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# THE GENERALIZED COHOMOLOGY THEORIES, BRUMFIEL-MADSEN FORMULA AND TOPOLOGICAL CONSTRUCTION OF BGG-TYPE OPERATORS

**Cenap ÖZEL**[1]

***ABSTRACT***

In this work, we investigate the topological construction of BGG-type operators, giving details about complex orientable theories, Becker-Gottlieb transfer and a formula of Brumfiel-Madsen.We generalize the BGG operators on the Morava K-theory and the others $F_p$ generalized cohomology theories.

**Key words:** Generalized cohomology theories, Bernstein-Gelfand-Gelfand operators.

## GENELLEŞTİRİLMİŞ KOHOMOLOJİ KURAMLARI, BRUMFIEL-MADSEN FORMÜLÜ VE BGG-TİPİ OPERATÖRLERİN TOPOLOJİK İNŞASI

***ÖZ***

Bu çalışmada, kompleks yönlendirelebilinir kuramlar, Brumfiel-Madsen formülü ve Becker-Gottlieb dönüşümü hakkındaki detayları vererek, BGG- tipi operatörlerin topolojik inşasını araştırıyoruz.BGG operatörlerini, Morava K-kuramı ve diğer $F_p$-genelleştirilmiş kohomoloji kuramlarına genelleştiriyoruz.

**Anahtar Kelimeler :** Genelleştirilmiş Kohomoloji Kuramları, BGG operatörleri.

## 1. INTRODUCTION

In this work, we will discuss the generalized complex-oriented cohomology theories of the flag space G/B, and the classical BGG and Kac operators will be constructed topologically using the transfer map for compact fibre bundles. Also we will generalize the BGG operators on the Morava K-theories.

In order to do this, in the first section we will give some topological notations.

In the second section, we will discuss the Becker-Gottlieb map and transfer map for a fiber bundle $\pi : E \rightarrow B$ with the fiber F, which is a compact differentiable G-manifold for a compact Lie group G.

In the third section, we will examine the Brumfiel-Madsen formula for the transfer map.

In the last section, we will give the main result of this work. Note that these results grew out a chapter of the author's thesis (Özel, 1998).

## 2. TOPOLOGICAL PRELIMINARIES

The general reference for this section is (Adams, 1974).

### 2.1. Generalities on Generalized Cohomology.

A generalized cohomology theory $h^*$ () is a contravariant functor from topological spaces to graded abelian groups which satisfies all the Eilenberg-Steenrod axioms except the dimension axiom. That is, the coefficients $h^* = h^*(pt)$ need not be concentrated in a single degree. We will always assume that $h^*$ is multiplicative, and that the associated ring structure is commutative in the graded sense. Then for a topological space X, $h^*(X)$ is a $h^*$ -module. The first example is ordinary cohomology with coefficients in $\mathbb{Z}$.

We take $H^i(X) = H^i(X,\mathbb{Z}) = [X, K(\mathbb{Z},i)]$ where $K(\mathbb{Z},i)$ is an Eilenberg Maclane space, and [X, Y]

1 Abant İzzet Baysal Üniversitesi, Gölköy Kampüsü, Bolu 14280; **E -mail:** cenap@ibu edu.tr
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denotes homotopy classes of based maps from X to Y for X and Y topological spaces with based points.

For a generalized theory $h^*()$, there is a sequence which computes $h^*(X)$ in terms of $H^*(X;h^*)$. This spectral sequence is called the *Atiyah-Hirzebruch spectral sequence*, and details can be found in (Adams, 1974).

**Theorem 2.1.** *There is a spectral sequence with $E_2$ term $H^p(X, h^q(pt)) \Rightarrow h^{p+q}(X)$. The differential $d_r$ is of bi-degree $(r, 1 - r)$.*

**Corollary 2.2.** *Suppose that X has no odd dimensional cells and $h^q(pt) = 0$ for q odd. Then the Atiyah Hirzebruch spectral sequence collapses at the $E_2$ term.*

Now we define reduced cohomology. Let i: pt $\to$X be the inclusion of a point and $\pi$ :X$\to$pt be the collapsing map. Then $\pi \circ i$ = id, so $i^* \circ \pi^*$= id on $h^*(pt)$. Let $h^*(X)$ = ker $i^*$ be the reduced cohomology of X. Then, as a $h^*$-module,

$$h^*(X) = \tilde{h}^*(X) \oplus h^*$$

## 2.2. Classifying Spaces.

In this section, we give some facts about the construction of universal bundles and classifying spaces of groups. The general reference for this sections is (Husemoller, 1975). Let G be a compact Lie group. There is a universal space EG with a free right G-action and $(\pi_i)(EG) = 0$ for all i >0. Moreover, EG is a limit of Stiefel manifolds with the inductive limit topology. For example, for G = U(n), the unitary group,

$$EU(n) = \lim_{m \to \infty} V_n(C^{n+m}),$$

where

$$V_n(C^{n+m}) = \frac{U(n+m)}{U(m)}.$$

is a Stiefel manifold. The classifying space BG is defined as EG/G. For G = U(n),

$$BG \cong \lim_{m \to \infty} G_n(C^{n+m}),$$

the Grassmannian manifold of n-planes.

We have the universal bundle (EG, p, BG) where $EG \xrightarrow{p} BG$ is the obvious projection map. Then BG has the following universal property.

**Theorem 2.3.** *Let $P \xrightarrow{p} B$ be a right G-principal bundle. Then there exist a unique (up to homotopy) classifying map f: B $\to$ BG such that $f^*(EG) \cong P$ as G-principal bundles over B.*

As a consequence,

**Corollary 1.4.** *BG is well-defined up to homotopy and classifies induced vector bundles.*

Let $P \xrightarrow{p} B$ be a right G-principal bundle. Then, if F is a f inite dimensional representation of G,E = P x $_G$ F is the associated vector bundle over B with structure group G, where

$$E = P \times_G F = P \times F / \sim$$

is the space obtained as the quotient of the product space Px F by the realition

$$(x, y) \sim (xt, t^{-1}y), t \in G, x \in P, y \in F.$$

**Theorem 2.5** *Let E $\to$ B be a vector bundle associated to the fibre F with structure group G. Then there exists f: B $\to$ BG with $f^*(EG \times_G F) \cong E$ as vector bundles over B.*

Consider the special case of the classifying space for a complex line bundle. The appropriate structure group is U(1), so the appropriate classifying space is BU(1). By the above construction,

$$BU(1) = \lim_{m \to \infty} CP^m = CP^\infty.$$

We know from (Husemoller, 1975) that

$$H^*(BU(1), \mathbb{Z}) = \mathbb{Z}[x],$$

where $\mathbb{Z}[x]$ is the graded ring of polynomials in one variable with coefficients in $\mathbb{Z}$ and degx = 2. Let

$$T = \prod_{i=1} U(1)$$

be a torus. Then,

$$BT = \prod_{i=1} BU(1),$$

and since $H^*(Bu(1), \mathbb{Z})$ is torsion-free, by the Kunneth formula, we have

$$H^*(BT, \mathbb{Z}) \cong \bigotimes_{i=1}^{l} H^*(BU(1), \mathbb{Z}) \cong \mathbb{Z}[x_1, ..., x_l],$$

where $\mathbb{Z}[x_1, ..., x_l]$ is the graded ring of polynomials in $l$ variables with coefficients in the ring $\mathbb{Z}$.

## 2.3. Complex Orientable Cohomology Theories.

We follow (Adams, 1974) in this discussion.

Let i: $CP^1 \to CP^\infty$ = BU(1) be the inclusion.

**Definition 2.6.** *We say that the multiplicative cohomology theory $h^*$ is complex oriented if there exists* *a class* $x \in \tilde{h}^*(CP^\infty)$ *such that* $i^*(x)$ *is a generator of* $\tilde{h}^*(CP^1)$ *over the ring* $h^*(pt)$*. Such a class x is called a complex orientation.*

$\tilde{h}^*(CP^1) \cong \tilde{h}^*(S^2)$ is generated by one element over $h^*(pt)$.

As an example, if $h^* = H^*$, then x can be taken as a ring generator of $H^*(CP^\infty, \mathbb{Z})$, so $x \in H^2(CP^\infty, \mathbb{Z})$. $CP^\infty$ has a universal line bundle $L_\lambda$ given as follows. Let $e^\lambda$ be the one-dimensional representation of $T = U(1)$ given by

$$e^\lambda(e^{i\theta}) \cdot v = e^{i\lambda\theta} \cdot v,$$

where $\lambda \in$ Lie (T) is a fundamental weight. Then, for a complex orientable theory $h^*$ with orientation given by x, the first Chern class is given by $x = c_1(L_\lambda)$, where $L_\lambda$ is the line bundle associated to $e^\lambda$. Let T be an l-dimensional torus.

**Theorem 2.7.** *With the above notation, we have isomorphisms of graded* $h^*$*-algebras*

$$h^*(CP^\infty) \cong h^*(pt)[[x]],$$

$$h^*(BT) \cong h^*(pt)[[x_1, \ldots, x_l]],$$

$$h^*(CP^n) \cong h^*(pt)[[x]]/(x^{n+1}),$$

$$h^*\left(\prod_{i=1}^{l} CP^{n_i}\right) \cong h^*(BT)/(x_1^{n_1+1}, \ldots, x_l^{n_l+1}).$$

Now let $\pi: L \to X$ be a line bundle over X. Then L induces a classifying map $\theta: X \to CP^\infty$. Then the first Chern class of L is $c_1(L) = \theta^*(x)$. Next we define the top Chern class of a vector bundle.

**Definition 2.8.** *Let* $\pi: E \to X$ *be a vector bundle. If there is a space Y and a map* $f: Y \to X$ *such that* $f^*(X) \to h^*(Y)$ *is injective and* $f^*(E) \cong \bigoplus L_i$*, where* $L_i$ *are line bundles on Y, f is called a splitting map for* $\pi$*.*

From (Husemoller, 1975),

**Theorem 2.9.** *If* $\pi: E \to X$ *is a vector bundle, there exists a splitting map of* $\pi$*.*

Then,

**Definition 2.10.** *The top Chern class* $c_n(E)$ *where* $\dim E = n$*, which also will be referred as the Euler class* $\chi(E)$*, is defined by the formula*

$$f^*(c_n(E)) = \prod_i c_1(L_i),$$

*where f is a splitting map for* $\pi$*.*

## 2.4. Formal Group Laws.

Let F be a commutative ring with unit.

**Definition 2.11.** *A formal group law over F is a power series F(x, y) over F that satisfies the following conditions:*

*1.* $F(x, 0) = F(0, x) = x$,

*2.* $F(x, y) = F(y, x)$,

*3.* $F(F(x, y), z) = F(x, F(y, z))$,

*4. there exists a series* $i(x)$ *such that* $F(x, i(x)) = 0$

From (Ravenel, 1992), we have

**Theorem 2.12.** *In complex oriented theory, for line bundles L, M we have*

$$c_1(L \otimes M) = F(c_1(L), c_1(M))$$

*where F is a formal group law over the coefficient ring* $h^*$*.*

Now, we will explain this. A line bundle L over space X is equivalent to a homotopy class of maps $f_L: X \to CP^\infty$. Let L and M be two line bundles. Then we have.

$$f_L \times f_M: X \to CP^\infty \times CP^\infty.$$

$CP^\infty$ has an H-space structure $m: CP^\infty \times CP^\infty \to CP^\infty$. Then homotopy class of $m \circ (f_L \times f_M)$ is then equivalent to the tensor product $L \otimes M$. There is an induced map $m^*: h^*(CP^\infty) \to h^*(CP^\infty \times CP^\infty)$. Since, $h^*(CP^\infty) \cong h^*(pt)[[x]]$ and $h^*(CP^\infty \times CP^\infty) \cong h^*(pt)[[x_1, x_2]]$, $m^*$ has the form,

$$m^*(x) = \sum a_{ij} x_1^i x_2^j = F(x_1, x_2).$$

Then $c_1(L \otimes M) = F(c_1(L), c_1(M))$. As an example, if L and M are line bundles, we see in ordinary cohomology $H^*(\ )$ that

$$c_1(L \otimes M) = c_1(L) + c_1(M).$$

The complex cobordism $MU^*$ is the universal cohomology thory with respect to push-forwards. From (Adams, 1974),

**Theorem 2.13.** *The formal group law of* $MU^*$ *is the Lazard's universal formal group law.*

## 3. THE BECKER-GOTTLIEB MAP AND TRANSFER

The general reference for this section is (Becker and Gottlieb, 1975).

Let $\pi: E \to B$ be a fiber bundle with the fiber F, which is a compact differentiable G-manifold for a compact Lie group G. For any cohomology theory $h^*$

we have the induced map $\pi^*: h^*(B) \to h^*(E)$. A transfer map is a backward map $h^*(E) \to h^*(B)$. Here, we will give a technique for producing a transfer map.

**Definition 3.1.** *Let $\xi \to B$ be a vector bundle. Let $D(\xi) = \{x \in \xi : |x| \leq 1\}$ and $S(\xi) = \{x \in \xi : |x| = 1\}$ be the disk and sphere bundles respectively. Then, $B\xi = D(\xi) / S(\xi)$ is called the Thom space of the vector bundle $\zeta$.*

Now we give the useful propositions from (Husemoller, 1975),

**Proposition 3.2.** *If $\xi \to B$ is a trivial n dimensional vector bundle, then the Thom space $B\xi = \Sigma^n B^+$, where $B^+$ is the union of B with point.*

**Proposition 3.3.** *If $\xi$ and $\eta$ are two vector bundles over B, then $B\xi \wedge B\eta = B(\xi \oplus \eta)$.*

We define transfer for the map from the fiber F to point. We can embed F equivariantly into a real G-representation V of dimension r such that r >> dim F. Let $N \to F$ be the normal bundle of the embedding. By the tubular neigbourhood theorem, we can identify the normal bundle N with a neigbourhood U of F by a differomorphism $\varphi$. The is an associated Pontryagin-Thom collapsing map c: $S_v \to F_N$, where $S_v$ is the one point compactification of V, defined by

$$c(x) = \begin{cases} \text{base point of } F_N & \text{if } x \notin U, \\ \varphi(x) & \text{f } x \in U. \end{cases}$$

Let T(F) be the tangent bundle of F. Then we can identify $T(F) \oplus N$ with the trivial bundle F x V. There is an inclusion i: $N \to N \oplus T(F) \cong F \times V$ and hence we have an inclusion of Thom spaces i: $F_N \to S_V \wedge F^+$.

**Definition 3.4.** *The transfer $\tau$ to a point is the composition $\tau = i \circ c$.*

Let $\pi: E \to B$ be a fiber bundle associated to the principle G-bundle $p: P \to B$. Then the transfer to a point gives a map

$$Id \times \tau: P \times_G S_V \to P \times_G (F \times V)^+.$$

When we collapse the section at $\infty$ to a point, which is equivalent to taking Thom spaces, we get a map t: $B\zeta \to B\pi^*(\xi)$ where $\xi$ is a vector bundle associated to the representation V. Then there is a map $t \wedge Id: B\xi \wedge B\bar{\xi} \to B\pi^*(\xi) \wedge B\bar{\xi}$, where $\bar{\xi}$ is the complementary bundle of $\xi$. If we restrict to the diagonal $\Delta$, in B x B, we have transfer map

$$\tau: (\pi): \Sigma^m B^+ \to \Sigma^m E^+.$$

## 4. THE BRUMFIEL-MADSEN FORMULA FOR TRANSFER

The general reference for this section is(Brumfiel and Madsen, 1976).

Let G be compact connected semi-simple Lie group with maximal torus T. Let $W_G$ and$W_H$ be the Weyl groups of G and H respectively. Suppose that $P \to B$ is a principal G-bundle. We have associated bundles

$$\pi_1: E_1 = P \times_G G / T \to B$$

$$\pi_2: E_2 = P \times_G G / T \to B.$$

Then there is a fibration $\pi_1 \to E_2$ with the fiber H /T. Since the Weyl group $W_G$ acts on G /T, $W_G$ also acts on $E_1$. The Weyl group $W_H$ of H also acts on $E_1$over $E_2$. Thus, cosets $\omega \in W_G / W_H$ define maps $\pi \circ \omega$ on $E_1$.

**Theorem 4.1.** We have,

$$\pi_1^* \circ \tau(\pi_2)^* = \sum_{\omega \in W_G/W_H} \omega \circ \pi^*.$$

**Corollary 4.2.** *If we choose H = T, we get*

$$\pi_1^* \circ \tau(\pi_1)^* = \sum_{\omega \in W_G} \omega.$$

Although Brumfiel and Madsen were the first to assert that Theorem 4.1 is true, there seems to be some problem with the proof. Feshbach (1979), and Lewis, et.al. (1986) have given different proofs of Theorem 4.1. Since EG is the universal space for G, we have the principle bundle $EG \to BG$.

**Corollary 4.3.** *Let $BT \to BG$ be the fiber bundle with the fiber G/T. Then*

$$\pi^* \circ \tau(\pi)^* = \sum_{\omega \in W_G} \omega$$

For a compact semi-simple Lie group G, any root $\alpha$ defines a subgroub $M_\alpha = K_\alpha . T$ such that the complexified Lie algebra $m_\alpha$ contains the root spaces $g_\alpha$ and $g_{-\alpha}$ where $K_\alpha$ is introduced in (Kac, 1985).The induced fiber bundle $\pi_i: BT \to BM_i$ has fiber $M_i / T \cong SU_2 / T \cong CP^1$. Then

**Corollary 4.4.**

$$\pi^* \circ \tau(\pi_i)^* = 1 + r_{\alpha_i},$$

*if $r_{\alpha i}$ is the reflection to corresponding to the simple root $\alpha_i$.*

## 5. THE TRANSFER AND THE GYSIN HOMOMORPHISM

Let $\xi: E \to X$ be a vector bundle and $h^*$ be the complex oriented theory. Then there is the associated Thom class $u \in h^*(X\xi)$. From (Dold, 1976), we have

**Theorem 5.1.** *The Thom map $\Phi: h^*(X) \to h^*(X\xi)$ given by $\Phi(x) = u.\pi^*(x)$ is an isomorphism.*

Let $\pi: E \to B$ be a fiber bundle with compact smooth f-dimensional fiber F. Suppose that the tangent bundle $TF \to F$ is a complex vector bundle. Then we have the Gysin homomorphism $\pi_* : h^k(E) \to h^{k-f}(B)$. Since the tangent bundle T(F) has a complex structure, so does the tangents space along the fibers $T_\pi$. Hence, in the complex orientable theory $h^*$,$T_\pi$ has an Euler class, so $\chi(T_\pi) = c_n(T_\pi)$.

**Theorem 5.2.** *(see (Becker and Gottlieb, 1975) The transfer $\tau(\pi)^*: h^k(E) \to h^k(B)$. is given by*

$$\tau(\pi)^*(x) = \pi_*(x.\chi(T_\pi)).$$

Let $\alpha$ be the line bundle on BT assciated to the character $e^\alpha$ where $\alpha$ is a root. We want to determine when its characteristic classes are not zero divisors. We know that the characters $e^\alpha$ do not usually generate the representation ring R(T). Let $\lambda_\iota$ be the fundamental weight corresponding to the simple root $\alpha_\iota$ such that $\lambda_\iota(h_{\alpha\iota}) = 1$, where $h_{\alpha\iota}$ is the coroot. Then

**Theorem 5.3.** *(see Husemoller, 1975) These $e^{\lambda\iota}$ generate the representation ring R(T).*

By Theorem 1.7,

$$h^*(BT) \cong h^*(pt)[[c_1(L_{\lambda 1}), ..., c_1(L_{\lambda l})]]$$

where l is the rank of the compact Lie group G. Since $c_1(L_{\lambda i})$ are generators of $h^*(BT)$ , the $c_1(L_{\lambda i})$ are not zero-divisors in $h^*(BT)$. This implies that $c_1(L_{\lambda i})$ is not nilpotent. We know that for any weigt $\lambda \in h^*$,$\lambda$ can be written as

$$\lambda = \sum_{i=1}^{l} n_i\lambda_i ,$$

where $n_i$ is the multiplicity number. Using the formal group law in $h^*$, the Euler class $\chi(L_\lambda)$ of the line bundle $L_\lambda$ in $h^*$ is equal to

$$\sum_{i=1}^{l} n_i c_1(L_{\lambda i}) + \text{higher order terms}.$$

If $n_\iota$ is not a zero-divisor in $h^*(pt)$, then $\chi(L_\lambda)$is not a zero-divisor in $h^*(BT)$, If the weight $\lambda$ is a root corresponding to the adjoint representation, the multiplicity numbers $n_i$ in the sum are the Cartan integers. By an examination of the Cartan matrices, we have

**Proposition 5.4.** *If $p > 3$ is a prime, there is some $n_i$ such that p does not divide $n_i$.*

Proof. It follows from the classification of complex semi-simple Lie algebras. If p divides $n_i$ for all i, then p divides all entries in the Cartan matrix. By examination of Cartan matrices, we see that p = 2 or p = 3.

**Corollary 5.5.** *If $h^*(pt)$ has no 2-torsion and 3-torsion, then the Euler class $\chi(L_{\alpha_i})$ is not a zero- divisor for any simple root $\alpha_i$.*

Since every root is the image of a simple root by an element of the Weyl group $W_G$ and the Weyl group acts by automorphism on $h^*(BT)$, we have

**Corollary 5.6.** *If $h^*(pt)$ has no 2-torsion and 3-torsion, then the Euler class $\chi(L_\alpha)$ is not a zero- divisor for any root $\alpha$.*

Now we want to give the Brumfiel-Madsen formula for the Gysin map of the fibration $\pi: BT \to BG$ with the fiber G/T.

We need a comlex structure on G/T. We know that the smooth manifold G/T is diffeomorphic to the complexified space $G_C/B$ where B is a Borel group. Then we can determine the tangent bundle of the fiber $G_C/B$. The tangent bundle $T(G_C/B)$ is isomorphic to $G \times_T g/b$, where g is the complexified Lie algebra of G and b is the Borel subalgebra of g. Using the adjoint representation of T, we have

$$g = b \oplus \bigoplus_{\alpha\in\Delta^+} g_{-\alpha}$$

where $\Delta^+$ is the set of positive roots corresponding to B. Thus

$$g / b = \bigoplus_{\alpha\in\Delta^+} g_{-\alpha} .$$

Therefore the tangent bundle along the fiber G/T is

$$T_\pi = EG \times_T g/b \cong \bigoplus_{\alpha\in\Delta^+} L_{-\alpha} ,$$

where $L_{-\alpha}$ is as above. We know that

$$X^n(T_\pi) = \prod_{\alpha\in\Delta^+} c_1(L_{-\alpha}) ,$$

where $\prod$ is the cup product in any complex orientable theory $h^*$. By Theorem 5.2, we have

$$\pi^* o\, \tau(\pi)^*(x) = \pi_* o\, \pi_*(x..\chi(T_\pi))$$

for $x \in h^*(BT)$. Since $\chi(T_\pi)$ is a product of the non-zero divisors in $h^*(BT)$, we have

**Theorem 5.7.** (see (Bressler and Evens, 1990)) *For $x \in h^*(BT)$,*

$$\pi^* o\, \pi_*(x) = \sum_{\omega\in W} \omega\left(\frac{x}{\Pi\chi(L_{-\alpha})}\right) ,$$

*here the right hand side is in a localization $h^*(BT)\left[\frac{1}{\Pi\chi(L_{-\alpha})}\right]$.*

But since the left hand side preserves the subring $h^*(BT)$, it may be regarded as a identity on $h^*(BT)$,

**Corollary 5.8.** *If $\chi(L_{-\alpha})$ is a non-zero divisor, for the fibration $\pi_i : BT \to BM_i$ with the fiber $M_i/T$,*

$$D_i(x) = \pi_i^* \circ \pi_{i*}(x) = (1 + r_i)\left(\frac{x}{\chi(L_{-\alpha})}\right).$$

Let $h^*$be the ordinary cohomology with complex coefficients. From Chapter 1 of (Özel, 1998), we know that there is an isomorphism $\Theta$: $h^* \to H^2(BT, C)$ given by $\lambda \to \chi(L_\lambda)$ , where $h^*$ is the dual Cartan subalgebra of semi-simple Lie algebra.

The isomorphism $\Theta$extends to an inclusion of the symmetric algebra $R = S(h^*)$ into $H^*(BT,C)$. Then

$$H^*(BT,C) \cong C[\lambda_1, ..., \lambda_l]$$

under the identification $\chi(L_{\lambda i}) = \lambda_i$. Bernstein et.al. (1973) introduced certain operators

$$\frac{1}{\alpha_i}(r_i-1): H^k(BT) \to H^{k-2}(BT)$$

where $r_i$ is the simple reflection associated to the simple root $\alpha_i$ .

When $G = M_i$ ,

**Corollary 5.9.**

$$D_i = \frac{1}{\alpha_i}(r_i-1)$$

*is just the classical BGG operator.*

If we apply Theorem 4.7 to K-theory, for $G = M_i$, the formula $D_i = \pi_i^* \circ \pi_{i*}$ in K-theory gives the Demazure operator. Now we prove this. We map the representation ring R(T) to K(BT) by mapping $e^\lambda$ to $[L(\lambda)]$, the class of the line bundle defined by $\lambda$. In K-theory, we can take $\chi(L) = [1] - [L]$, where [1] is the class of the trivial line bundle. In the case where $G = M_i$ wis rank one, $D_i$ is the Demazure operator. It has the form

$$D_i = \frac{1}{1 - e^{-\alpha_i}}\left(1 - e^{-\alpha_i} r_i\right).$$

Now, we will apply this result to BP-theory and Morava K-theory. In order to do this, we will give some definitions. let F be a formal group law over commutative ring with unit R.

**Definition 5.10.** *For each n, the n-seriels [n] (x) of F is given by*

$[1](x) = x$,

$[n](x) = F(x, [n-1](x))$ for $n > 1$,

$[-n](x) = i([n](x))$.

Of particular interest is the p-series, where p is a prime. In characteristic p it always has leading term $ax^q$ where $q = p^h$ for some integer h. This leads to the following.

**Definition 5.11.** *Let F(x, y) be a formal group law over an $F_p$ -algebra. If [p] (x) has the form*

$$[p](x) = ax^{p^h} + \text{higher terms}$$

*with a invertible, then we say that F has height h at p. If [p] (x) = 0 then the heigt is infinity.*

Suppose that $h^*$ is an $F_p$ -algebra and the formal group law has the heigt h. since the elements $x = \chi(L_{\lambda i})$ $\in h^*(BT)$ are non-zero divisors, [p] (x) has the form

$$[p](x) = ax^{p^h} + \text{higher terms}, \quad \text{(a is a unit.)}$$

This lead us to mod p K-theory and the Morava K-theories. The Morava K-theory $K(n)^*$ for any prime p has the heigt n. The reference for these cohomology theories is (Ravenel, 1992). By Proposition 4.4, we generalize Corollary 5.5 and 5.6.

**Theorem 5.12.** *For any prime $p > 3$, in $K(n)^*(BT)$, the Euler class $\chi(L_{\alpha_i})$ is not a zero divisor for any simple root $\alpha_i$.*

**Theorem 5.13.** *For any prime $p > 3$, in $K(n)^*(BT)$, the Euler class $\chi(L_{\alpha_i})$ is not a zero divisor for any simple root $\alpha$.*

Let $\pi : BT \to BG$ is a fiber bundle with the fiber G/T. By Theorem 5.13 and 5.7, we have

**Theorem 5.14.** *For $x \in K(n)^*(BT)$,*

$$\pi^* \circ \pi_*(x) = \sum_{\omega \in W} \omega\left(\frac{x}{\Pi\chi(L_{-\alpha})}\right),$$

*here the right hand side is in a localization*

$$K(n)^*(BT)\left[\frac{1}{\Pi\chi(L_{-\alpha})}\right].$$

**Corollary 5.15.** *Let $\pi_i : BT \to BM_i$ be a fiber bundle with the fiber $M_i/T$. For $x \in K(n)^*(BT)$,*

$$D_i(x) = \pi_i^* \circ \pi_{i*}(x) = (1 + r_i)\left(\frac{x}{\chi(L_{-\alpha})}\right).$$

Of course, these results can be generalized to $F_p$ -algebra $h^*$which has a formal group law F with the height n. In this section, so far we have concentrated our attention on BT. Now, we will give some interesting results about the flag variety G/T. Since the cohomology of G/T vanishes in odd degrees, Corollary 2.2 gives

**Corollary 5.16.** *Let $h^*$ be any complex oriented cohomology theory. Then the Atiyah-Hirzebruch spectral sequence for G/T collapses at the $E_2$-term.*

Let $\pi_i : BT \to BM_i$ . Since G/T is a T-principal bundle, there is a classifying map $\theta$: $G/T \to BT$. Similarly there is a classifying $\theta_i$: $G/M_i \to BM_i$. The following diagram is a cartesian square.

$$\begin{array}{ccc} G/T & \xrightarrow{\ \theta\ } & BT \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \pi_i} \\ G/M_i & \xrightarrow{\ \theta_i\ } & BM_i \end{array}$$

Let $C_i = p^* \circ p_{i*}$ .Then $\theta^* \circ D_i = C_i \circ \theta^*$ .The following theorem gives a topological description of the operator $C_i$ . From (Evens, 1988),

**Theorem 5.17.** *If $h^*(pt)$ contains $\mathbb{Z}\left[\frac{1}{W_G}\right]$, then $\theta^*$ is surjective.*

**Definition 5.18.** *For $i = 1, \ldots, l$, let $D_i$ be the linear operator associated to the simple root $\alpha_i$ . Then we say that $D_i$ satisfy braid realitons if*

$$(D_i D_j D_i)^{m_{ij}} = (D_j D_i D_j)^{m_{ij}},$$

*where $m_{ij}$ is the number of factors in each side for all pairs i and j.*

Now we will give our result about the infinite dimensional flag variety. Let G be an affine Kac-Moody group and K be the unitary form of G. For every simple root $\alpha_i$, let $M_i = K_i . T$. We have a principal $M_i$ -bundle $K \to K/M_i$, and the associated fiber bundle $K/T \to K/M_i$ with fiber $M_i/T$. $M_i/T$ is diffeomorpich to complex projective space $CP^1$.

**Theorem 5.19.** *Let $\pi_i : K/T \to K/M_i$ be the fiber bundle with the compact fiber $CP^1$ and F be a commulative ring ith unit. For $x \in H^*(K/T, F)$,*

$$O_i(x) = \pi_i^* \circ \pi_{i*}(x) = -(1 + r_i)\left(\frac{x}{\varepsilon^{r_i}}\right),$$

*here the right hand side is in the localization $H^*(BT) = \left[\frac{1}{\Pi \chi(L_{-\alpha})}\right]$ . In fact $O_i$ is the Kac operator which was introduced in (Kac, 1985).*

Proof. By the Burmfiel-Madsen formula and Theorem 4.2, we have the fallowing identiy.

$$\pi_i^* \circ \tau(\pi_i)^*(x) = \pi_i^* \circ \pi_{i*}\left(\psi(-X_i) . x\right) = (1 + r_i)(x),$$

where $r_i$ is the simple reflection associated to $\alpha_i$ and $X_i$ is the fundamental weight corresponding to the simple root $\alpha_i$ . Let $x \in H^*(K/T, F)$. We know from (Konstant and Kumar, 1986) that $\psi(X_i) = \varepsilon^{r_i}$ where $\psi$: $S(h^*) \to H^*(K/T, F)$. In $H^*(K/T, F)$, we know that the element $\varepsilon^{r_i}$ is a non zero-divisor, so we can define the local ring $H^*(K/T, F)\left[\frac{1}{\varepsilon^{r_i}}\right]$.

Then, we have the folloving identity in the local ring $H^*(K/T, F)\left[\frac{1}{\varepsilon^{r_i}}\right]$,

$$\pi_i^* \circ \pi_{i*}(x) = -(1 + r_i)\left(\frac{x}{\varepsilon^{r_i}}\right).$$

Since the left hand side of the identity is an element of $H^*(K/T, F)$, we are done .

We know from (Kac, 1985) that the Kac operators statisfy braid relations for all affine Kac-Moody group.

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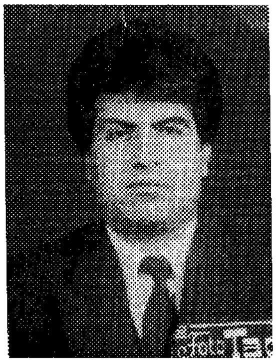

**Cenap Özel** received B.S. degree in Mathematics from Selçuk University, Konya, in 1990, M.S. degree from Cumhuriyet University, Sivas, in 1993, and Ph.D. degree from University of Glasgow in 1998. He worked at Sivas Anadolu Öğretmen Lycée between 1990-1993. He has been with the Department of Mathematics, Abant İzzet Baysal University, Bolu, since 1993.