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Project-Based Learning to Explore Taxicab Geometry

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Abstract: In Turkey, the content of the geometry course in the Primary School Mathematics Education, which is developed by The Council of Higher Education (YOK), comprises Euclidean and non-Euclidean types of geometry. In this study, primary mathematics teacher candidates compared these two geometries by focusing on Taxicab geometry among non-Euclidean geometries so that they could comprehend Euclidean geometry better. They were asked to perform this comparison through a project-based learning method, due to its student-centered and interdisciplinary nature and association with real life [10]. The findings were collected through the study groups' project reports and the teacher candidates' written opinion statements. By the end of the study period, the teacher candidates involved in the current study discovered that Taxicab geometry was a kind of geometry used in daily life. They further realized that the concept of distance used in real life sometimes emerged as Euclidean distance and sometimes as Taxicab distance.

Keywords: Euclidean geometry, Non-Euclidean geometry, Taxicab geometry, project-based learning.

1. INTRODUCTION

The main objective of geometry education can be summarized as making "students have an ability to employ geometry in explaining their physical environment and the universe and in problem solving process" [1–3]. In many countries, the preferred curriculum to achieve this aim covers the abstract geometrical shapes of Euclidean geometry not found in nature.

However, evidence suggests that the theoretical knowledge covered by Euclidean geometry alone cannot provide students with an understanding of the physical world [1,2]. For this reason, it seems inevitable that a new menu of geometries, including non-Euclidean geometries, must be designed [4, 5]. The

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first signs of this appeared in the standards [8] published by the 1989 National Council of Teachers of Mathematics (NCTM). These standards emphasized that it was vital that students study different geometries so that axiomatic systems are understood. The standards contain the following:

> Students should be able to see that Euclidean geometry is just one of different axiomatic systems. This aim can only be achieved by having students researching different geometries. Therefore, students figure out how different axioms and definitions can produce different geometries [8].

For example, in this study, the axiom describing the congruence of triangles known as Side Angle Side (SAS) in Euclidean geometry is not valid in Taxicab Geometry [7]. Those geometries not obeying at least one of the axioms from Euclidean Geometry are called non-Euclidean geometries. Because of this, Taxicab Geometry is also a non-Euclidean geometry. Like the NCTM standards, in Turkey, the content of the geometry course in the Primary School Mathematics Education, designed by The Council of Higher Education (YOK), covers Euclidean and non-Euclidean geometries.

The general aim of geometry education can be described as having students learn about the environment, and use geometry in the problem solving process. Does Euclidean geometry, alone, help students understand their environment? Euclidean geometry, with its theoretical knowledge, fails to make connections and integrations in real life. For example, considering that roads are generally horizontal and vertical, a taxi driver may not always use Euclidean distance in real life. Therefore, it could be suggested that Taxicab geometry provides students with more opportunities than Euclidean geometry does in terms of making meaning out of the real world [6]. What is targeted through project-based learning, which presents real life problems, is students' recognition that, in real life, Taxicab geometry may be more practical than Euclidean.

The teacher candidates involved in the study were required to prepare projects, and compare and contrast Euclidean and non-Euclidean geometries. This research shows that students often cannot transfer their mathematical knowledge into situations outside the classroom. These projects engage students in applications of mathematics, which may help them transfer their mathematical skills into other disciplines and to real-world problems. Using significant problems often increases student motivation, in turn promoting learning.

In line with the relevant research, the aim of this study is to have students get to know Euclidean geometry better, by performing group activities and comparing and contrasting group results. At the end of the study, the students involved in the study submitted written reports on the results of their comparisons and gave oral presentations. Moreover, their opinions concerning the project were solicited.

2. EUCLIDEAN AND NON-EUCLIDEAN GEOMETRIES

2.1. Euclidean Geometry and Distance

Euclidean geometry is a mathematical system in which all theorems ("true statements") are derived from a finite number of axioms.

The distance between two points $A = (a_1, a_2), B = (b_1, b_2)$ in the Euclidean plane is defined as $d(A, B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.

2.2. Non-Euclidean Geometry

Non-Euclidean geometry is any geometry different from Euclidean geometry—for example, hyperbolic geometry, projective geometry, Taxicab geometry, etc. These geometries differ in at least one axiom from Euclidean geometry.

2.3. Taxicab Geometry and Distance

E. F. Krause has defined a new geometry, the *Taxicab geometry*, by using the metric $d_T(A, B) = |a_1 - b_1| + |a_2 - b_2|$ for $A = (a_1, a_2), B = (b_1, b_2)$ in the Euclidean plane [6].

2.4. Comparing Euclidean and Taxicab Geometries

Consider the following questions:

(1) In the graph in Figure 1, which is closer to *A*, point *B* or point *C* in Euclidean geometry?
(2) On the same graph, (Figure 1) which is closer to *A*, point *B* or point *C* in Taxicab geometry?

In Euclidean geometry, with the measurements given in the Cartesian plane, as in Figure 1, we would simply state that since $d(A,C) = 5$ and $d(A,B) = 6$, that $d(A,C) < d(A,B)$. Hence, *C* is closer to *A* than *B*.

In Taxicab geometry, consider the points (oriented in a similar fashion) in the Figure 2 map. If you are driving and plan to keep your license (!), your distances are measured in a very different fashion.

It now seems that the taxi distance from *A* to *C* is 7 blocks, while the distance from *A* to *B* is 6 blocks. Unless we choose to go off-road, *B* is now closer to *A* than *C*. Therefore,

Euclidean distance < Taxicab distance.

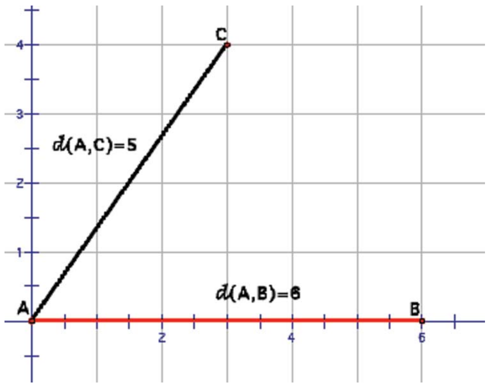

Figure 1. The points are $A = (0,0)$, $B = (6,0)$, $C = (3,4)$ on the Cartesian plane (color figure available online).

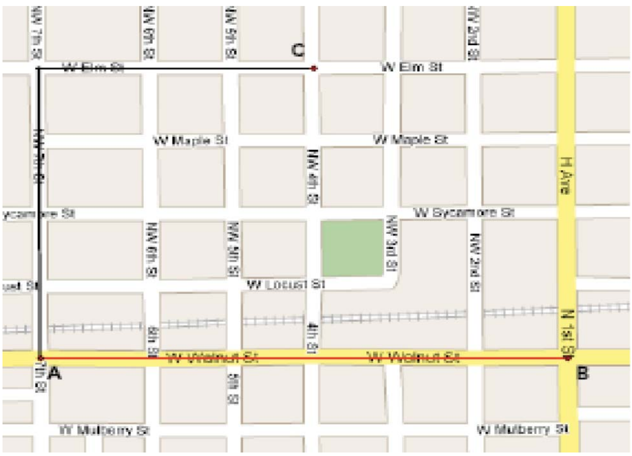

Figure 2. The map consists of the points *A, B, C* (color figure available online).

3. GEOMETRY COURSE

Geometry is a compulsory subject that primary education mathematics teacher candidates are required to take in their first year at the teaching programs. The content is developed by YOK. The course is offered in a total of 12 weeks, at three hours of class each week throughout a term. In our case, the lecturers were the researchers themselves.

The main objective of this course is to have teacher candidates learn about Taxicab geometry, among non-Euclidean geometries as well as Euclidean geometry, and examine comparisons and contrasts. Taxicab geometry was chosen in this study because it is quite similar to Euclidean geometry with its axiomatic pattern, meaningful applications, and comprehensibility for people beginning to study non-Euclidean geometries. The study had the teacher candidates discover the similarities and differences between these two geometries by means of the project-based learning method.

3.1. Course Structure

For two weeks during the early stage of the course, the axiomatic pattern of Euclidean geometry and relevant theorems were presented together. During the following two weeks, after Euclidean distance was defined, Taxicab geometry was defined, together with Taxicab distance. Following this four-week period, the students were divided into four six-member groups. Each group was assigned project topics with different problems. Samples of the groups' research process and one of the project problems are presented in Section 3.3.

3.2. Goal of Project

The project's goal can be summarized as follows: Euclidean geometry appears to be a good model of the "natural" world, but Taxicab geometry is a better model of the artificial urban world that humans have built.

3.3. Problem of Project

Ideal City has three high schools: Atatürk at (−6, 4), Cumhuriyet at (1, 6), and 100.Yıl at (−1, –3). You must answer the following questions using Taxicab geometry:

a) Draw in school district boundary lines so that each student in Ideal City attends the high school nearest his or her home.
b) If a hospital is at (1, 1), in which region from part (a) should the hospital be located?
c) If a bookstore wants to open in a location that is equidistant from each of three high schools, where should it be located?
d) Belma's mother is a teacher, and her father is a doctor. Her family is looking for an apartment in Ideal City. Her mother works at Atatürk High School and her father works at the hospital. They both want to be the same distance from their jobs. Where should they locate?
e) Find a pharmacy address which should be less than four blocks from Belma's apartment and more than two blocks to the hospital.
f) Solve this problem using Euclidean geometry and then compare with solutions from Taxicab geometry.

Table 1. Calendar of the Study

Procedure levels	Days
1. Inform them about their project, and determine projects groups	2
2. Planning of study groups	1
3. Collecting data about their projects	8
4. Solving problems	10
5. Controlling solutions and commenting on them	2
6. Preparing presentation and report	3
7. Discussing results with other project groups	1
8. Modeling their project results as a product	5
9. Presentations	1

3.4. Process of Project

Thirty-three days were given to mathematics teachers candidates to prepare their project. The calendar of the study is summarized in Table 1.

3.5. Presentations

Students prepared their reports of project results, and made verbal presentations using PowerPoint. The researchers examined and compared each group's reports and products to each of the other groups at the end of the project.

4. FINDINGS

The students' answers to questions above are given below. Respectively, A, C, and Y points show Atatürk, Cumhuriyet, and the 100.Yıl high schools.

a) We want to draw school district boundaries so that every student is going to the closest school. There are three schools: Atatürk at $A = (-6, 4)$, Cumhuriyet at $C = (1, 6)$, and 100.Yıl at $Y = (-1, -3)$.

   I) Let us first focus on a boundary between A point and C point. The boundary line needs to fall on the points whose distance between A and C are the same. So we need:

$$d_T(P, A) = d_T(P, C),$$

$$|x + 6| + |y - 4| = |x - 1| + |y - 6| .$$

Solving for x and y becomes more difficult with the absolute values. This solution has different cases:

Case 1: Let $x < -6$.

(i) For $y < 4$; $-x - 6 - y + 4 = -x + 1 - y + 6$
$-2 \neq 7$; so there is no solution for $y < 4$.
(ii) For $4 \leq y < 6$; $-x - 6 + y - 4 = -x + 1 - y + 6$
$2y = 17$, $y = 17/2$; so there is no solution for $4 \leq y < 6$.
(iii) For $y \geq 6$; $-x - 6 + y - 4 = -x + 1 + y - 6$
$-10 \neq -5$; so there is no solution for $y \geq 6$.

Case 2: Let $-6 \leq x < 1$.

(i) For $y < 4$; $x + 6 - y + 4 = -x + 1 - y + 6$
$2x = -3$, $x = -3/2$.
(ii) For $4 \leq y < 6$; $x + 6 + y - 4 = -x + 1 - y + 6$
$2x + 2y = 5$, $x + y = 5/2$.
(iii) For $y \geq 6$; $x + 6 + y - 4 = -x + 1 + y - 6$
$x = -7/2$.

Case 3: Let $x \geq 1$.

(i) For $y < 4$; $x + 6 - y + 4 = x - 1 - y + 6$
$10 \neq 5$; so there is no solution for $y < 4$.
(ii) For $4 \leq y < 6$; $x + 6 + y - 4 = x - 1 - y + 6$
$2y = 3$, $y = 3/2$; so there is no solution for $4 \leq y < 6$.
(iii) For $y \geq 6$; $x + 6 + y - 4 = x - 1 + y - 6$
$2 \neq -7$; so there is no solution for $y \geq 6$.

Figure 3 shows the points and lines satisfying $d_T(P, A) = d_T(P, C)$.

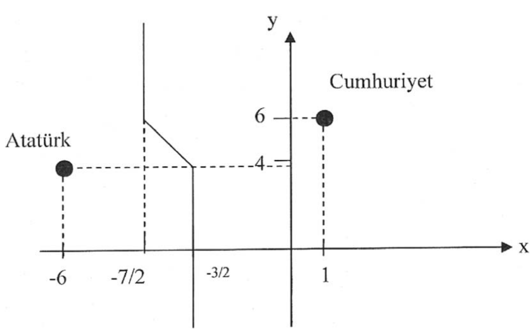

Figure 3. The figure shows the points and lines that so far satisfy points and lines that follow $d_T(P, A) = d_T(P, C)$.

II) We will now look at the boundary between C school and Y school. Again we want to find the points that are equidistant from the two schools, to create our boundary. We are looking for:

$$d_T(P,C) = d_T(P,Y),$$

$$|x+1| + |y+3| = |x-1| + |y-6|\,.$$

Case 1: Let $x < -1$.

(i) For $y < -3$; $-x-1-y-3 = -x+1-y+6$
$-4 \neq 7$; so there is no solution for $y < -3$.
(ii) For $-3 \leq y < 6$; $-x-1+y+3 = -x+1-y+6$
$2y = 5$, $y = 5/2$.
(iii) For $y \geq 6$; $-x-1+y+3 = -x+1+y-6$
$2 \neq -5$; so there is no solution for $y \geq 6$.

Case 2: Let $-1 \leq x < 1$.

(i) For $y < -3$; $x+1-y-3 = -x+1-y+6$
$2x = 9$, $x = 9/2$; so there is no solution for $y < -3$.
(ii) For $-3 \leq y < 6$; $x+1+y+3 = -x+1-y+6$
$2x+2y = 3$, $x+y = 3/2$.
(iii) For $y \geq 6$; $x+1+y+3 = -x+1+y-6$
$2x = -9$, $x = -9/2$; so there is no solution for $y \geq 6$.

Case 3: Let $x \geq 1$.

(i) For $y < -3$; $x+1-y-3 = x-1-y+6$
$-2 \neq 5$; so there is no solution for $y < -3$.
(ii) For $-3 \leq y < 6$; $x+1+y+3 = x-1-y+6$
$2y = 1$, $y = 1/2$.
(iii) For $y \geq 6$; $x+1+y+3 = x-1+y-6$
$4 \neq -7$; so there is no solution for $y \geq 6$.

Figure 4 shows the points and lines satisfying $d_T(P,C) = d_T(P,Y)$.

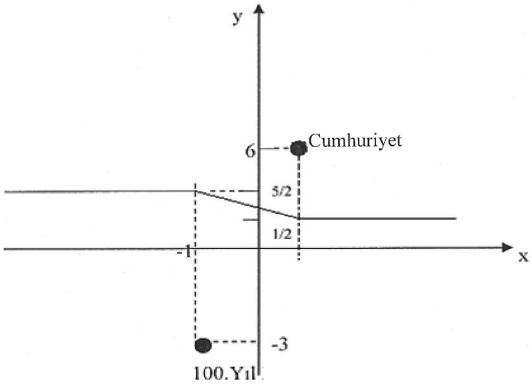

Figure 4. The figure shows the points and lines that so far satisfy $d_T(P, C) = d_T(P, Y)$.

III) Let us now focus on a boundary between A school and Y school. The boundary line needs to fall on the points for which the distances between the two schools are the same. So we need:

$$d_T(P, A) = d_T(P, Y),$$

$$|x + 6| + |y - 4| = |x + 1| + |y + 3| .$$

Case 1: Let $x < -6$.

(i) For $y < -3$; $-x - 6 - y + 4 = -x - 1 - y - 3$
$-2 \neq -4$; so there is no solution for $y < -3$.
(ii) For $-3 \leq y < 4$; $-x - 6 - y + 4 = -x - 1 + y + 3$
$2y = -4, \quad y = -2.$
(iii) For $y \geq 4$; $-x - 6 + y - 4 = -x - 1 + y + 3$
$-10 \neq 2$; so there is no solution for $y \geq 4$.

Case 2: Let $-6 \leq x < -1$.

(i) For $y < -3$; $x + 6 - y + 4 = -x - 1 - y - 3$
$2x = -14, \quad x = -7$; so there is no solution for $y < -3$.
(ii) For $-3 \leq y < 4$; $x + 6 - y + 4 = -x - 1 + y + 3$
$2x - 2y = -8, \quad x - y = -4.$
(iii) For $y \geq 4$; $x + 6 + y - 4 = -x - 1 + y + 3$
$2x = 2, \quad x = 1$; so there is no solution for $y \geq 4$.

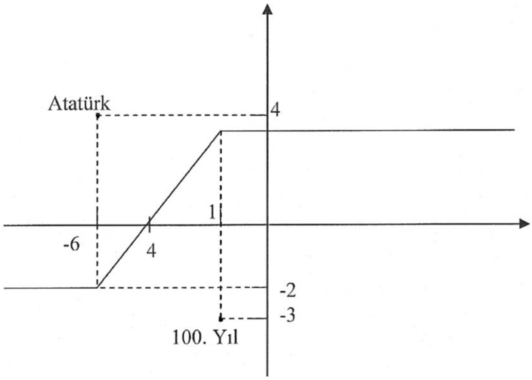

Figure 5. The figure shows the points and lines so far that satisfy $d_T(P, C) = d_T(P, Y)$.

Case 3: Let $x \geq -1$.

(i) For $y < -3$; $x + 6 - y + 4 = x + 1 - y - 3$
$10 \neq -2$; so there is no solution for $y < -3$.
(ii) For $-3 \leq y < 4$; $x + 6 - y + 4 = x + 1 + y + 3$
$2y = 6, \quad y = 3.$
(iii) For $y \geq 4$; $x + 6 + y - 4 = x + 1 + y + 3$
$2 \neq 4$; so there is no solution for $y \geq 4$.

Figure 5 shows the points and lines satisfying $d_T(P, C) = d_T(P, Y)$.

At the end of the analysis the Ideal City is divided into three regions.

The next intersection region indicates the status of Cumhuriyet High School by Atatürk and 100.Yıl high schools. This region can be defined as the first region (see Figure 6).

The next intersection region indicates the status of Atatürk High School by 100.Yıl and Cumhuriyet high schools. This region can be defined as the second region (see Figure 7).

The next intersection region indicates the status of 100.Yıl High School by Cumhuriyet and Atatürk high schools. This defines the third region (see Figure 8).

Now, we must first find the point where the three boundaries intersect. This point is at $M = (-3/2, 5/2)$, so M will be considered as the center of the Ideal City, which is divided into three regions as shown in Figure 9. Thus, each student goes to the school closest to his or her house. Now, we will look at each boundary from this point. We will label the boundaries for ease of explanation as follows:

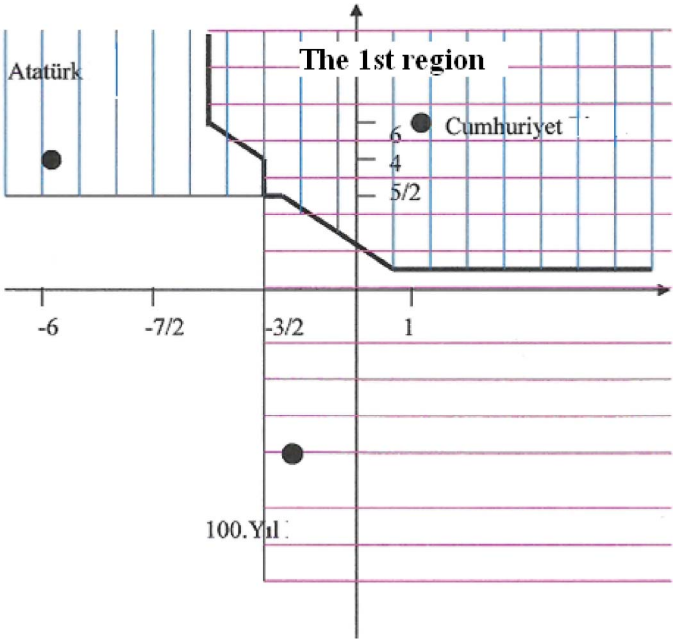

Figure 6. The first region (color figure available online).

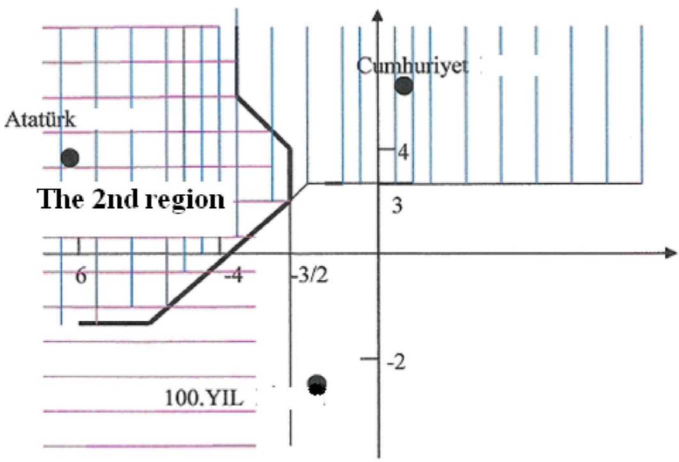

Figure 7. The second region (color figure available online).

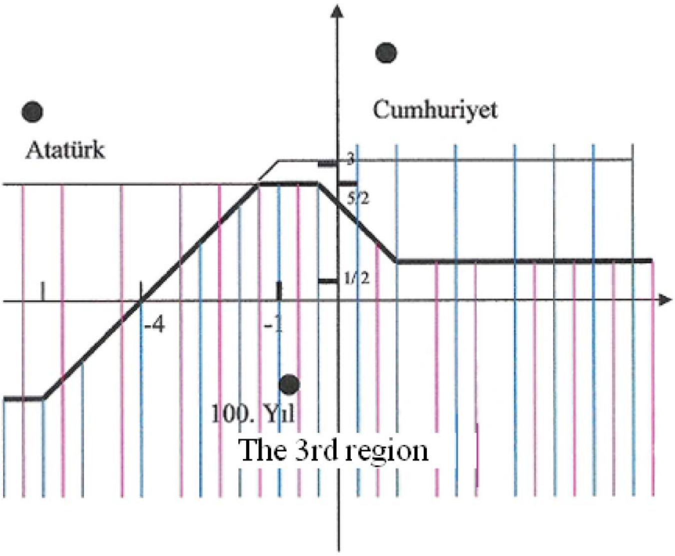

Figure 8. The third region (color figure available online).

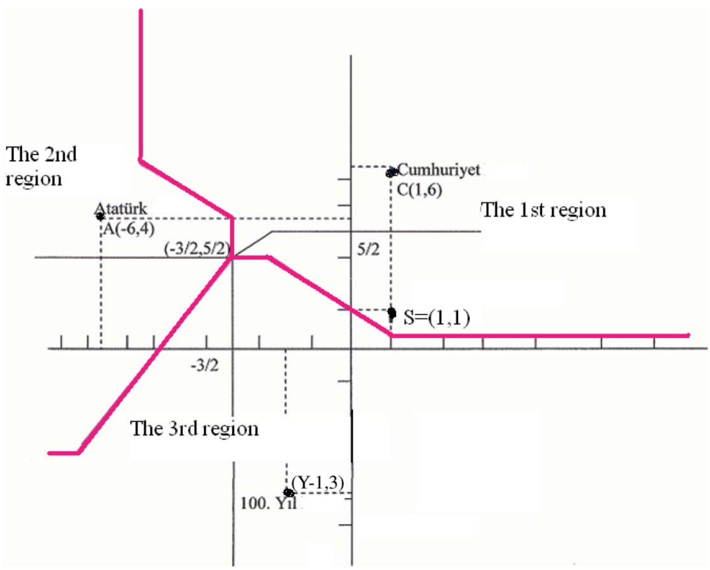

Figure 9. District boundary lines in Taxicab geometry (color figure available online).

b) If the hospital is at $S = (1{,}1)$, it should be located in the first region (see Figure 9).
c) A bookstore wants to open in a location that is equidistant from each of three high schools. $M = (-3/2, 5/2)$ is equidistant from each of three high schools. The bookstore should be located at this point.
d) Belma's mother is a teacher, and her father is a doctor. Her mother works at Atatürk High School, and her father works at the hospital.

The family is looking for an apartment in Ideal City. They both want to be the same distance from their jobs. A and S show Ataturk High School and the hospital, respectively, in the following solutions. We suppose that $B = (x,y)$ is Belma's apartment's coordinate;

$$d_T(A, B) = d_T(A, S),$$

$$|x + 6| + |y - 4| = |x - 1| + |y - 1|\,.$$

Case 1: Let $x < 6$.

(i) For $y < 1$; $-6 - x + 4 - y = 1 - x + 1 - y$
$-2 \neq 2$; so there is no solution for $y < 1$.
(ii) For $1 \leq y < 4$; $-6 - x + 4 - y = 1 - x - 1 + y$
$-2 = 2y$, $y = -1$; so there is no solution for $1 \leq y < 4$.
(iii) For $y \geq 4$; $-6 - x - 4 + y = 1 - x - 1 + y$
$-10 \neq 0$; so there is no solution for $y \geq 4$.

Case 2: Let $-6 \leq x < 1$.

(i) For $y < 1$; $6 + x + 4 - y = 1 - x + 1 - y$
$2x = 8$, $x = -4$.
(ii) For $1 \leq y < 4$; $6 + x + 4 - y = 1 - x - 1 + y$
$2x - 2y = -10$, $x - y = -5$.
(iii) For $y \geq 4$; $6 + x - 4 + y = 1 - x - 1 + y$
$2x = -2$, $x = -1$.

Case 3: Let $x \geq 1$.

(i) For $y < 1$; $6 + x + 4 - y = -1 + x + 1 - y$
$-10 \neq 0$; so there is no solution for $y < 1$.
(ii) For $1 \leq y < 4$; $6 + x + 4 - y = -1 + x - 1 + y$
$12 = 2y$, $y \neq 6$; so there is no solution for $1 \leq y < 4$.
(iii) For $y \geq 4$; $6 + x - 4 + y = -1 + x - 1 + y$
$2 \neq -2$; so there is no solution for $y \geq 4$.

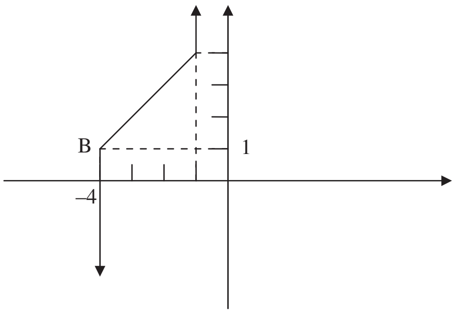

Figure 10. $y < 1, x = -4$; $1 \leq y < 4$, $y - x = 5$; $y \geq 4, x = -1$ lines. The location of Belma's apartment is B = (−4, 1).

Belma's apartment should be on the combination of the set of lines which is $y < 1$, $x = -4$; $1 \leq y < 4$, $y - x = 5$; and $y \geq 4, x = -1$ obtained from Case 2. We suppose that Belma's apartment is $B = (-4, 1)$, which is at the intersection point of $x = -4$ and $y - x = 5$ (see Figure 10).

e) A pharmacy's address should be less than four blocks from Belma's apartment, and more than two blocks from the hospital. Suppose E is the pharmacy's address. Let $E = (x,y)$:

$$d_T(B, E) \geq 4, d_T(E, S) \leq 2,$$

$$|x + 4| + |y - 1| \geq 4, |x - 1| + |y - 1| \leq 2.$$

If $|x + 4| + |y - 1| \geq 4$ is, there are two cases:

Case 1: Let $x \geq 4$.

(i) For $y \geq 1$; $x + 4 + y - 1 \geq 4$
$x + y \geq 1.$
(ii) For $y < 1$; $x + 4 - y + 1 \geq 4$
$x - y \geq -1.$

Case 2: Let $x \leq 4$.

(i) For $y \geq 1$; $-x - 4 + y - 1 \geq 4$
$-x + y \geq 9.$
(ii) For $y < 1$; $-x - 4 - y + 1 \geq 4$
$-x - y \geq 7$
$x + y \leq -7.$

If $|x-1|+|y-1| \leq 2$ then there are two cases:

Case 1: Let $x \geq 1$.

(i) For $y \geq 1$; $x - 1 + y - 1 \leq 2$
$x + y \leq 4$.
(ii) For $y < 1$; $x - 1 - y + 1 \leq 2$.
$x - y \leq 2$.

Case 2: Let $x < 1$.

(i) For $y \geq 1$; $-x + 1 + y - 1 \leq 2$,
$-x + y \leq 2$.
(ii) For $y < 1$; $-x + 1 - y + 1 \leq 2$
$x + y \geq 0$.

The pharmacy address, which should be less than four blocks from Belma's apartment, and more than two blocks to the hospital, must be in the intersection region of the solutions (see Figure 11).

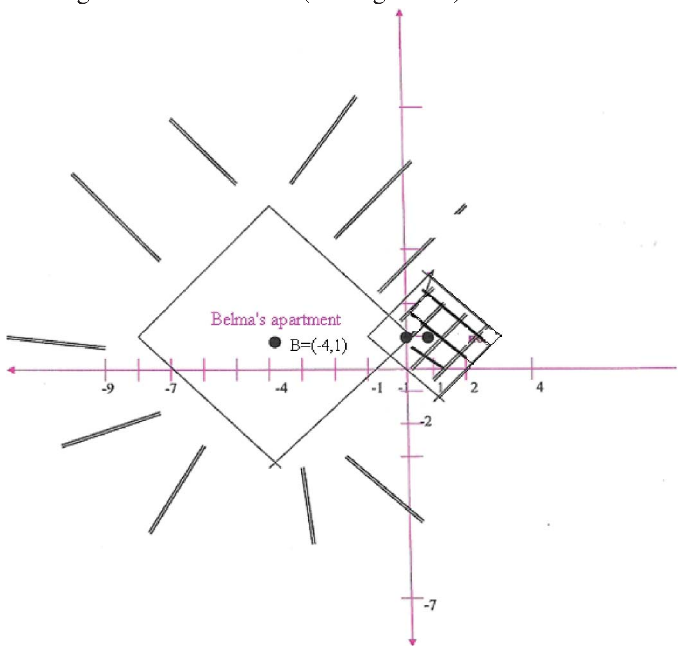

Figure 11. The pharmacy address is the intersection region of the solutions (color figure available online).

f) This problem was solved, also using Euclidean geometry, and then compared, by the students, with solutions using Taxicab geometry. In this section, graphics of solutions were only given in Euclidean geometry. Because the algebraic solutions were so long, they were not given here.

The solution to a), using Euclidean geometry:

We want to draw school district boundaries such that every student is going to the school closest to him or her. There are three schools: Atatürk at $A = (-6, 4)$, Cumhuriyet at $C = (1, 6)$, and 100.Yıl at $Y = (-1, -3)$.

I) Let us first focus on a boundary between A and C points. The boundary line needs to fall on the points such that the distances between A and C are the same. So we need:

$$d(P, A) = d(P, C),$$

$$\begin{aligned} \sqrt{(x+6)^2 + (y-4)^2} &= \sqrt{(x-1)^2 + (y-6)^2} \\ x^2 + 12x + 36 + y^2 - 8y + 16 &= x^2 - 2x + 1 + y^2 - 12y + 36 \\ 14x + 4y + 15 &= 0 \\ y &= -\frac{7}{2}x - \frac{15}{4}. \end{aligned}$$

II) We will now look at the boundary between C school and Y school. Again, we want to find the points that are equidistant from the two schools to create our boundary. We are looking for:

$$d(P, C) = d(P, Y),$$

$$\begin{aligned} \sqrt{(x+6)^2 + (y-4)^2} &= \sqrt{(x+1)^2 + (y+3)^2} \\ x^2 + 12x + 36 + y^2 - 8y + 16 &= x^2 + 2x + 1 + y^2 + 6y + 9 \\ 10x - 14y + 42 &= 0 \\ y &= \frac{5}{7}x + 3. \end{aligned}$$

III) Let us now focus on a boundary between A school and Y school. The boundary line needs to consist of points for which the distances between the two schools are the same. So we need:

$$d(P, A) = d(P, Y),$$

$$\begin{aligned} \sqrt{(x-1)^2 + (y-6)^2} &= \sqrt{(x+1)^2 + (y+3)^2} \\ x^2 - 2x + 1 + y^2 - 12y + 36 &= x^2 + 2x + 1 + y^2 + 6y + 9 \\ 4x + 18y &= 27 \\ y &= -\frac{2}{9}x + \frac{3}{2}. \end{aligned}$$

The point $K = (-\frac{189}{118}, \frac{219}{118})$ lies on the intersection of the lines from I, II, III. We will label the boundaries for ease of explanation as follows:

The solutions in Taxicab geometry, compared to solutions in Euclidean geometry, while in Taxicab geometry the intersection of the boundary lines was $M\,(-3/2, 5/2)$ which is the central point of Ideal City at Figure 9, in Euclidean geometry the intersection of the boundary lines was $K = (-\frac{189}{118}, \frac{219}{118})$ which is the central area of the city at Figure 12. Then, the center and the boundaries of two cities, which were constructed by using different geometries, are different.

The solution to b), using Euclidean geometry:

If the hospital is at $S = (1,1)$, it should be located in the third region. It was located in two different regions in Taxicab and Euclidean geometries (see Figures 9 and 12).

The solution to c) using Euclidean geometry:

The bookstore wants to open in a location that is equidistant from each of three high schools. The intersection of the following three lines is $K = (-\frac{189}{118}, \frac{219}{118})$ point (see Figure 12).

$$y = \frac{-7}{2}x - \frac{15}{4},$$
$$y = \frac{5}{7}x + 3,$$
$$y = \frac{-2}{9}x + \frac{3}{2}.$$

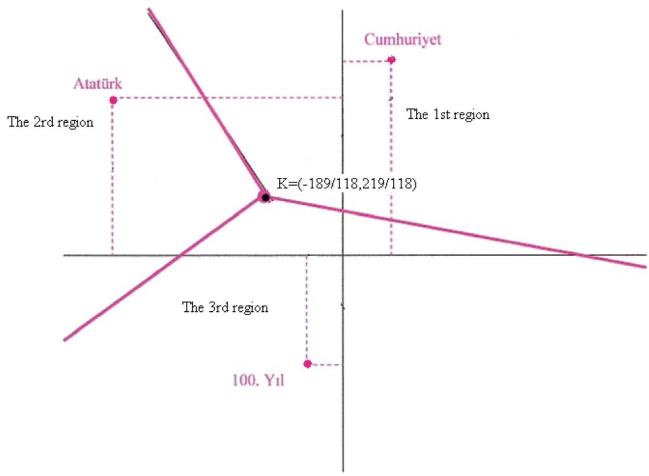

Figure 12. District boundary lines in Euclidean geometry (color figure available online).

$K = (-\frac{189}{118}, \frac{219}{118})$ is equidistant from each of three high schools. The bookstore should be located at this point.

The solution of d) using Euclidean geometry:

Belma's mother is a teacher, and her father is a doctor. Her mother works at Atatürk High School, and her father works at the hospital. The family is looking for an apartment in Ideal City. They both want to be the same distance from their jobs. Respectively, *A, S,* and *B* points show Atatürk High School, the hospital, and Belma's apartment with the following solutions: Let $B = (x,y)$, $A = (-6, 4)$, and $S = (1, 1)$.

$$d(A, B) = d(A, S),$$

$$\begin{aligned} \sqrt{(x+6)^2 + (y-4)^2} &= \sqrt{(x-1)^2 + (y-1)^2} \\ x^2 + 12x + 36 + y^2 - 8y + 16 &= x^2 - 2x + 1 + y^2 - 2y + 1 \\ 14x - 6y + 50 &= 0 \\ y &= \frac{7}{3}x + \frac{25}{3}. \end{aligned}$$

Belma's apartment should be on the line, $y = 7x/3 + 25/3$. We suppose that Belma's apartment is $B = (-3, 4/3)$, which is on the following line (Figure 13):

The solution of e) using Euclidean geometry:

The pharmacy should be less than four blocks from Belma's apartment, and more than two blocks from the hospital. *E* shows the pharmacy's address in the following solution. Let $E = (x,y)$, $B(-3, 4/3)$, and $S = (1, 1)$.

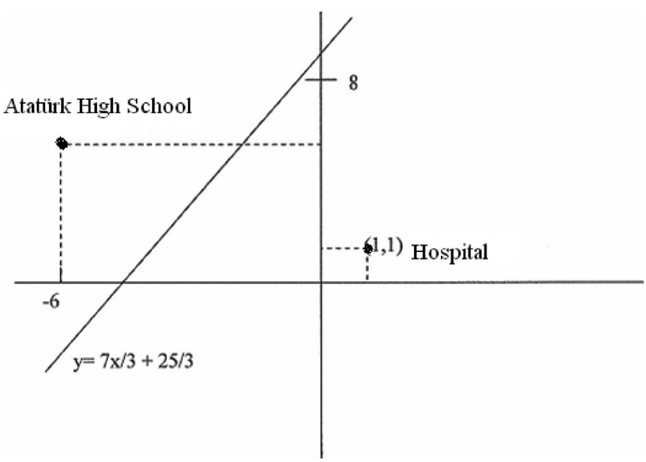

Figure 13. Belma's apartment coordinates on $y = 7x/3 + 25/3$ line.

For $d(B,E) \geq 4$,

$$\sqrt{(-3-x)^2 + (\tfrac{4}{3} - y)^2} \geq 4 \Rightarrow (-3-x)^2 + (\tfrac{4}{3} - y)^2 \geq 16 \text{ (I)}.$$

For $d(E,S) \leq 2$,

$$\sqrt{(x-1)^2 + (y-1)^2} \leq 2 \Rightarrow (x-1)^2 + (y-1)^2 \leq 4 \text{ (II)}.$$

The pharmacy should be at the intersection of (I) and (II) regions (see Figure 14).

When the students solved their given questions in Euclidean geometry, they said they saw that ideal solutions could not be rendered in Euclidean geometry. This geometry did not always apply in real life because the Euclidean distance between two points is "as the crow flies."

After students answered all questions they formed a physical model by adding essential buildings (see Figure 15). These buildings were located in the Taxicab circle with radius 4. M (−3/2, 5/2) point, which is the center of Ideal City, will be considered the center of the Taxicab circle. The buildings in this circle are the shopping center, the sports center, banks, theaters, museums, cafes, the library, restaurants, parks, and other facilities.

The students compared Euclidean geometry to Taxicab geometry in these project problems. One of the projects contained some questions that made it possible for the students to explore the notion of a Taxicab circle, and then compare it to a Euclidean circle. One of these questions asked, "City planners want to set up primary schools so that each student living within 18 blocks of the center of city is within six blocks of a school. How few schools can they

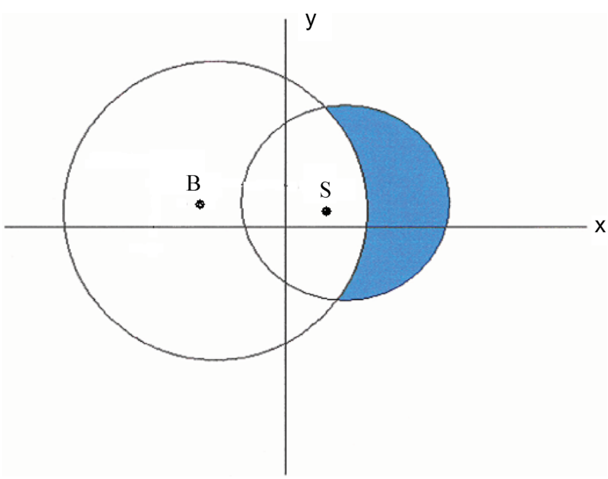

Figure 14. The pharmacy's address is in the scanned area (color figure available online).

Figure 15. The model of the given graphic prepared by the students (color figure available online).

get by with, and where should they be located?" Figures 16 and 17 illustrate the graphic of the solution provided by the students. They found the Taxicab circle and placed the desired settlement locations on the proper coordinates on the graphic (see Figure 16).

When they solved this question in Euclidean geometry, they saw that the number of circles in Taxicab geometry is not enough for a solution in Euclidean geometry (see Figure 17).

In Figure 17, if the areas outside the circular regions are taken into account, the number of schools providing the required conditions is much higher in Euclidean geometry than that required in Taxicab geometry.

Figure 18 shows a model of the given graphic, prepared by the students. They used the models in their presentations. During the project process, the students interviewed urban and regional planners who said they did not prefer Euclidean geometry, while planning an ideal city, because it was not really suitable. Instead, they said, made use of Taxicab geometry because Taxicab distance was more applicable to real life. The students said that they reached the same conclusion when they completed their projects.

The students made use of The Geometer's Sketchpad to draw graphics. They managed to better interpret the results they obtained, in solving the problems, because of the dynamic nature of this software program (Figure 19).

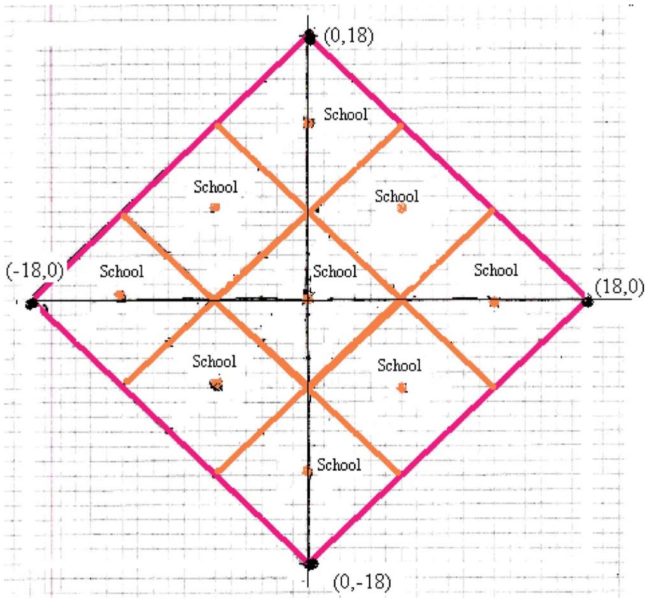

Figure 16. Students placed the schools on proper coordinates on the taxicab circles (color figure available online).

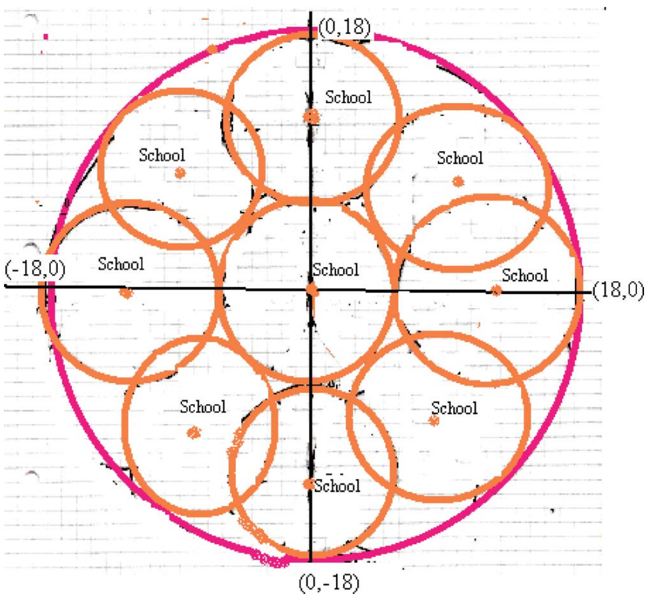

Figure 17. Students placed the schools on proper coordinates on the Euclidean circles (color figure available online).

Figure 18. The physical model of the given graphic prepared by the students (color figure available online).

One of the study groups used SimCity, an online computer game, that was very appropriate for Taxicab geometry. They built the project city in the desired pattern by using this game.

Students submitted written reports on their projects. They were required to state the positive and negative experiences during the project process, and explain the positive experiences they gleaned from the projects. According to the teacher candidates, this project proved useful in that it yielded permanent learning. Students found it appealing as well as motivating, and the questions encouraged students to do research. The statements expressing the candidates' opinions are given below. Among the teacher candidates, Meral said:

> I quickly got involved in Taxicab geometry, though I hadn't had the slightest idea about it before. I realized that we had always made use of it. We visited some urban and regional planners in order to get information,

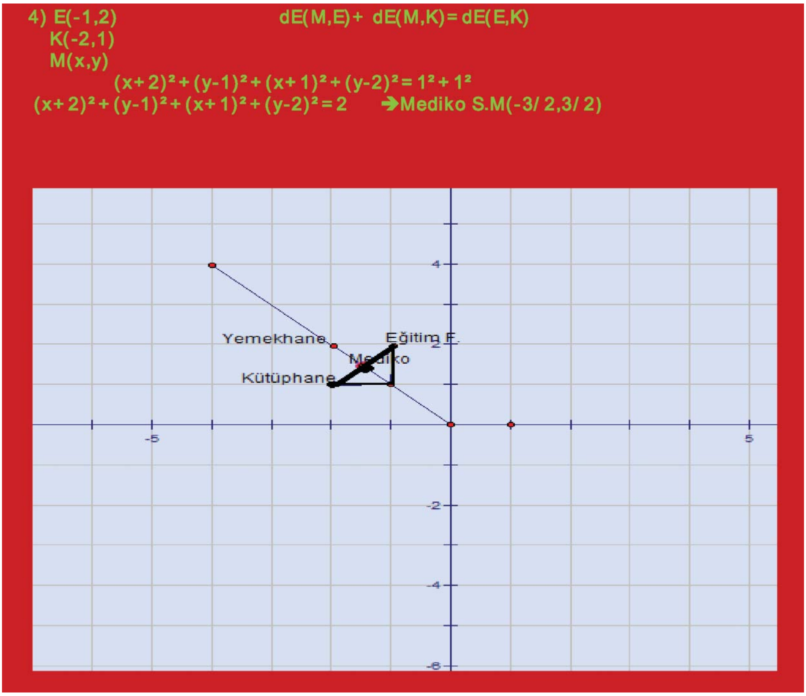

Figure 19. Students' graphics on the Geometer's Sketchpad (color figure available online).

> and learned from the experts about the points to consider while designing a city. Every time we learned something new, I felt as if we had just found a treasure. This study helped me better see the immensity of geometry. Now, I am aware of the existence of non-Euclidean geometries, and I made use of Taxicab geometry among them. I understood by the end of this study that just learning and reading or writing about something alone did not necessarily bring about permanent knowledge. I acquired more lasting knowledge thanks to the projects.

Through this study, Meral learned about Taxicab geometry, which she normally used in her daily life, albeit unconsciously. She emphasized that she had acquired permanent knowledge, with the help of project-based learning, by means of discovering where and how Taxicab geometry was used.

Similarly, Ali said it was the first time he had ever come across non-Euclidean geometry with practical usage in everyday life. It is also clear from his statements that he became conscious of the fact that a city to be designed in accordance with this geometry, not that common in Turkey, would be ideal in terms of both arrangement and transportation:

> I did not know that there was another type of geometry apart from Euclidean geometry. The more research I did, the more interested I became. When I saw that some cities were planned in this way, I realized that this type of geometry was not utopian, but a wonderful model when given a chance. Obviously, Taxicab geometry brought about both efficient arrangement and ideality through fewer structures, in terms of closeness to the ground. Therefore, it also led to less cost.

Ayşe also had ideas similar to Ali's:

> We realized that Taxicab geometry was an ideal geometry for designing an ideal city. In Taxicab geometry, we used fewer buildings than in Euclidean geometry, and we achieved a more orderly city. Also, now I look at the streets and city centers with completely different and more critical eyes.

Murat stated that the use of a non-Euclidean geometry widened his perspective on geometry itself. He also stressed the efficiency of working as a group:

> My perspective on geometry changed and expanded. That I learned about a non-Euclidean type of geometry led me to attach a different meaning to geometry. In addition, I learned how efficient group activities could be.

Elif, on the other hand, highlighted the importance of research she did for the project. She also stated it was through this study that she realized mathematics was a natural part of life.

> I understood that a person's horizons broadened only through research. We acquired creative knowledge. If only our country had the suitable geographical and economic conditions needed for implementing this project, and if only future generations would see, just like us, that mathematics is not just in theory, but with us everywhere and at all times.

Demet said that she would have her future students compare and contrast the two types of geometries by using Taxicab geometry, which she learned about through this study, so that they could better learn Euclidean geometry:

> I took great pleasure in learning Taxicab geometry, because people become happier when they learn things that they themselves apply and that exist within life itself. Although Taxicab geometry is not that common in our country, learning this subject at this age made me really happy. I'm sure that, as a future teacher, I will tell my future students

> about Taxicab geometry, and have them compare these two types of geometry.

Mehmet, on the other hand, emphasized group tasks promoting self-development, and that group members produced a nice project product using creative and different opinions:

> Throughout the project, everyone's creativity was in the forefront, and we were able to value each other's ideas, criticize constructively, and transform all that was learned into a product.

Finally, Oya stated that project-based learning rendered their study more effective:

> Perhaps, the numbers, solutions, and shapes of this assignment would disappear on papers after all. However, group awareness, investigation, consulting, accessing knowledge, being able to create a product, and present it in front of an audience, will never leave our minds.

5. CONCLUSIONS

The student opinions collected at the end of this project suggest that they took great pleasure in participating in the study. The study attracted students' attention, and strengthened their interpersonal relationships. They realized there were other types of geometries apart from Euclidean geometry, and, as a result their perspectives on geometry were broadened. They also saw that Taxicab geometry was quite a practical type of geometry in urban geography, occupying an important place in their daily lives. They further realized that the concept of distance, used in real life, sometimes emerged as Euclidean distance, and sometimes as Taxicab distance.

In field courses in Turkey, Mathematics teacher candidates are generally taught using a teacher-centered approach. Project-based learning makes it possible for students to see the practical usage of the mathematical topics they learn in everyday life [9]. In this study, the students became conscious of the importance of Taxicab geometry by means of project-based learning. Therefore, we believe project-based learning should be used more in mathematics teacher education for other mathematics subjects as well.

Since the content designed by YOK had to be followed strictly, the time duration assigned for this study was limited. The effect of project-based learning on student achievement, in teaching non-Euclidean types of geometry, can be further investigated through experimental studies increasing study time duration.

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BIOGRAPHICAL SKETCHES

Tuba Ada received her B.S. degree in mathematics from Anadolu University, Eskişehir, Turkey, in 1994. She received her Ph.D. degree in mathematics from Eskişehir Osmangazi University, Eskişehir, Turkey, in 2002. She studied trigonometry in Iso-Taxicab geometry in her Ph.D. dissertation. She has been assistant professor in the education faculty at Anadolu University from 2005 to the present. She has taught geometry, syntetic geometry, analytic geometry, and advanced calculus courses for eight years. She is interested in Euclidean and non-Euclidean geometry teaching.

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