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How Do Students Prove Their Learning and Teachers Their Teaching? Do Teachers Make a Difference?

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Abstract

Problem Statement: Gaining reasoning skills in early years affects the formal proving skills in the following years, thus it is quite significant. The acquiring of this skill is only possible with the approaches that the teachers used in the process. At this point, the problem to be researched in terms of making proofs is seen in how middle school students prove a mathematical expression; what kinds of reasoning and proof types they use in this process; how the teachers of these students prove the same expression; and how they reflect it to their instruction.

Purpose of the Study: The purpose of this study is to investigate the middle school students' and their teachers' reasoning types and proof methods while proving a mathematical expression.

Method: A basic qualitative research design was conducted to investigate the research problems. Participants in this study were two middle school mathematics teachers who have different professional experiences, and 18 students from 6th, 7th and 8th grades. A clinical interview technique was used to collect data and the interviews were video recorded. A thematic analysis method was used to analyze the data.

Findings and Results: The middle school students tried to decide on the argument by following specific cases in order to verify a mathematical expression, and in this context they performed several actions, such as pattern recognition, seeking the relationship between two variables, and making conjectures. They have performed three types of actions, namely verification, explanation and abstraction during the proving of a mathematical expression. Moreover, they have provided some arguments which were not accepted as proof, by offering experimental, intuitive or

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illogical justification. On the other hand, it has been observed that the middle school mathematic teachers thought in the same way that their students thought while proving a given mathematical expression.

Discussion, Conclusion and Recommendations: As a result of this study, it has been found that students had difficulties in proving mathematical statements; they preferred to use experimental proofs and mostly adapted an inductive approach. On the other hand, the proving tendency of the teachers was mostly at a verification and explanation level; they have a similar structure of thinking with their students in the process of proving mathematical expressions. Reasoning and proof should be the fundamental aspects of mathematics teaching, should play a significant role in mathematical contents without taking it independently, and should be developed in the earlier years. In addition, to what extent mathematics textbooks and mathematics curriculum in each grade level support the reasoning and proof standards should be investigated.

Keywords: Mathematics education; generalization; making conjecture; reasoning and proof.

Introduction

Proof as fundamental to mathematical understanding is needed for the construction and transmission of mathematical information. At the same time, it is an important tool in learning mathematics, as well. Hence, proof is an important concept in the way of mathematics and mathematics education. For this reason, in school mathematics, in the early years, it is suggested that the proof teaching should be disseminated in the mathematical experiences at the proper grade level of students (Healy & Hoyles, 2000; NCTM, 2000; Yackel & Hanna, 2003). However, it is a fact that the focus of mathematics lessons in primary school is arithmetical concepts, equations and algorithms; on the other hand, in the middle school; the teaching of proof is mostly found in geometry lessons (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). This quick transition to proof is indicated as the possible reason that students experience many difficulties during the making of proof (Healy & Hoyles, 2000).

The studies revealed that all students from elementary school to the higher education have difficulties in reasoning and proving exercises. Most of these studies showed that students tended to exemplify and verify (Harel & Sowder, 1998; Knuth, Slaughter, Chooppin, & Sutherland, 2002; Stylianides & Stylianides, 2009; Aylar, 2014, Uygan, Tanisli, & Kose, 2014; Stylianou, Blanton, & Rotou, 2015; Guler & Ekmekci, 2016), and they mostly preferred inductive reasoning (Harel, 2001). On the other hand, as pointed out by Knuth and Sutherland (2004), and Reid and Knipping (2010), many students thought that verifying with an example was enough to prove a statement. Hence, the process of the development of proof is handled from the beginning of the elementary school to high school and students are required to see proof as a fundamental element in the learning of mathematics. Furthermore, proof should have a place in the process of teaching mathematics in the natural flow; it should be placed as not handling an independent subject area in the mathematical content of center (NCTM, 2000). In this process, the teachers play a big role. However, the studies performed with the teachers who play a significant role in this state of the students showed similar results. The researches revealed that teachers were experiencing difficulties in writing proof (Jones, 2000; Knuth, 2002; Iskenderoglu & Baki, 2011) and they had similar thinking structures with their students in the process of proving mathematical expression. On the other hand, it has been found that the opinions, beliefs, and knowledge of the teachers also affected their students' proof performance (Knuth, 2002).

In Turkey in 2003 and in 2005, proof is emphasized in process standards in high school mathematics curricular programs where formal proof takes place. On the other hand, in other mathematics programs, the proof concept is not mentioned directly. However, proof is indirectly mentioned as a part of the ability to reason, making generalizations, making inference, defending, verifying a mathematical statement and constructing an argument (Ministry of National Education, 2013). Developing students' reasoning and supporting them to develop proof within their early years play an important role in formal proof development. In addition, the acquirement of this skill is only possible with the approaches that the teachers used in the process. Teachers should develop rich environments where how and why are discussed, which enhance thinking skills of the students, instead of offering pre-prepared solutions in the process of verification of a mathematical statement, which requires making a proof and expecting students to apply the same logic in similar situations. At this point, in Turkey, the problem to be researched in terms of making proof is seen as how middle school students prove a mathematical expression, what types of reasoning and proof that they use in this process, how the teachers of these students prove the same expression, and how they reflect it to their instruction.

Purpose of the Study

The purpose of this study is to investigate the middle school students' and their teachers' reasoning types and proof methods while proving a mathematical expression. For this purpose, the following questions were addressed:

1. What are the middle school students' types of reasoning and methods of proof while proving a mathematical expression?
2. What are the middle school mathematics teachers' types of reasoning and methods of proof while proving a mathematical expression?

This study is significant because it emphasizes how middle school students and teachers prove the mathematical expressions and the difficulties students have in writing proof, and stresses the role of teachers in the proving process by determining the relationship between teachers' and students' types of reasoning and methods of proof.

Theoretical Framework

Reasoning can be defined as the coordinated process of the evidences, beliefs and ideas resulting from the conclusion of what reality is (Leighton, 2003). From a different point of view, reasoning is a process of producing new knowledge from preliminary thoughts (Rips, 1994). In this paper, "reasoning" is simply considered to be the ways of thinking that were adopted to produce statements and in seeking the results.

There are various ways of thinking or types of reasoning while writing a mathematical argument. For example, Reid and Knipping (2010) defined reasoning types, such as induction, deduction, abduction, reasoning by analogy and others. An inductive reasoning occurs when an appropriate subset of an event is examined and proceeds to a generalized conclusion. Deductive reasoning is observed when the statements are correlated with the data by using one or more logical deduction rules, whereas analogic reasoning is observed if a statement is developed or revealed by considering the similarities between mathematical events. Finally, abductive reasoning typically occurs with the observations of a specific case and the discovery of an inference allowing the formation of a statement.

During the reasoning process, two types of actions occur, namely discovering and justifying. In the process of discovering, new knowledge is investigated and explained; whereas in the process of justifying, mathematical statements are verified or proven (Ball & Bass, 2003). New knowledge is investigated and explained within the reasoning aroused in the process of discovery, whereas mathematical statements are verified or proven by the reasoning aroused in the process of justification (Ball & Bass, 2003). The exploratory aspect of reasoning requires making generalization, which includes paying attention to the pattern and order, making conjecture and testing; whereas the defense of the reasoning requires explaining the meaning by developing arguments. Argument is a verification, which is a part of the reasoning that aims to self-persuade or persuade others (Bergqvist, Lithner, & Sumpter, 2006). However, not all but only some arguments can fulfill the standards of a proof. Therefore, the proof is usually the end product of the process and it can be supported by activities such as pattern recognition, making conjecture and arguments that are included in the process, but not in the scope of proof. Thus, both reasoning and proof require each other, as shown in Figure 1 (Stylianides, 2010).

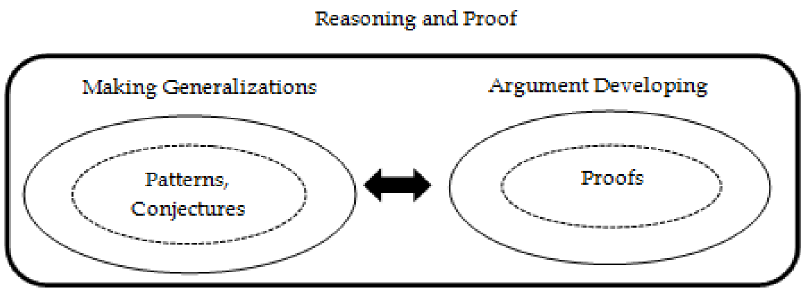


Figure 1. Generating and validating new knowledge in mathematics

Proof consists of the conjectures that use mathematical language and definitions, logical arguments that carefully express the premises, and the reasoning used to reach a valid conclusion. In other words, it can be defined as a valid argument against/for a mathematical statement (Stylianides, 2008). The term "valid argument" refers to the content that is agreed on by mathematicians. Within this paper, considering that the participants are at middle school level, it was expected that these students, who were supposed to have abstract thinking capability, should test their mathematical statements through various types of reasoning and proof, and express their statements using mathematical language.

On the other hand, during the recent discussions about the level of making proof at elementary and high school levels, the discussion of proof and verification came to the forefront and the difference between them has been revealed by emphasizing that the generalization tendency of early year students should not be accepted as proving (Stylianides & Stylianides, 2009). Sharing the same view, experimental verifications were not considered as proofs within this paper; it has been assumed that there are three stages of students' proof, "verification" where they investigate the validity of their statements, "explanation" where they explain why their statement is true and "abstraction" where they follow the shortest path for their abstraction using mathematical language (Iskenderoglu & Baki, 2011).

Method

Research Design

A basic qualitative research design is particularly well suited to obtaining an in-depth understanding of effective educational processes (Merriam, 2009). Because the purpose is to obtain an in-depth understanding of the middle school students' and their teachers' reasoning types and proof methods, this research design was conducted to collect, analyze, and interpret data. In basic qualitative research design, questions, focus points and established relationships in the interviews, observations and document analysis are performed by depending on the theoretical framework of the study (Merriam, 2009).

Participants

The participants of this study, where a basic qualitative approach was adopted (Merriam, 2009), consisted of two middle school mathematic teachers and 18 students of these teachers attending 6^{th}, 7^{th} and 8^{th} grades, three students from each grade. "Criteria sampling," which is one of the purposive selection methods, was used to select the participants (Yildirim & Simsek, 2011), the seniority of the teachers (five years of experience and 30 years of experience) and students' achievement levels (low, medium, high) were set as the criteria of the sample, based on volunteerism.

Research Instrument and Procedure

For the purposes of the study, the data was collected using clinical interviews. On account of the purpose of providing detailed knowledge and identifying thinking structure (Clement, 2000), clinical interview is used. Before the interviews, clinical interview questions, which consist of one open-ended question for each content domain, namely numbers and operations, geometry and measurement, and algebra, were prepared. The researcher and a field expert evaluated the interview questions and made necessary revisions. The pilot study of clinical interview questions was conducted with a similar group representing the participants. As shown in Figure 2, clinical interview questions were prepared considering the grade levels of students. These questions were also conducted to teachers, who were asked to prove them.

Data Analysis

The thematic analysis method that is widely used in qualitative research was used to analyze data (Liamputtong, 2009). Two experts independently defined first starting codes. The reliability of coding was calculated and the rate was found to be 90%. After the coding process, experts determined the themes and sub-themes together with a consensus on them. Based on the indicators of the process of reasoning and proving, the following themes emerged: *middle school students' process of reasoning and proving mathematical statements* and *middle school teachers' process of reasoning and proving mathematical statements.* It is determined that subthemes which belong to these themes are *making generalizations* and *evidences supporting mathematical statement.* All of these processes were analyzed in terms of *inductive, deductive, abductive, analogy* and *other* types of reasoning. Then the themes, which were defined and named in detail, were interpreted; the findings of the research were interpreted under these themes and presented with direct quotations from the dialogues.

Validity and Reliability

All research phases were reported in detail in order to ensure the validity and reliability of the research. The purposeful sampling method was used to select participants. While a data collection instrument was being prepared, the field experts evaluated the questions and the instrument was piloted with a similar group representing the participants. Data which were obtained was analyzed with the researcher and field expert. To calculate the inter-coder reliability rate, Miles and Hubermans' (1994) formula (reliability= number of agreements/(total number of agreements) + (disagreements)) was used. Obtained data were presented under the themes and subthemes in detail and the findings were supported with direct quotations without ruining originality.

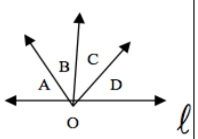
Number and Operations

1.Prove that the sum of three consecutive numbers is divisible by 3. (Grade 6th, 7th, and 8th).

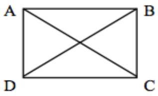
Geometry and Measurement

Geometry

2. Given the line l, $\wedge A \cong \wedge B$ ve $\wedge C \cong \wedge D$. Show that $m(\wedge B) + m(\wedge C) = 90$. (Grade 6th and 7th)



3.

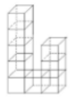


Show that the diagonals of a rectangle are congruent to each other. It means show that $|AC| = |BD|$. (Grade 8th)

Measurement

4. If the length of a rectangle is doubled, what will happen to its area? Show your work. (Grade 6th)

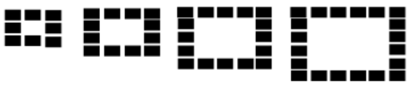
5. Show that the area of a parallelogram is multiplication of its base and altitude. (Grade 7th)



6. Find the whole surface area formula of the geometrical object that is constructed by unit cubes. (Grade 8th)

Algebra

7.



1 2 3 4

a) How many squares will be in the 6th step of the pattern? Explain.
b) How many squares will be in the 24th step of the pattern? Explain.
c) Write the pattern rule to calculate the number of squares in any number of steps? (Grade 6th, 7th, and 8th)

Figure 2. Clinical interview questions

Results

Middle School Students' Process of Reasoning and Proving Mathematical Statements

Making generalizations

The middle school students tried to solve the argument by following specific cases in order to verify a mathematical expression, and in this context they performed several actions, such as pattern recognition, seeking the relationship between two variables and making conjectures, as shown in Table 1.

Table 1.
Middle School Students' Processes of Generalizing Mathematical Expressions as Justification

						6.-7.-8 *Grade Level*
Verification of the Hypothesis	*Generalization*	*Pattern Recognition*	Converting to an Arithmetic Pattern	*Type of Generalization*	Arithmetic	L(6), M(6), H(6)
		Seeking the Relationship Between Two Variables	Recursive Relationship		Arithmetic	L(6), M(6), H(6)
			Functional Relationship		Algebraic/ Visual	L(1), M(2), H(2)
			Proportional Relationship		Arithmetic	L(3), M(1), H(2)

Table 1 Continue

		Random Example	Arithmetic	L(6), M(6), H(5)
		Specific Example	Arithmetic	L(2),H(1)
	Making Conjectures	Mathematical Conjectures	Verbal/ Visual/ Arithmetic/ Algebraic	L(6), M(6), H(6)
		Prototype Shape	Visual	L(6), M(6), H(6)
		Trial/Error	Arithmetic/ Algebraic	L(6), M(6), H(6)
		Ratio/ Proportion	Arithmetic	L(6), M(6), H(6)
		Type of Reasoning		
Abduction	Deduction	Analogy	Induction *From specific to general *Not specific, probable	Other Reasoning *Erroneous reasoning *Referring to an authority

After seeing the pattern question seen in Figure 3, the first action of all students, independent of their grade and achievement levels, was to convert the visual pattern to an arithmetic pattern, and then focus on the recursive relationship between two variables (obtaining the next term by adding a constant to the previous one). In this process, students generalized the relationship arithmetically by observing the constant difference between the terms of the pattern.

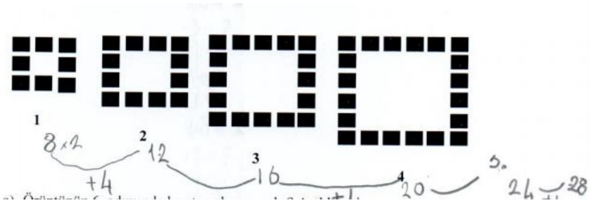
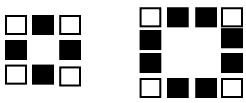


Figure 3. The approach of 6th grade students with medium achievement level

On the other hand, the functional relationship between two variables (the step number and the term), was only recognized by five students with medium and high achievement levels. For example, an 8th grade student with medium achievement level explained this relationship as "*One is added to each* (corner). *In fact, the initial shape, in other words step zero is this* (he formed the shape below). *We take the corners as +4 … I mean if n=0, we get +4 … I mean step zero is +4… The first step 4+4, second step 2.4+4, and nth step is 4n+4*".

Results revealed that students achieved a visual and algebraic generalization using an inductive approach by analyzing the constant and changing terms in each step.



There are students who have reached the wrong conclusion by erroneous reasoning while searching for the relationship between two variables. In general, these students, who mostly had low and medium achievement levels, had difficulties in continuing the pattern to the 24^{th} step because they have mainly focused on the recursive relationship. In this case, they tried to use the multiplicative relation between six and 24; and they incorrectly used proportional reasoning by attempting to apply a similar reasoning to the terms corresponding to these steps.

In the process of generalizing, middle school students have made verbal, arithmetic, visual and algebraic conjectures that were not proven yet. While making conjectures, they guessed that the given hypothesis might be true and they attempted to show their statements via exemplifying and testing and especially in geometry by trial/error, ratio and formulation based on the prototype shape, which was the concept with maximum examples.

Nearly all students who argued that "*The sum of three consecutive numbers is divisible by three*" attempted to verify this statement by giving random examples. Only three students with high and low achievement levels checked the validity of their conjectures by selecting particular cases. In this process, students used inductive, analogic, and abductive types of reasoning. In addition, it has been observed that some students have made erroneous reasoning or authoritative reasoning, such as teacher, textbook, which were considered as other reasoning. For example, a 6^{th} grade student with a medium achievement level exhibited an inductive approach, "*First, I have to give some examples, my numbers are 4+5+6=15, 15 is divisible by 3, 7+8+9=24, is divisible by three, therefore the answer of this question is yes it is divisible*", by forming a rule by considering some particular cases.

In addition to inductive reasoning, some students have been observed to make conjectures using an analogic approach by considering the similarities of two cases or using an abductive approach that is related to inductive and deductive reasoning, which is the observation of a specific case that leads to a familiar result. To give examples, "*…10, 12, 14 yes it can be divided. One of these is divisible by three. The others are complement of three. Their arithmetical mean is 12 …*" *(M_6)* or "*I'll check if 12, 13, 14, are divisible by three. For example 12, 1+2=3, it is divisible. 13, 1+3=4, it isn't. 14, 1+4=5, it cannot be divided. …3+4+5=12, it is divisible*" *(H_6).*

In addition to students who make conjectures through exemplification, some students, especially those with low or medium achievement level, have given unsatisfactory answers, either erroneously or by trial/error, in the process of investigating the verification of their statements. For example, in the case of geometry questions, some students attempted to make conjectures using prototype shapes, or they referred to an authority to verify their statements.

The seventh grade student with medium achievement level attempted to obtain the general formula of the given pattern by trial/error, which is a deductive approach "*now, 8, 12, 16, 20 …* (he wrote the number of squares of each step), *at the 6^{th} step, first we can say n+4, no n+4 doesn't work …*". On the other hand, some of the students who focused on the difference between terms also investigated the rule of the pattern by trial/error. First, they multiplied the common difference by n (4n), then they found the constant added to 4n using the number of the first step (8) and generated the rule of the pattern through an abductive approach. 10 students, who were experiencing medium or high achievement levels about the hypothesis in the areas of numbers and operations, algebra, geometry and learning to measure, have attempted to express their conjectures mathematically using verbal, arithmetic, visual and algebraic generalizations. In this process, they have used inductive, deductive and abductive types of reasoning. For example, a 7th grade student with medium achievement level mathematically expressed that "*Based on the identical angles given on line I, the sum of two angles is 90 degree*" as shown in Figure 4, by making algebraic generalizations with a deductive approach.

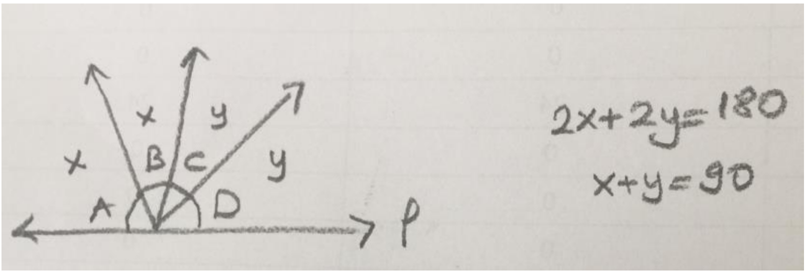


Figure 4. The approach of a 7th grade student with medium achievement level

Similarly, an 8th grade student with high achievement level explained his conjecture about "*In a rectangle, the length of the diagonals are equal*" verbally as "*since long and short sides of a rectangle are equal … I use Pythagoras. The square of this (DB – diagonal) is the square of (DC) plus the square of (BC); the square of this (AC – other diagonal) is the square of (AD) plus the square of (DC).*"

Evidences supporting mathematical statement

As can be seen from Table 2, in the process of proving a mathematical expression, middle school students have performed three types of actions, namely verification, explanation and abstraction; moreover, they have formed some arguments, which were not accepted as proof, by offering experimental, intuitive or illogical justification.

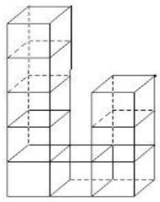
During the proving, students with medium or high level of achievement have first investigated the verification of the hypothesis arithmetically, algebraically and geometrically/visually, and then explained why it is true, thus the reasoning types that they have selected and used in this process were deduction and abduction. Finally, they made the abstraction by using mathematical language and checking the conditions of the generalization through the shortest path. For example, in the geometry question displayed in Figure 2, a 6th grade student with a high achievement level has made a mathematically valid proof without using any arithmetical variable, in other words without using direct variables, to verify that the sum of B and C angles is 90 degrees through a deductive approach: "*Since A is equal to B, and C is equal to D, and A, B, C, D is equal to 180, let's consider these two as a group (A and B), (C and D). We have two groups and if we divide 180 by two and we take one element from each group, they* *are 90. For example, let's say A is 50, B is also 50, their sum is 100, the others are 40, 40 from 80. The sum of 50 and 40 is 90. I mean, whatever we assign to A, it will be 90. Whatever."*

On the other hand, six of the 7th and 8th grade students with medium and high achievement have made algebraic demonstration through a deductive approach by using variables, whereas five students have made geometrical and visual demonstration by using geometric shapes or visual representations and in this process they have made abstraction using mathematic language. An 8th grade student with a high achievement level has algebraically shown why three consecutive numbers are divisible by three, as displayed in Figure 5.

$$\frac{x+x+1+x+2}{3}=\frac{3x+3}{3}=\frac{\cancel{3}x}{\cancel{3}}+\frac{3}{3}=x+1$$

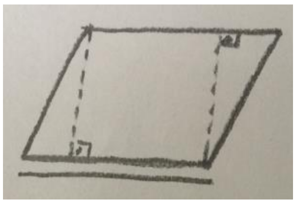
Figure 5. The approach of an 8th grade student with high achievement level

Similarly, for calculating the surface area of the geometric shape given below, an 8th grade student with medium achievement has first counted the unit cubes and overlapping surfaces, then he attempted to algebraically prove the surface area formula given for the shape.



The seventh grade student with medium achievement has demonstrated the formula of the area of parallelogram by using the area of rectangle through geometric proof as below:

M_7: …This parallelogram (he sketched), I've got a right triangle when I drew the height like this. If I move this part there, this portion of the base (the base of the triangle) *will move here, meaning that the base will not be changed. Then we get a rectangle. For the area, we will multiply the sides, and we will find the area of the rectangle. This is how this happens for the parallelogram.*

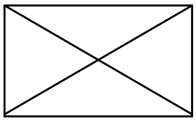


On the other hand, middle school students have had some arguments that could not be considered as proof. As can be seen from Table 2, these arguments were classified as experimental, intuitive and illogical justification. While performing a verification or explanation, students from all grade and achievement levels tended to apply exemplification or trial/error methods first; however, some students with low and medium achievement levels followed the wrong direction.

Table 2.
Middle School Students' Processes of Proving Mathematical Expressions as Justification

					6.-7.-8. Grade Level
Justification of the Hypothesis	*Evidences that Support Mathematical Statement*	*Proving a Statement* Verification, Explanation and Abstraction (Using Mathematic Language)	*Type of Proof*	Arithmetic	M(2), H(2)
				Algebraic	M(3), H(3)
				Geometric Visual	M(3), H(2)
		Non-proof Argument	Experimental	Verification by Example	L(6),M(5),H(4)
				Trial/Error	L(2),M(2),H(2)
				Ratio/ Proportion	L(3),M(2),H(1)
				Dynamic Thinking	M(1)
			Intuitive		L(1), M(1)
			Presenting Illogical Justification		L(6), M(2)
			Type of Reasoning		
Deduction *Testing		Induction	Other Reasoning *Referring to an authority *Erroneous reasoning		Abduction

To answer the question asking to prove that the diagonals of a rectangle are equal, two 8th grade students with low and medium achievement level acted intuitively and expressed it as: "*… The length of the diagonals starts here* (mutual corners). *If we turn it and D replaces C and C replaces D, diagonals would be the same … I mean D will replace C. I reverse it.*"



As can be seen from Table 2, in all grade levels, students with low achievement level have presented illogical justifications and they attempted to justify their answers by referring to an authority or in an erroneous manner.

Middle School Teachers' Process of Reasoning and Proving Mathematical Statements

Making generalizations

As shown in Table 3, it has been observed that the thinking structure of middle school mathematic teachers was similar to their students while proving a given mathematical expression. Teachers' acts of this process are pattern recognition, seeking the relationship between two variables and making conjectures. For example, to solve the pattern question, both teachers transformed the shape pattern into a numeric pattern without analyzing the shape and generalized the pattern to the next step through an inductive approach, by focusing on the difference between terms, in other words using the recursive relationship.

When teachers were asked to extend the pattern to the further steps (24th step); the junior teacher declared that he can extend the pattern by using a formula, "*… I can find it by using a formula I mean I can find it by writing the formula of the pattern … I'm trying to memorize the general formula … we had such a formula … an=a_1+ (n-1).r, a_1 is the first term; I put four as the common difference. If I take eight for a_1 and four for r, we find 8+ (1-1).4=8 at the first step, 8+ (2-1).4= 12 at the second step. I can find 24th step using this … an=8+ (24-1).4=8+23.4=8+92=100*", and he algebraically generalized the functional relationship between two variables through a deductive approach, by using arithmetic series rule. When the teacher was asked to analyze the shape, he could only generate the functional relationship visually after analyzing the structure of the shape through an inductive way.

The teacher with more professional experience has generalized the pattern by focusing on a recursive relationship to extend the pattern to further steps, thus he made a conjecture as below by using trial/error with an abductive approach:

Experienced Teacher (ET): … for the rule of the pattern, we find the difference between them, 4n here. Now, at the first step, we have a total of eight squares, we write one for n, we get four, therefore I have to add four to obtain eight. For 24th step, we will write 24 for n. We find 4.24+4= 100.

Similarly, when this teacher was asked to analyze the structure of the shape, he examined the shape and generalized the functional relationship arithmetically by approaching the shape through an inductive approach: "*In each step, here we have five* (squares) *at the third stage (*upper side)*, then three, three* (left and right sides)*. Let's think in this way. At the 3rd step, there are three squares in each side, whereas at the 4th step the number of squares is four. Four times four is 16, when we add corners it makes 20. At the 6th step there will be six inside. Six times four, 24, plus four from the corners, it makes 28."* Both teachers have made mathematical algebraic conjectures with a deductive approach while generalizing the given mathematical explanations as seen in Table 3.

Table 3.
Middle School Mathematic Teachers' Processes of Generalizing Mathematical Expressions

Justification of the Hypothesis	*Generalization*			*Type of Generalization*		*Professional Experience*
		Pattern Recognition	Converting to an arithmetic pattern		Arithmetic	ET (Experienced Teacher)
						LET (Less Experienced Teacher)
		Seeking the Relationship Between Two Variables	Recursive Relationship		Arithmetic	ET, LET
			Functional Relationship		Algebraic/ Visual/ Arithmetic	ET, LET
		Making Conjectures	Mathematical Conjectures		Algebraic	ET, LET
			Random Example		Arithmetic	ET
			Trial/ Error		Arithmetic/ Algebraic	ET

Type of Reasoning			
Deduction	Induction	Abduction	Other Reasoning
	*From specific to general		*Erroneous reasoning

However, less-experienced teachers made an erroneous reasoning while making conjectures about "*The sum of three consecutive numbers is divisible by three*" and he stated that the sum of all consecutive numbers, except -1, 0, +1, can be divided by three. On the other hand, the experienced teacher has tried to show his statements with inductive and abductive approaches, such as giving examples, testing, and trial/error, and with various arithmetical and algebraic actions while making conjectures about some hypotheses.

Evidences supporting mathematical statements

In the process of proving a mathematical expression, middle school mathematic teachers have performed three types of actions, namely verification, explanation and abstraction; moreover, they have formed some arguments which were not accepted as proof (see Table 4). While proving, they have explained why the statement is true, and they have made abstraction by selecting algebraic, geometric and visual evidences and by using a deductive approach. Both teachers gave similar answers to the questions in the areas of numbers and operations, geometry and learning to measure and they have proven their statements by using mathematic language.

Table 4.
Middle School Mathematic Teachers' Processes of Proving Mathematical Expressions

				Professional Experience
Evidences that Support Mathematical	*Proving a Statement* Verification, Explanation and Abstraction (Using Mathematic Language)	*Type of Proof*	Algebraic/ Geometric/ Visual	ET (Experienced Teacher) LET (Less Experienced Teacher)
			Trial/Error	ET
	Non-proof Argument	Experimental	Dynamic Thinking	ET, LET
			Verification by Example	LET
		Type of Reasoning		
	Deduction	Abduction	Other Reasoning	
	*Testing		* Erroneous reasoning	

On the other hand, the less experienced teacher showed and explained the hypothesis, "*If the sides of a rectangle are doubled, its area increases by four times*" algebraically, with a deductive approach as follows: "*The easiest way of proving it is drawing a rectangle* (he drew one). *Now, I do it or show it to my pupils with multiple variables. I'll call long side as a and short side as b. Thus, the area of the rectangle is A=a. b. If I double both sides* (he drew another rectangle) *this side becomes 2a, and this one becomes 2b and … A=2a.2b … when I multiply it makes 4ab. We have calculated the area of the first rectangle as ab, since the second one was found to be 4ab, I did prove that its area was increased by four times. Whatever number we use for a and b, it will always be four times bigger.* On the other hand, the more experienced teacher explained the same hypothesis using "*the ratio of the areas of similar shapes is equal to the square of their similarity ratio;*" however, he showed its verification by assigning numbers. The verification and explanation of this teacher can be considered as an argument that is non-proof because the teacher did not completely use mathematic language while verifying the hypothesis. For instance; "… (he drew two rectangles with sides three, four and six, eight) *the ratio of the areas is equal to the square of their similarity ratio. Their similarity ratio is two; the square of two is four. Let's call similar rectangles as A_1 and A_2, $A_1/A_2=k^2$. We calculated A_1 as 48, and A_2 as 12, the ratio of these is four … let's find the similarity ratio, 8/4=2, the square of two is equal to four.*"

Regarding the pattern question, the less experienced teacher has generalized the rule of the pattern by using the arithmetic series formula and then he has proven the validity of the rule visually; however, he tested both rules by assigning numbers to the variables. On the other hand, as explained in the generalization part, the experienced teacher has found the rule as 4n+4 with an abductive approach by using the common difference between terms through trial/error, and he attempted to test the validity of the rule by assigning numbers and also by visually examining the shape and using the inference that he has revealed.

Finally, while verifying the hypothesis where the area of a parallelogram was questioned, both teachers attempted to show the area of the parallelogram experimentally, using the area of a rectangle (similar to some students).

Discussion and Conclusion

It has been observed that most of the students (regardless of the grade) had difficulties in determining the functional relationship between two variables (Zazkis & Liljedahl, 2002; Becker & Rivera, 2006). The few students who have identified a functional relationship were those of medium or high achievement levels. Even though teachers did not encounter similar problems, it has been noticed that their approaches were also similar to the ones of their students and seeking the functional relationship was not among their first choices within the process of generalization. Therefore, it can be said that the students' tendency towards arithmetical generalization rather than algebraic and visual generalization may be a result of this fact.

At the same time, it has been observed that nearly all students have made illogical conjectures such as giving examples of trial/error, and these students are generally with low or medium achievement levels (Aylar, 2014). On the other hand, the presence of the students, who made mathematically meaningful conjectures and algebraic generalization, is also important. These students are generally from 7th and 8th grades, with medium and high achievement levels, which can be interpreted as reaching a generalized conclusion is apprehended with the increase of grade and achievement level (Knuth & Sutherland, 2004). Regarding the teachers, it has been observed that while making conjectures, the first priority of the teacher with more professional experience was giving examples; thus the same tendency was observed in his students as well. Considering that proving is the next step in making conjecture, it can be said that this teacher believes that showing with an example is a valid proof and reflects this idea to his students as well.

It has been observed that in the process of generalization, teachers and students predominantly preferred inductive reasoning (Harel, 2001); however deductive, analogical and abductive types of reasoning were also chosen and used. There are, though, students making erroneous reasoning. It should be noted that students' ability to use inductive and deductive reasoning from an early age is especially important for the development of proving skills (NCTM, 2000).

Regarding proving related to supporting a mathematical statement, it has been observed that most of the students were not at the desired level in terms of verifying, explaining and abstracting the hypothesis (regardless of the grade); they were generally making verification based on experimental arguments, especially with the help of the examples; in this process, they have mostly used inductive or erroneous reasoning and sometimes they have referred to an authority for justification. There are many studies supporting this fact (Reid & Knipping, 2010; Stylianides & Stylianides, 2009; Knuth & Sutherland, 2004; Knuth, Slaughter, Chooppin, & Sutherland, 2002). Knuth et al. (2002) stated that students might have used experimental arguments as proofs because their teachers have directed them as they could use well-chosen examples for this purpose. Hence, the teacher with more professional experience has preferred to give examples while proving, which seems to have triggered this fact. On the other hand, it has been observed that students' achievement levels and their proving and reasoning skills are correlated (regardless of the grade). Students with high achievement level can make arithmetic, algebraic, geometric/visual proofs and they can think deductively. This result is similar to some research findings (Arslan, 2007).

Students have encountered difficulties in some areas, especially in algebra and geometry, even though their tendency to reach a generalization depends on their grade level, and they cannot carry out algebraic proofs. This fact can be one of the factors that affect their proving performance negatively. These outcomes are in line with the results of some studies (Aylar, 2014). In addition, students are unfamiliar with the terminology of proving, in other words, the use of mathematical language. Some of the reasons for this might be as follows: students don't know what it means to convince someone or how to do it; class discussion might be ignored or students might not be allowed to talk during the lessons. Therefore, it is evident that the approaches that teachers apply in the classroom influence the reasoning and proving skills of the students. Hence, the verification, explanation and abstractions of the students and teachers were similar and correlated, which is an indicator of this fact.

It has been observed that teachers' tendencies to make proofs was mostly at the level of verification and explanation. Thereby, it can be said that teachers are not at the desired level in terms of verifying, explaining and abstracting the hypothesis (regardless of the experience) (Jones, 2000; Knuth, 2002; Iskenderoglu & Baki, 2011). It is known that teachers' views, beliefs and knowledge affect students' proving abilities (Knuth, 2002), which makes this result challenging.

As a result, it can be said that students encounter difficulties in proving mathematical statements. Since showing the verification of a mathematical proof using examples seems to be a valid proof for them, they prefer to use experimental evidences in this process. The reason leading to this situation is that teachers don't know what a proof means and what is needed to make a proof. Thereby, teachers tend to teach existing proofs instead of making them.

We presented a number of future research directions based on the results obtained from this research. Firstly, proof should be included within the natural flow of the mathematic teaching process and be placed at the center of the mathematical content without being considered a separate field. The proof activities can be used as a tool in all content domains; the purpose of proving and its significance for mathematics should be underlined. The importance of proof should be highlighted starting from the early years and experimental arguments should not be accepted as a proof at any grade level. In addition, we can explore how mathematics textbooks and education programs support the standards of reasoning and proving for each grade level. Moreover, considering that students predominantly tend to make inductive reasoning, they should be engaged in the activities requiring deductive reasoning. On the other hand, teachers should be involved in discussions where students' proofing skills are deeply discussed. In teacher education, the purpose of the proof and its mathematical significance should be explicitly emphasized; more importance should be assigned to the instruction of proof. A similar study examining teachers' and students' reasoning and proving tendencies in secondary education should be conducted.

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Öğrenciler Öğrendiklerini Öğretmenler Öğrettiklerini Nasıl Kanıtlar? : Öğretmen Bir Fark Yaratır mı?

Atıf:

Tanisli, D. (2016). How do students prove their learning and teachers their teaching? Do teachers make a difference? *Eurasian Journal of Educational Research*, 66, 47-70, http://dx.doi.org/10.14689/ejer.2016.66.3

Özet

Problem Durumu: Erken yaşlardan itibaren muhakeme becerisinin kazanımı daha sonraki yıllarda formel anlamda kanıt yapma becerisini etkileyeceğinden oldukça önemlidir. Bu becerinin kazanımı süreçte ancak öğretmenlerin kullandıkları yaklaşımlar ile mümkündür. Öğretmenlerin kanıt yapmak için gerekli olan matematiksel bir iddiayı doğrulama ya da çürütme sürecinde yapılması gerekenleri hazır olarak sunmak ve bir başka durumda öğrencilerden de benzer mantığı uygulamalarını beklemek yerine öğrencilerin düşünme becerilerini geliştirecek, nasıl ve nedenin sorgulandığı, tartışıldığı zengin ortamlar hazırlamaları gereklidir. Bu noktada Türkiye'de öğretim programlarının yeniden yapılanması ile birlikte, ortaöğretim öncesi öğrencilerinin kanıt yapma bağlamında, matematiksel bir ifadeyi nasıl kanıtladıkları, bu süreçte hangi muhakeme ve kanıt türlerini kullandıkları, bu öğrencilerin öğretmenlerinin de aynı ifadeyi nasıl kanıtladıkları ve öğretimlerine nasıl yansıttıkları araştırılması gereken bir problem olarak görülmektedir.

Araştırmanın Amacı: Bu araştırmanın amacı, ortaokul öğrencilerinin ve öğretmenlerinin verilen matematiksel ifadelere ilişkin muhakeme etme ve kanıtlama süreçlerini belirlemektir. Araştırmanın, ortaokul düzeyinde öğrencilerin ve öğretmenlerinin kanıt yapma bağlamında matematiksel bir ifadeyi nasıl kanıtladıklarına, bu süreçte öğrencilerin yaşadıkları zorluklara aynı zamanda öğretmenlerin ve öğrencilerin muhakeme etme ve kanıtlama süreçleri aralarındaki

ilişkiyi belirleyerek öğretmenlerin de bu süreçteki rollerine dikkat çekme açısından önemli olduğu söylenebilir.

Araştırmanın Yöntemi: Bu çalışmada temel nitel araştırma yaklaşımı benimsenmiştir. Çalışmanın katılımcılarını farklı mesleki deneyimlere sahip 2 ortaokul matematik öğretmeni ile bu öğretmenlerin 6., 7., 8. sınıfına devam eden ve her sınıftan üç öğrenci olmak üzere toplam 18 öğrenciden oluşturmaktadır. Zengin bilgiye sahip olduğu düşünülen durumlar üzerinde çalışma olanağı verdiğinden, bu çalışmada amaçlı örnekleme yöntemi çeşitlerinden 'ölçüt örnekleme' kullanılmıştır. Öğretmenlerin çalışma süreleri (5 yıl ile 30 yıl), öğrencilerin başarı düzeyleri(yüksek, orta, düşük) örneklem ölçütü olarak belirlenmiş, gönüllülük esas alınmıştır.

Araştırma verilerinin toplanmasında nitel araştırma yöntemlerinden biri olan klinik görüşme tekniği kullanılmış ve görüşmeler video kameraya çekilmiştir. Verilerin analizinde tematik analiz yöntemi kullanılmıştır. Verilerin analizi yapılırken öncelikle başlangıç kodları iki alan uzmanı tarafından bağımsız şekilde belirlenmiş ve araştırmacılar bir araya gelerek belirlenen kodları karşılaştırmıştır. Kodlar konusunda görüş birliğine varıldıktan sonra temaların oluşturulması için araştırmacılar yeniden önce bağımsız sonra birlikte çalışarak temaların da tutarlı olmasını sağlamışlardır. Kodlar ve temaların oluşturulması sürecinde iki araştırmacı arasında görüş birliğine varılarak ana temalar ve alt temalar belirlenmiştir. Daha sonra ayrıntılı bir biçimde tanımlanan ve adlandırılan tema ve alt temalar yorumlanmıştır.

Araştırmanın Bulguları: Araştırmada ortaokul öğrencileri matematiksel bir ifadeyi doğrularken belli sayıdaki adımlardan hareketle iddia hakkında karar vermeye çalışmışlar ve bu bağlamda örüntü tanımlama, iki değişken arasındaki ilişkiyi arama ve varsayımda bulunma şeklinde eylemler gerçekleştirmişlerdir. Verilen matematiksel ifadeleri genelleme sürecinde ise henüz kanıtlanmamış aritmetiksel, sözel, görsel, cebirsel çeşitli varsayımlarda bulunmuşlardır. Varsayımda bulunurken verilen önermelerin doğru olabileceğini tahmin ederek, iddialarını örnek verme ve test etme, özellikle geometride kavramı temsil eden en fazla örnek olma özelliğine sahip prototip şekle dayalı olarak, deneme/yanılma, oran/orantı ve formüle etme gibi çeşitli eylemlerle göstermeye çalışmışlardır. Bu süreçte öğrenciler tümevarım, analojik, geri çıkarım muhakeme türlerini kullanmışlardır. Yanı sıra bazı öğrencilerin de hatalı ya da öğretmen, ders kitabı gibi bir otoriteyi referans göstererek muhakeme yoluna gittikleri gözlenmiştir. Matematiksel bir ifadenin kanıtlanması sürecinde ise öğrenciler doğrulama, açıklama ve soyutlama olmak üzere üç eylem gerçekleştirmişler yanı sıra deneysel, sezgisel ya da mantıklı olmayan gerekçeler sunarak kanıt kapsamına alınmayan argümanlar oluşturmuşlardır. Kanıtlama sırasında genel olarak da orta ve yüksek başarı düzeyine sahip öğrenciler öncelikle bir önermenin doğruluğunu aritmetik, cebirsel ve geometrik/görsel olarak araştırmışlar daha sonra neden doğru olduğunu açıklayarak bu süreçte genel olarak tümdengelim ve geri çıkarım muhakeme türlerini seçme ve kullanma eylemlerini gerçekleştirmişlerdir. Diğer taraftan matematiksel bir iddiayı kanıtlarken ortaokul öğrencilerinin kanıt olarak ele alınamayan argümanları da söz konusu olmuştur. Bu argümanlar deneysel, sezgisel ve mantıklı olmayan gerekçeler şeklinde ele alınmıştır. Tüm sınıf ve başarı düzeyinden öğrencilerin doğrulama ve açıklama yaparken öncelikle ağırlıklı olarak örnek verme ya da deneme/yanılma yoluna gittikleri, yanı sıra genel olarak düşük ve orta başarı düzeyinden bazı öğrencilerin de doğrulama yaparken hatalı yol izledikleri görülmüştür. Özellikle tüm sınıf düzeylerinde düşük başarı düzeyine sahip öğrenciler kanıtlama yaparken mantıklı olmayan gerekçeler sunmuşlar ve bu süreçte hatalı ya da bir otoriteyi referans göstererek gerekçelerini savunmaya çalışmışlardır. Diğer taraftan ortaokul matematik öğretmenlerinin verilen matematiksel bir ifadeyi doğrularken öğrencileri ile benzer düşünme yapılarına sahip oldukları gözlenmiştir. Öğretmenler bu süreçte örüntü tanımlama, iki değişken arasındaki ilişkiyi arama ve varsayımda bulunma şeklinde eylemler gerçekleştirmişlerdir. Verilen tüm matematiksel ifadeleri genelleme sürecinde her iki öğretmen tümdengelim bir yaklaşımla cebirsel olarak matematiksel varsayımlarda bulunmuşlardır. Matematiksel bir ifadeyi kanıtlama sürecinde ise doğrulama, açıklama ve soyutlama olmak üzere üç eylem gerçekleştirmişler yanı sıra deneysel gerekçeler sunarak kanıt kapsamına alınmayan argümanlar da oluşturmuşlardır. Kanıtlama sırasında iddiaların neden doğru olduğunu açıklayarak cebirsel, geometrik ve görsel kanıt türlerini seçerek ve tümdengelim bir yaklaşım kullanarak soyutlama yapmışlardır. Ancak öğretmenlerin de deneyimleri fark etmeksizin matematiksel ifadeleri doğrulama, açıklama ve soyutlama boyutunda istenilen düzeyde olmadıkları söylenebilir.

Araştırmanın sonuçları ve öneriler: Araştırma sonucunda, öğrencilerin matematiksel bir iddiayı kanıtlarken zorlandıkları, süreçte deneysel kanıtları kullanmayı tercih ettikleri ve daha çok tümevarım yaklaşımını benimsedikleri görülmüştür. Diğer taraftan öğretmenlerin ise genel olarak kanıt yapma eğilimlerinin daha çok doğrulama ve açıklama düzeyinde yer aldığı ve matematiksel ifadeleri kanıtlama sürecinde öğrencileri ile benzer düşünme yapılarına sahip oldukları belirlenmiştir. Sonuç olarak, öğrenciler matematiksel bir iddiayı kanıtlarken zorlanmakta, süreçte deneysel delilleri ve deneysel kanıtları kullanmayı tercih etmektedirler. Çünkü matematiksel bir ifadenin doğruluğunu örnek kullanarak göstermek onlar için geçerli bir kanıt anlamına gelmektedir. Bu durum öğretmenlerin kanıtın ne anlama geldiğini, kanıt yapma için neye gereksinim olduğunu bilmemelerinin bir sonucudur. Dolayısıyla öğretmenler kanıt yapabilmeye değil, var olan kanıtları öğretmeye eğilimlidir.

Bu bağlamda araştırma sonuçlarına dayalı olarak şu öneriler getirilebilir. Öncelikle muhakeme ve kanıt matematik öğretiminin doğal akışı içine dâhil edilmelidir. Ayrı bir konu alanı olarak ele alınmadan matematiksel içeriğin merkezine konulmalıdır. Aynı zamanda öğrencilere kanıt yapma etkinliklerinin her öğrenme alanında araç olarak kullanılabileceği vurgulanmalı, kanıtın amacının ve matematik için öneminin altı çizilmelidir. Öğrencilerin çoğunlukla tümevarım muhakemeyi kullanmaya eğilimli oldukları göz önüne alındığında ise, tümdengelim muhakemeyi gerektiren etkinliklerle çalışmaları sağlanmalıdır. Öte yandan deneysel argümanlar hiçbir sınıf seviyesinde kanıt olarak kabul edilmemelidir. Öğretmenlerin birincil kaynaklarının ders kitapları ve öğretim programları olduğu dikkate alındığında yapılacak araştırmalar bağlamında her sınıf düzeyi için matematik ders kitaplarının ve öğretim programlarının muhakeme ve kanıt standartlarını ne kadar desteklediği incelenebilir.

Anahtar Sözcükler: Matematik eğitimi, genelleme, varsayımda bulunma, muhakeme ve kanıt.