teachers stated that they did not remember encountering translations during their own school mathematics experiences including undergraduate studies. However, they added that they could still explain geometric translations using what they learned from an introductory physics course. Consequently, prospective teachers' explanations appeared to rely mainly on the examples chosen from physics and the physical world.

5.1 Prospective teachers' reasoning categories regarding geometric translations

The analyses of the clinical interviews revealed that with the exception of two participants, all the prospective teachers considered the motion itself as translation. In general, prospective teachers exhibited two different motion conceptions of translations: translation as rotational motion and translation as translational motion (see Tables 2 and 4).

Table 4 Teacher candidates' preconceptions of geometric translations

Motion conceptions of translations		Mapping conception of translations
Translation as rotational motion	Translation as translational motion	
Translation achieved through rotational motion of a single geometric figure on top of the plane	Translation achieved through translational motion of a single geometric figure on top of the plane	Translation achieved through mapping of a single geometric figure (motion is not an aspect of translation)
Translation preserves the shape and size of a figure	Translation preserves the shape and size of a figure	Translation preserves the shape and size of a figure
The plane is a flat background	The plane is a flat background	Plane consists of an infinite number of discrete points
Geometric figures and the plane are independent from each other	Geometric figures and the plane are independent from each other	Geometric figures consist of infinite number of points and they are part of the plane
Translation applies only to a single geometric object	Translation applies only to a single geometric object	Translation applies only to a single geometric object
External force is an essential aspect of translation	External force is an essential aspect of translation	External force is not an essential aspect of translation

5.1.1 Translation as rotational motion

During the clinical interviews, for type-1 tasks which required prospective teachers to describe translations (i.e., Can you provide an example and a nonexample of a translation?; How would you define a geometric translation?), 20 out of 44 participants considered translations as rotational motion of a single object (see Table 3). The participants stated that the word "translation" reminded them of examples they had encountered in physics course. A typical example provided by these participants was a disc or a wheel rolling down an inclined surface or rolling along a horizontal surface (see Fig. 2). Aside from these examples, eight out of 20 prospective teachers provided noncircular rolling shapes as examples of translations (see Fig. 3).

Participants who held this conception considered that translation would be achieved through rotational motion of a single geometric figure on top of the plane. These participants indicated that the rolling was a key aspect of the translation in addition to displacement. Almost half of them stated that if the object was carried on a smooth plane without rolling (like Ali's example, see Fig. 4), it would be a nonexample of translation.

All the participants who held this conception stated that the translation would preserve the shape and size of a figure. When asked to explain what changes occur as a result of

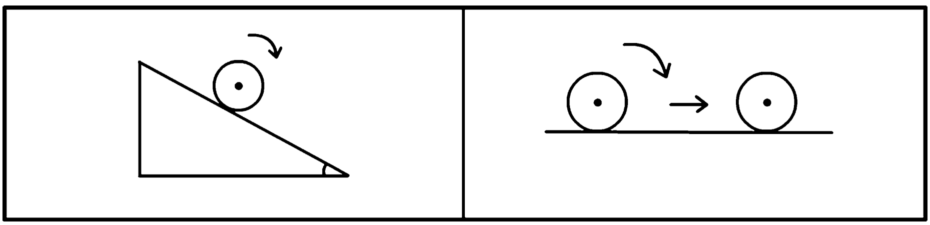


Fig. 2 Prospective teachers' examples of translations

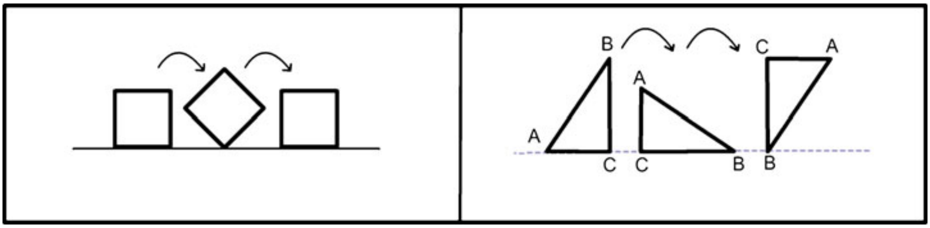


Fig. 3 Prospective teachers' examples of translations

translation, the participants stated that there would only be changes of the location of the pre-image and that the image might point in a different direction.

All the participants who held this conception conceived the plane as a background where objects could be manipulated upon it. All the examples (e.g., Figs. 2 and 3) and statements showed that they considered geometric figures as separate entities from the plane.

Furthermore, all participants held the idea that translation applies only to a single geometric object and that an external force was necessary to rotate and move an object from one point to another point on the plane.

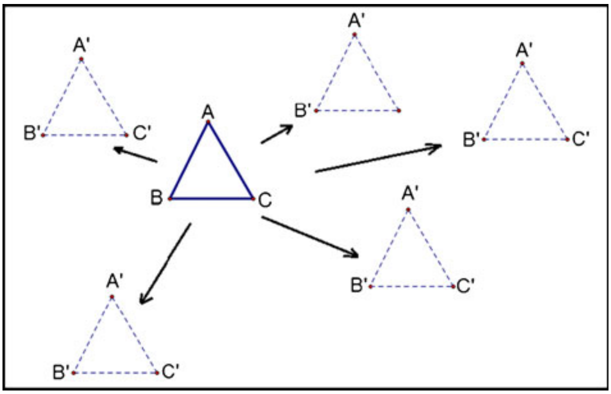
5.1.2 Translation as translational motion

Contrary to the conception of translation as rotational motion, 22 prospective teachers considered translation as translational motion (see Table 3). These participants indicated that linear motion and displacement were the major characteristics of the translation. When they were asked whether the rotational motion of a figure would be an example of a translation, prospective teachers indicated that translation would not involve rotation. Participants who held this conception explained translations with various expressions such as, "moving an object away", "sliding an object", and "carrying an object onto another location without rotating" (see Table 2).

Furthermore, these prospective teachers stated that translation would not cause any changes of the shape and size of a figure. For instance, Ali (all participants are given pseudonyms in this article) stated that the figure could be moved in any direction and its size and shape would not change as a result of translation (see Fig. 4).

Participants' examples (see Figs. 4 and 5) further indicated that they considered the plane as a flat background. One of the participants, Kerem stated that one could think about

Fig. 4 Ali's example of translation



a plane as an *ice skating rink* upon which the figures can be moved. The participants conceived the figures as independent of the plane upon which they were located.

Although the prospective teachers conceived translation as linear motion, they did not specify what defines a translation. In their explanations, participants did not provide any information regarding a vector or an indicator that specifies the direction of the translation and the distance between pre-image and image points. In this respect, participants' understanding of translations seemed to be an undefined motion of a single object on the plane, which could be anywhere on the plane (Edwards & Zazkis, 1993; Yanik & Flores, 2009).

Prospective teachers' responses to the interview tasks further revealed that they considered an external force (e.g., pushing or pulling) as a prerequisite to move a figure on the plane. While five of the participants explicitly mentioned forces, others used body language and their gestures to show that by applying a force the figure could be moved from one point to another. For instance, one of the participants, Elif, drew a figure (see Fig. 5) and stated that "you need to apply a force to the figure to push it or drag it to be able to move and displace it to another location." Since Elif conceived translation as motion, she appeared to believe that she had to apply a force to the figure in order to produce motion. In her description, rolling was not part of translation. Elif considered the force as a motion initiator and thought that the motion and the accompanying displacement were the result of applied force.

5.1.3 Mapping conception of translations

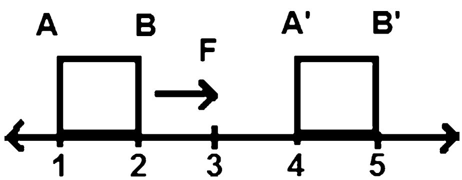
Among all the prospective teachers, only one of them, Alp, had a mapping conception of translations. He considered the domain as a single object and translation applied to only one object rather than all points in the plane. When asked to provide an example and a description for a translation, Alp stated that the points of a figure would be mapped onto other points in the plane based on a specific distance and direction (see Fig. 6). When asked to explain what he meant about mapping, he stated that translation would not include a movement of an object like taking an object and locating it on another place on the plane. He said: "It is like constructing the same shape in some other place in the plane. You don't take the shape and move it to another location. You have this figure here and you construct another one there" (see Fig. 6).

Alp stated that both plane and geometric figures consisted of an infinite number of points. He considered the geometric figures as part of the plane rather than separate entities. According to Alp, force was not related to geometric translation.

5.2 Prospective teachers' reasoning categories regarding translation vector

The results of the clinical interviews revealed that although the prospective teachers had some ideas regarding translations (i.e., translations preserve the shape, size, and angle measure of a figure), the majority of them seemed unsure about the role of vectors in translations (see Table 3). Findings indicated that only 12 out of 44 participants (27%) held views consistent with vector concepts in geometric translations. In other words, these

Fig. 5 Elif's example of a translation



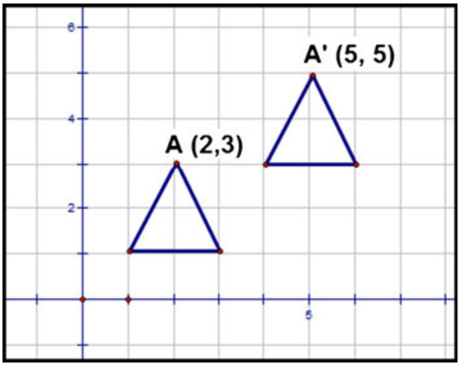


Fig. 6 Alp's example of a translation

prospective teachers considered the direction and magnitude of the vector when executing geometric translations. While nine of these participants had the conception of translation as translational motion, two participants conceived translation as rotational motion and only one participant had a mapping conception of translations. For many prospective teachers, a vector seemed to be an abstract tool that has some role in translations about which they seemed unsure. Overall, four types of conceptions of vectors in translations that arose from the data were (1) vector as a force, (2) vector as a line of symmetry, (3) vector as a direction indicator, and (4) vector as a displacement (see Tables 2 and 3).

5.2.1 Vector as a force

A total of six prospective teachers conceived of a vector as a force. In this consideration, a vector was used as an external force to move a figure. For instance, for the question that required the use of a vector to perform a translation (see Table 1, task 5), Ece stated that she would place the vector on point C to drag the triangle (see Fig. 7). She also showed that during the dragging process, the triangle would rotate a little bit clockwise. Once she executed the translation, she admitted that the figure was actually rotated around point B, but she thought that it would still be a translation. Ece's main goal seemed to be to move the triangle although the end result looked more like a rotation rather than a translation. For Ece, it seemed that force was a prerequisite for translation and motion was the result of that

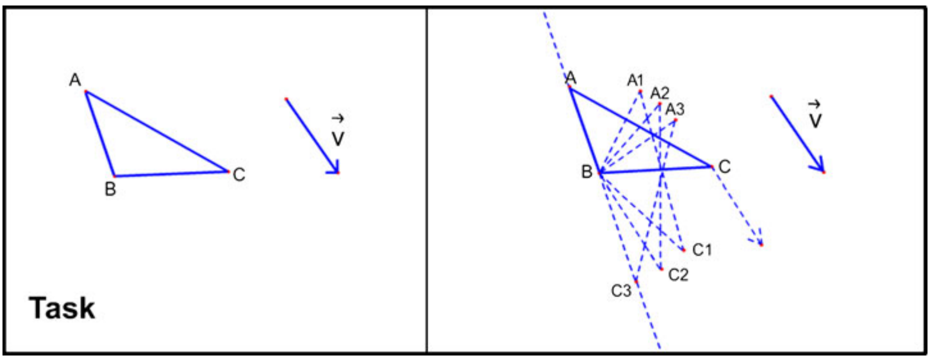


Fig. 7 Ece's use of vector for translation

applied force. Ece's answer seemed to indicate that the type of motion was not an essential part of translation as long as the figure was placed on another location.

Similar to Ece, Ata, another participant who considered the vector as a force, thought that vectors could be used to move geometric figures. For instance, when asked to predict the location of the pre-image based on the given vector and image figure (see Fig. 8), Ata drew the pre-image figure (see Fig. 8) and stated, "I thought how I could apply the vector to point K so I could get this image figure." Ata further explained that the vector could be considered as a force that changed the location of geometric figures. As was shown in Ece's example, Ata also rotated the pre-image around the point Z to get the image figure.

Furthermore, during the interviews prospective teachers initially struggled for a while regarding where on the object they needed to apply the force. Based on their conceptions about translation (i.e., rotational versus translational motion) while one teacher candidate stated that she would apply the force to the center of mass so she could produce a translational motion, five of the participants stated that they would apply the force on the specific points on the object where they could rotate the figure. All the participants stated that the direction of force would determine the direction of translation and the location of the figure after translation would be somewhere in the direction of the force.

5.2.2 Vector as a line of symmetry

The most common interpretation among prospective teachers was considering the vector as a symmetry line. For instance, when asked to translate the triangle using the given vector (see Table 1, task 5), Aisha extended the vector as shown in Fig. 9 and performed a reflection instead of a translation. This is illustrated in the following excerpt from the interview: (R: Researcher, A: Aisha)

R: Can you draw the image triangle using the vector (V)?

A: [She drew it and then explained how she figured it out]. I took the vector as a symmetry line. First, I determined three image points (A', B', and C'). Then I connected them with line segments.

R: How did you decide where to locate the image points (A', B', and C')?

A: I took equal distance from the vector [She connected pre-image and image points with line segments and showed that they were perpendicular to the extended vector].

When Aisha was asked to identify the translation vector for a given pre-image-image pair, (see Table 1, task 7), her answer was consistent with her previous response. She drew

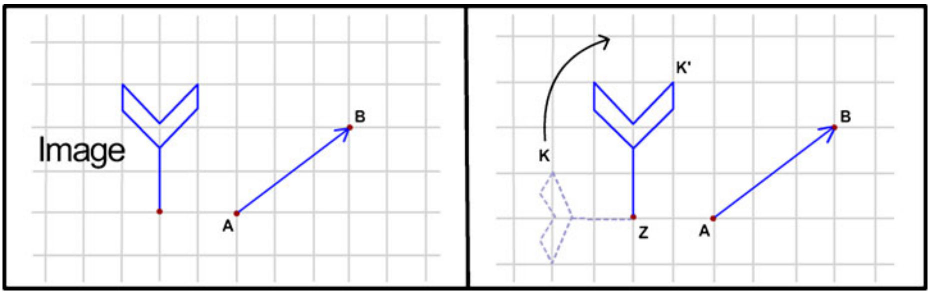


Fig. 8 Ata's use of vector for translation

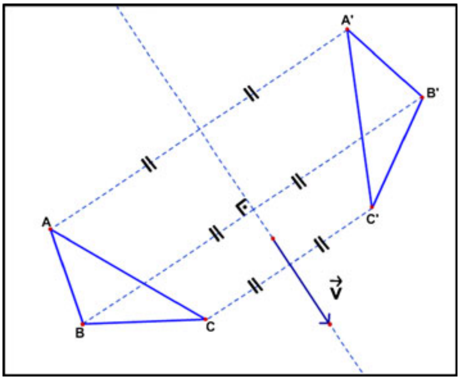


Fig. 9 Aisha's use of vector for translation

a line between the figures and stated that this would be the only vector that could be drawn (see Fig. 10). She showed that the point Y would be the image point of M, and the point Z would be the corresponding point of L (see Fig. 10).

Aisha's response indicated that she had some difficulty in interpreting the vector representation. She considered the vector as a line and seemed unsure about the differences between a vector and a line. Although a vector has a magnitude and end points (i.e., the tail and head), Aisha thought that she could extend the vector forever in both directions. This might be because of the similarity between a ray and a vector. While a ray begins at a certain point and extends forever in one direction, a vector has a beginning and an ending point. This might have given Aisha the impression that she could extend the end points of a vector forever in both directions.

5.2.3 Vector as a direction indicator

Clinical interviews revealed that four out of 44 prospective teachers considered the vector as a direction indicator for the translation. All but one of these prospective teachers conceived translations as rotational motion and the vector as an abstract tool that shows the direction of the translation (see Tables 2 and 3). For instance, Fig. 11 illustrates Suzan's use of vector in performing a translation. Suzan explained that she moved the triangle while rotating it in the direction of the vector. She further stated that the vector showed which

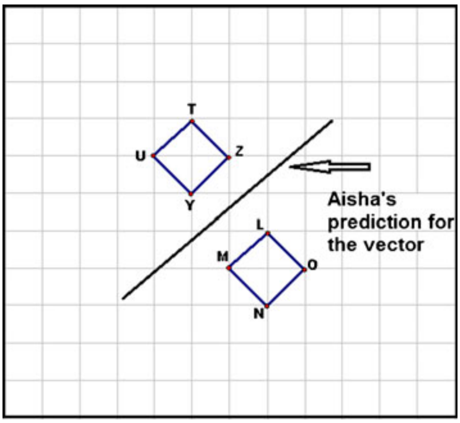


Fig. 10 Aisha's identification of vector for a translation

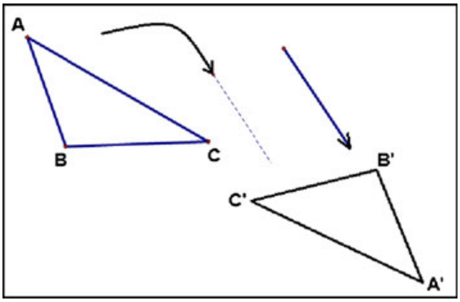


Fig. 11 Suzan's use of vector in a translation

direction the triangle would be moved, but she did not provide any indication about what defines the distance between pre-image and image points.

5.2.4 Vector as a displacement

While in their definitions or examples none of the participants specifically mentioned the vector or any indicator about how they specified the distance between pre-image and image points, when asked to perform translations that required the use of a vector, 12 out of 44 prospective teachers took into consideration the direction and the magnitude of the vector to find out the new location of the image points. Nine of the participants conceived translations as translational motion, two participants considered translations as rotational motion, and one participant had a mapping conception of translation. One example is illustrated in the following excerpt from the interview:

(R: Researcher, S: Selin)

R: Can you draw the image triangle?

S: [She drew it] (see Fig. 12):

R: Could you please explain it to me?

S: I placed the vector on each corner of the triangle and then carried the points (A, B and C). After that I joined them (with line segments) and constructed the (image) triangle.

R: What can you say about the pre-image and image triangles?

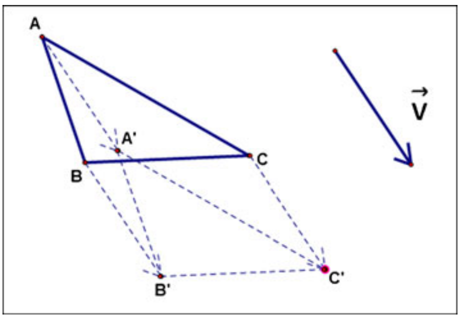


Fig. 12 Selin's execution of a translation using the vector

S: Those sides are parallel [AB//A'B'; AC//A'C'; BC//B'C'], those angles and the lengths are the same [angles A and A'; B and B'; C and C; AB=A'B'; AC=A'C'; BC=B'C'). The only thing has changed is the location of the pre-image.

R: Can you tell me the role of vector in translation?

S: Geometric figures are translated based on the vector. The vector determines the location of the image figure. That is known with the help of the length and the direction of the vector.

In another example, when asked to predict the location of the pre-image based on the given vector and image figure (see Table 1, task 6), Hakan used the length and the opposite direction of the vector to determine the location of the pre-image (see Fig. 13). He stated that vector would specify the distance and direction of the translation and every point of the figure would be translated based on these aspects. Both Selin and Hakan specified the displacement of the figure based on the vector's direction and the magnitude. Later, when the participants were asked to explain how they knew to use a vector in this way, they said that they remembered using (displacement) vectors to figure out final positions of points from their previous physics courses that they took in high school or the teacher preparation program at the university (see Fig. 14).

Prospective teachers' reasoning suggested that they were able to see the relationships between the length and the direction of the vector and the pre-image and image points. However, there were still a few participants who conceived of translation as rotational motion. It seemed that those participants' conception of translation was independent from their thinking about vectors.

5.3 Some possible mixed conceptions of translations and vectors

Although the prospective teachers' major conceptions of translations and vectors have been discussed so far, further inferences might be drawn from the data (see Table 3). For instance, initially, eight prospective teachers seemed to hold both notions of rotational and translational motion. For these participants, it seemed that motion and displacement were major aspects of geometric translations. When asked to think about situations involving rotational and translational motion, they seemed unsure about whether the type of motion would be a major characteristic of translations. It seemed that these prospective teachers were using nonmathematical clues and their reasoning was based on real-life experiences that made them think that translation was a movement of an object on the plane. Since none of the participants explicitly stated that both types of motion would be called translation, it was not included in Table 3 as a mixed conception of translations. However, it might still be possible for some of the participants to hold mixed motion conceptions of translations since they seemed uncertain in their final decisions.

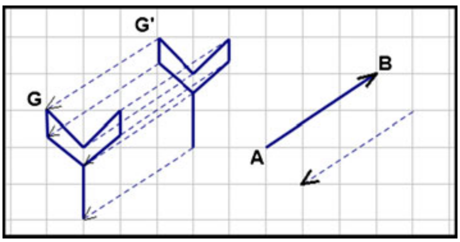


Fig. 13 Hakan's use of vector for a translation

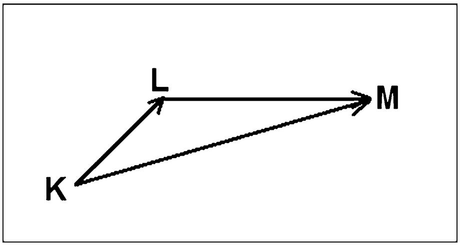


Fig. 14 Displacement vector

Another inference that might be drawn from the data is regarding the notion of a vector. It might be possible that some participants consider a vector as both a force and a direction indicator. For instance, three of the participants who considered translation as rotational motion applied the vector as a pushing or dragging force and moved the figure in the direction of the vector (see Fig. 11). Although in their explanations, these participants mentioned the vector as a direction indicator, by their body movements, it seemed that they might also consider the vector as a force that moved the object.

6 Discussion

The goal of this study was to explore prospective middle school mathematics teachers' initial understandings of geometric translations. The analyses of data revealed that prospective teachers reasoned about translations and vectors in a variety of ways. Being aware of these preconceptions and taking them into consideration during instruction can enhance teacher candidates' subject-matter knowledge and future classroom practices. Furthermore, it is crucial for instructors to know possible sources of learners' preconceptions so they can make them part of their instruction or exclude them to prevent possible misconceptions. According to Biemans and Simons (1995), being aware of learners' misconceptions and addressing them in instruction may help individuals reorganize their knowledge structure and correct their naïve conceptions.

6.1 The role of physics course in teacher candidates' conceptualizations of geometric translations

The physics course was one of the resources that seemed to affect teacher candidates' conceptualizations of translation as motion. Translational motion in this course is explained as the movement of an object from one position to another along a straight path without rotation (Halliday, Resnick & Walker, 1993). In this type of motion, every point of a figure

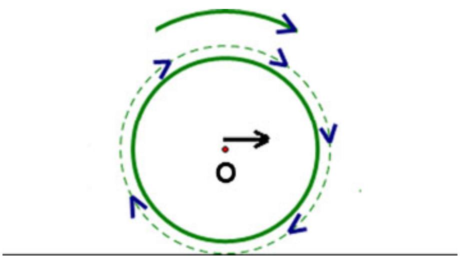


Fig. 15 Example of a rotational motion

follows a linear path unlike the points of a figure in rotational motion. Rotational motion includes combined motions of pure rotation and translation (Halliday et al., 1993). Halliday et al. (1993) describe rotational motion through an example of a rolling wheel along a street at a constant speed without slipping. While every point on the edge of the wheel traces a cycloid path, the center of mass (O) follows a linear path (see Fig. 15). Participants who held motion conceptions of translations particularly focused on the type of motion (i.e., translational versus rotational) while translating an object from one point to another and stated that the geometric figures would preserve their size and shapes as a result of such a translation.

During the physics course, force–motion relationships are also discussed. In particular, the resource book provides examples regarding how force causes a change in motion of a body (mass). Examples include how a force initiates motion, makes an object move faster or slower, changes the direction of motion or stops motion. In the resource book the body is generally represented by a dot and each external force is represented by a vector with its tail on the dot. Participants who considered the external force as an initiator of motion generally applied pulling or pushing force on an object to show how the translation would be executed.

The resource book further discusses the motion–displacement relationship and explains how a vector can be used to represent a displacement. In particular, motion of an object along a straight line is described by a displacement vector. During the interviews, some of the participants stated that they also remembered using displacement vectors (see Fig. 14) in their high school physics course such as using head to tail and parallelogram methods to calculate a resultant vector. These examples and representations might have affected teacher candidates' interpretations of geometric translations and vector concepts when they first encountered the translations tasks. The findings revealed that the participants who held motion conceptions of translations appeared to consider that translation could only be achieved through the motion of an object and therefore an external force was needed to generate the motion.

6.2 Teacher candidates' understanding of geometric translations from the p-prim perspective

The results of the study showed that prospective teachers conceived translations as mainly motions and held various views about the possible roles of vectors in translation. The findings further revealed a fundamental misconception in physics- that force causes motion (Clement, 1982; Megowan, 2005; Sadanand & Kess, 1990). Some of the participants believed that force was necessary for motion and for that reason they considered it as an essential part of geometric translations. From diSessa's (1993) perspective, teacher candidates' view of force-translation relationship can be explained through p-prims. During the interviews it seemed that some of the p-prims (e.g., *force as a mover* and *dying away*) were activated when the participants attempted to describe what a translation was. For instance, several teacher candidates seemed to consider that there needs to be some form of force in order to produce rotational or translational motion. Regarding rotational motion, although not all participants explicitly mentioned force in their statements, their body movements, such as using their hands to show how noncircular shapes could be rotated (see Fig. 3) revealing that they actually considered the external force as an impetus to start moving the figures. From diSessa's (1993) perspective, teacher candidates' view that force is an essential part of motion would be an example of the p-prim: *force as a mover.* When teacher candidates were asked to explain what specified the figure's end location, some of the participants stated that "it depends on when you stop rotating the figure" and some of them said that "the object would stop eventually." While some of them considered that the external (hand) force controlled the motion, others focused on the frictional force that would cause the figure stops. It might be possible that the teacher candidates' reasoning was mediated by the p-prims: *maintaining agency* (Hammer, 1996), *resistance, interference,* and *dying away* (diSessa, 1993). *Maintaining agency* involves the idea that continuing cause (e.g., pushing or dragging) maintains motion (Hammer, 1996). Participants who considered that the motion of an object depends on an external force might activate the *maintaining agency* p-prim. "*Resistance* and *interference* are two p-prims pertaining to causes that impede an effect" (Hammer, 1996, pp. 103–104). Participants might consider the frictional force as the major cause that would stop the figure after awhile. *Dying away* may underlie the participants' expectation of why the figure slows down and stops as the influence of the *actuating agency* (e.g., external force) (Hammer, 1996) fades over time.

Analysis of the data further indicated that although many prospective teachers knew that a vector has a magnitude and a direction, this knowledge did not generally lead them to conclude that vectors define translations. For instance, some of the teacher candidates who held the *force as mover* view conceived the vector as a force and the *initiator of motion*. This was particularly observed when the participants applied the vector on an object as a pushing or pulling force (see Fig. 5).

Findings further revealed that some teacher candidates held the conception of *vector as a direction indicator*. The reasoning behind this conception might be related to the direction signs we see in our daily lives. For instance, many direction signs that look like a vector (e.g., see Fig. 16) are used for various purposes such as indicating the direction of the traffic or showing the way to find a hospital or parking spot. In our daily life an arrow on a line generally indicates the *direction one needs to follow*. During the interviews, participants who conceived the vector as a direction indicator pointed out the arrowhead on the vector and stated that the figure should be moved in that direction.

Prospective teachers also conceived the vector as a displacement. One possible reason for this interpretation might be related to conceiving the *vector as a journey* (Watson, Spyrou & Tall, 2002). According this view, vector may have a meaning that includes "an action of translating a shape on a plane (Watson et al., 2002, p. 85). Watson et al. (2002) state that school mathematics generally represents the vector as an arrow, which refers to the idea of journey from one point to another point. Participants who held this conception considered both the direction and the length of the vector when executing a translation. When they were asked to explain how they knew that they need to consider the direction and the magnitude of the vector, they referred back to the physics courses that they took in the teacher education program and high school.

Findings also revealed that teacher candidates considered vectors as symmetry lines. This result showed a remarkable similarity with the author's previous study (Yanik & Flores, 2009). Participants' understandings of vectors as symmetry lines might have been related to the lack of knowledge of the difference between a line and a vector. Several responses to the task (see Table 1, task 5) during the interviews indicated that participants

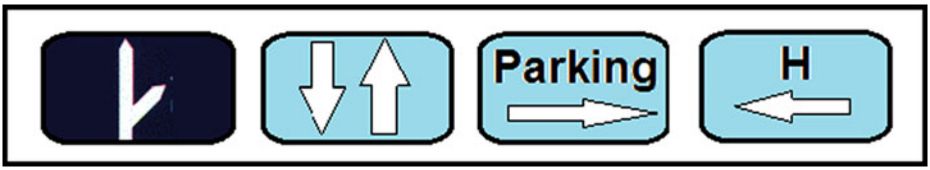


Fig. 16 Direction signs

interchangeably used the word "line" and "vector" and considered the vector as if it could be extended infinitely in both directions. Although participants who considered the vector as a reflection line did not know the role of the vector in translations, they all seemed to have an expectation that translation would alter the location of the given figure in some way. This expectation might also be related to colloquial uses of language which will be discussed next as a potential factor that might influence teacher candidates' reasoning about translations.

6.3 The effect of language in interpretation of translation tasks

One possible source of teacher candidates' thinking about translations is related to language. According to Ozdemir (2006), "words gain different meaning depending on situations and activities in which they are used" (p. 21). Prospective teachers' reasoning about translation as motion might be related to the meaning of the word "translate" in Turkish (*ötele*). It is possible that the conventional meaning of "*ötele*" in the Turkish language may affect participants' normative conceptualization of the term "translation." In Turkish, the word *translate* means an action that includes both motion and the displacement. In everyday language, *ötele* is used to move (something) away. In the Turkish dictionary, *ötele* is defined as the linear motion of an object. In other words, *ötele* occurs when all points of an object move the same distance in the same direction. Most of the teacher candidates stated that they did not remember whether they studied geometric transformations in their past school years. Therefore, teacher candidates who had few if any experiences with transformational geometry might have used the conventional meaning of *ötele* when they were asked to perform translation tasks. Past research (e.g., Kotsopoulos, 2007; Nesher, Hershkovitz & Novotná, 2003) discussed the role of language in learners' interpretations of mathematical concepts and processes. For instance, Kotsopoulos (2007) found that "students experience interference when language is borrowed from their everyday lives and used in their mathematics world" (p. 301). This interference can potentially undermine students' ability to learn more abstract mathematical concepts (Kotsopoulos, 2007; Nesher et al., 2003).

7 Implications

7.1 Implications for teaching

This study was conducted with second-year prospective teachers and revealed their motion conceptions of geometric translations. Past research (e.g., Edwards, 2003; Glass, 2001; Harper, 2003; Hollebrands, 2003; Jung, 2002; Yanik & Flores, 2009) also found that students of different ages and backgrounds conceived transformations as mainly motion and had incomplete understanding of geometric translations even after taking some courses or participating in activities related to geometric transformations. Past research further revealed that some "conceptions are deeply embedded in students' minds" and can cause resistance for future learning (Aguirre, 1988, p. 216). For instance, if prospective teachers maintain their belief that translation could only be understood through motion, they will not easily accept the mapping understanding of translations and teach it in the future in their own classrooms. Therefore, prospective teachers' somewhat naïve ideas regarding translations and vectors need to be expanded in order to form a foundation for complex ideas.

The teacher candidates' different levels of sophistication in their understanding of translations as motion in this study correspond to some extent with the findings of the author's previous research (Yanik & Flores, 2009). For instance, in both studies almost half of the participants conceived translations as undefined motion of a single geometric object. Furthermore, in both studies teacher candidates had an incomplete understanding of the plane, the domain for transformations, and the parameter (i.e., vector) that defines translations. These concepts seemed to be critical in terms of developing a thorough understanding of geometric translations. The next section presents some ideas that would be helpful for classroom instruction.

7.1.1 Understanding of the plane

Their understanding of the plane seemed to be one of the crucial factors that shaped teacher candidates' conceptions of translations. The results of this study revealed that participants who held motion and mapping conceptions of translations had different conceptualizations of the plane. For instance, participants who held a motion conception of translations (i.e., translational or rotational motion) considered the plane as a background surface upon which geometric figures can be manipulated. These teacher candidates considered translations as physical motions of geometric figures *on top* of the plane. The only participant who held a mapping conceptualization of translations, Alp, considered that the plane consists of an infinite number of discrete points and the geometric figures are subsets of points within the plane rather than separate entities independent of the plane. Furthermore, Alp considered the points in the plane as specific locations rather than physical entities that can be carried to other locations. This conceptualization of the plane seemed to be necessary for Alp to consider translations as mappings rather than motions of geometric figures. When asked to explain what was the same and what was different before and after a geometric translation, Alp explained that there would be only changes of the coordinates of the figure, but that the size and the shape of the figure would not change as a result of translation (see Fig. 6). Instruction that emphasizes the meaning of points in the plane as locations and conceiving numbers as point locations (Lakoff & Núñez, 2000) would help learners understand transformations as mappings.

7.1.2 Understanding of domain for geometric translations

All participants, regardless of their conceptions of translations considered the domain as a single geometric figure rather than all points in the plane. While teacher candidates who held a motion conception of translations considered the domain as independent from the plane, the participant who held a mapping conception of translations considered the domain as part of the plane. In their explanations, participants provided physical representations (e.g., drawings) of only one figure to translate rather than considering all points in the plane. It seemed that the knowledge that translation applied to all points in the plane was not familiar to any of the participants. Technology (e.g., dynamic geometry programs) would be helpful for instructors to create environments in which learners could explore the relationships between the plane and the geometric figures. Specifically, activities that include multiple figures to be translated can be designed through using dynamic geometry programs. These experiences would provide learners opportunities to see that rigid translations not only preserve the relative distances and angles within a figure but also preserve them among all points in the plane.

7.1.3 Understanding of the vector

Past research (e.g., Harper, 2003; Hollebrands, 2003; Watson et al., 2002; Yanik & Flores, 2009) revealed that learners had difficulties with understanding vectors in geometric translations. This study supports the findings of the prior studies and suggests a special emphasis should be placed on the vector characteristics during the teaching of geometric translations. diSessa (1993) argues that instruction can exploit learners' naïve ideas to help them understand more complex ideas. "The teacher may schematize instruction as promoting appropriate aspects of students' knowledge and reasoning" (Hammer, 1996, p. 119). According to Poynter and Tall (2005), instruction should take into account the cognitive development of students that "builds from perception and action through reflection to higher theoretical conceptions" (p. 1266). Understanding a vector as a journey can be observable and might be a starting point for the instruction. For instance, initially, students might translate an object on a table and focus on the *action* (Poynter & Tall, 2005). However, students should go beyond sensori-motor actions (Piaget & Inhelder, 1956).

It is important for students to see the relationship between pre-image and image points and see the role of vector in translations (Poynter & Tall, 2005; Yanik & Flores, 2009). Instruction that focuses on the effect of translation may have students realize these relationships and reach a more abstract level of understanding (Poynter & Tall, 2005). diSessa (1988) discusses the use of computers, specifically designing learning environments which are based on experiential learning. According to diSessa (1988), computer-based learning environments provide students with various opportunities to meet their naïve ideas and beliefs, and help them develop more integrated ideas. For instance, dynamic geometry programs (e.g., GeoGebra, Cabri, and The Geometer's Sketchpad) can be used to design activities that focus on the role of vectors in geometric translations. Specifically, manipulating the vector and asking students to figure out the effect of that manipulation would be helpful for them to explore the relationships between vector and pre-image-image figures (Hollebrands, 2007; Yanik & Flores, 2009).

7.2 Implications for future research

7.2.1 Future directions for research for teacher education

This study revealed teacher candidates' naïve ideas which are potentially useful for teacher educators to understand prospective teachers' underlying reasoning processes. According to Edwards (2003), learners' and contemporary mathematicians' conceptualizations of transformational geometry might be quite different. Exploring teacher candidates' conceptualizations of geometric transformations and describing the nature of their understandings may provide an opportunity to find ways of filling the gap between two conceptions. However, it is still unclear whether these participants' preconceptions are held by other prospective teachers with different levels of understanding and backgrounds. Also, we do not know much about whether there would be other preconceptions held by teacher candidates. Further research is needed to identify the prevalence and the possible sources of the preconceptions identified in this study. Furthermore, future research needs to be conducted to explore what other preconceptions are involved in conceiving geometric translations.

7.2.2 Future directions for research in relation to teaching and learning geometric translations for teachers and students

The results of the study revealed that preconceptions might be rooted in early school experiences. Further research is needed to explore how instructional experiences shape pupils' conceptions of geometric transformations and whether or not these conceptions are altered throughout the school years. Past research (e.g., Cai, 2005; Leinhardt, 2001) revealed various impacts of the mathematical representations on students' mathematical thinking and reasoning. Specifically, it might be useful to investigate how teachers and students use pedagogical representations (Huang & Cai, 2007; Leinhardt, 2001) in their classroom as expressions of geometric transformations and how those influence students' conceptions of transformational geometry.

Furthermore, we do not know much about how classroom teaching emphasizes the different conceptions of geometric transformations (i.e., motion versus mapping conceptions) before prospective teachers enter the teacher education program. It is still unclear how students progress from motion conceptions to mapping conceptions of transformations.

The data suggested that the use of language has potential to affect the learning of geometric transformations. Specifically, the results indicated that the conventional meaning of words may interfere with the learning of normative conceptualization of translations. Future research may focus on how mathematics teachers use conventional meaning of words to help students make sense of transformations and how the use of language influences the students' conception of geometric transformations.

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