Self-duality in higher dimensions

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Abstract. Let ω be a 2-form on a $2n$ dimensional manifold. In previous work, we called ω "strong self-dual, if the eigenvalues of its matrix with respect to an orthonormal frame are equal in absolute value. In a series of papers, we showed that strong self-duality agrees with previous definitions; in particular if ω is strong self-dual, then, in $2n$ dimensions, ω^n is proportional to its Hodge dual ω and in $4n$ dimensions, ω^n is Hodge self-dual. We also obtained a local expression of the Bonan 4-form on 8 manifolds with $Spin_7$ holonomy, as the sum of the squares of any orthonormal basis of a maximal linear subspace of strong self-dual 2-forms. In the present work we generalize the notion of strong self-duality to odd dimensional manifolds and we express the dual of the Fundamental 3-form 7 manifolds with G_2 holonomy, as a sum of the squares of an orthonormal basis of a maximal linear subspace of strong self-dual 2-forms.

1. Introduction

The mathematical framework of gauge theories consists of vector bundles over differentiable manifolds. These vector bundles are equipped with a connection that takes values in the Lie algebra of the structure group of the vector bundle. "Action integrals" are expressed in terms of the inner products of the components of the curvature 2-form. The variation of the action integral leads to the field equations of the theory. Yang-Mills theory on 4-manifolds is the prototype of this scheme. In Yang-Mills theories, the action integral is just the norm of the Lie algebra valued curvature 2-form of this vector bundle. Yang-Mills equations obtained by the variation of the action integral are satisfied provided that the curvature 2-form is self dual.

In a series of papers we defined the notion of "strong self-duality" in even dimensions and we studied the minimization of various action integrals, algebraically, by comparing inner products and wedge products of 2-forms. Strong self-dual forms turned out to be the building blocks for the curvature 2-forms that minimize various action integrals. [1],[2],[3],[4],[5] In the present work, we generalize the notion of strong self-duality to odd dimensions. This generalization relies on the results obtained in [10] pertaining skew-symmetric matrices in odd dimensions. We present a brief overview of the properties of strong self-dual 2-forms in even dimensions and we state without proof, relevant results in odd dimensions.

2. Self-duality and Strong Self-duality

The motivation of our definition of strong self-duality is the eigenvalue structure of self-dual 2-forms in 4-dimensions. We start by a description of their eigenvalues.

2.1. Self-duality in 4-dimensions

Let e^i, $i = 1, \ldots, 4$ be a local, positively oriented, orthonormal basis for the cotangent bundle T^*M of a 4-dimentional M be a 4-dimensional, oriented Riemannian manifold M. The local expression of a 2-form is given by

$$\omega = \sum_{i<j} \omega_{ij} e^{ij}, \tag{1}$$

where $e^{ij} = e^i \wedge e^j$. The self-dual ($*\omega = \omega$) and anti-self-dual ($*\omega = -\omega$) 2-forms are eigenvectors of the Hodge duality map, $*$, and in 4-dimensions, the 6-dimensional linear space of 2-forms is the orthogonal direct sum of the 3-dimensional self-dual and anti self-dual 2-forms.

The ω_{ij}'s can be viewed as a skew-symmetric matrix whose eigenvalues are pure imaginary and occur in conjugate pairs. Denoting them as $\pm i\lambda_1$ and $\pm i\lambda_2$, it can be seen that they satisfy

$$\begin{aligned} \lambda_1^2 + \lambda_2^2 &= \omega_{12}^2 + \omega_{13}^2 + \omega_{14}^2 + \omega_{23}^2 + \omega_{24}^2 + \omega_{34}^2, \\ \lambda_1^2 \lambda_2^2 &= (\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23})^2. \end{aligned}$$

Hence

$$\lambda_1 \mp \lambda_2 = \sqrt{(\omega_{12} \mp \omega_{34})^2 + (\omega_{13} \pm \omega_{24})^2 + (\omega_{14} \mp \omega_{23})^2}.$$

For self-duality $\lambda_1 = \lambda_2$, while for anti-self-duality $\lambda_1 = -\lambda_2$. These two cases are distinguished by the sign of the Pfaffian, $\text{Pf}(\omega) = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23}$. Thus in four dimensions, the equality of the absolute values of the eigenvalues gives the usual notion of self-duality in the Hodge sense.

2.2. Strong self-duality in even dimensions

In higher (even) dimensions, we proposed the equality of the eigenvalues in absolute value as a definition of the self-duality of 2-forms. Let ω be a 2-form on a $2n$ dimensional manifold M. If $\{e^1, e^2, \ldots, e^{2n}\}$ is a local orthonormal basis for the cotangent bundle of M, then ω is expressed as in (1) and the corresponding skew-symmetric matrix has eigenvalues $\{\pm i\lambda_1, \ldots, \pm i\lambda_n\}$. We define n ω to be strong self-dual (respectively, strong anti-self-dual) if

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n| \tag{2}$$

and $*\omega^n > 0$ (respectively $*\omega^n < 0$). [1] Note that this is equivalent to the statement that the distinction is based on the sign of the Pfaffian of ω with respect to a positively oriented orthonormal basis.

In terms of matrices, strong self-duality or anti self-duality can also be expressed by the minimal polynomial requirement

$$\omega^2 + \lambda^2 I = 0 \tag{3}$$

where $\lambda^2 = -\frac{1}{2n} Tr\omega^2$.

In four dimensions, the strong self-duality coincides with usual Hodge duality. More precisely, the matrices satisfying $\omega^2 + \lambda^2 I = 0$ consist of the union of the usual self-dual and anti-self-dual 2-forms (including the zero form).

2.3. Strong self-duality in odd dimensions

In odd dimensions, the right way to generalize the notion of strong self-duality is via the minimal polynomial of the corresponding skew-symmetric matrix. In odd dimension, every skew-symmetric matrix is necessarily singular. Thus, the closest analogue of (2) and (3) would be to require that the kernel be 1-dimensional and the non-zero eigenvalues be equal in absolute value. Equivalently be can require that the minimal polynomial of the corresponding matrix be

$$(\omega^2 + \lambda^2 I)\omega = 0, \tag{4}$$

with $\lambda^2 = -\frac{1}{2n}Tr\omega^2$ as before. Note that a strong self-dual 2-form in $2n+1$ dimension restricted to the complement of its kernel satisfies (3), hence it is essentially a strong self-dual 2 form in $2n$ dimensions. The complication arises when we algebraic variety defined by (3), and in particular linear subspaces lying in this variety.

The structure of matrices satisfying (4) has been analyzed in [10]. We adopt the notation of [10] and denote the rank, trace, kernel and orthogonal complement of the kernel of a matrix A by, r_A, $Tr(A)$, $Ker(A)$ and W_A. We denote the set of $(2n+1) \times (2n+1)$ matrices with minimal polynomial (4) by $\mathcal{S}_{2n+1}$. Note that the elements of $\mathcal{S}_{2n+1}$ need not have constant rank and its structure as an algebraic variety can be complicated. On the other hand, linear subspaces of $\mathcal{S}_{2n+1}$, that we denote as $\mathcal{L}$ have a more uniform structure. We briefly quote relevant results form [10]. If A and B are two matrices belonging to a linear subspace of $\mathcal{S}_{2n+1}$, then any linear combination of these matrices should satisfy (4). This gives,

$$A^2B + ABA + BA^2 + \lambda_A^2 B = 0, \qquad AB^2 + BAB + B + \lambda_B^2 A = 0. \qquad (5)$$

By Remark 1, in[10], If a vector X belongs to $Ker(A)$, then $Y = ABX$ belongs to $Ker(B)$ and it follows that matrices belonging to the same linear subspace have constant rank. The structure of linear subspaces is determined by the Lemma below (Lemma 5 in [10]).

Lemma A. *Let $\{A, B\}$ be a basis for a 2-dimensional subspace in $\mathcal{S}_{2n+1}$ such that $A \perp B$.*

(i)If $Ker(A) = Ker(B)$, then $AB + BA = 0$.

(ii) If $Ker(A) \cup Ker(B) = \{0\}$, them $KerA() \perp Ker(B)$.

Linear subspaces of $\mathcal{S}_{2n+1}$ are called to be of Type 1 or Type 2, according as pairs of matrices satisfy (i) or (ii) of Lemma A. We quote below Lemma 8 in [10].

Lemma B. *Let $\mathcal{L}$ be a linear subspace of $\mathcal{S}_{2n+1}$.*

(i) If $\mathcal{L}$ is of the first type, then there exists a subspace $\mathcal{K} \subset \mathcal{S}_{2n}$ such that $dim(\mathcal{L}) = dim(\mathcal{K})$.

(ii) If $\mathcal{L}$ is of the second type, then there exists a subspace $\mathcal{M} \subset \mathcal{S}_{2n+2}$ such that $dim(\mathcal{L}) = dim(\mathcal{M})$.

3. Properties of strong self-dual 2 forms

In this section we give a brief review of the basic properties of strong self-dual 2-forms in even dimensions and we state without proof thse properties that generalize to odd dimensions.

3.1. Eigenvalue Inequalities

Well known inequalities between elementary symmetric functions of the eigenvalues of a skew symmetric matrix leads to inequalities between the norms of the powers of a 2-form ω. In this setting, strong self dual 2 forms are the ones for which the norms of ω^k are maximal. In $2n$ dimensions, all inequalities are saturated, as shown in previous work. We start by quoting a result from [9]

Lemma C. *Let s_k be the kth elementary symmetric function of the numbers $\{\alpha_1, \alpha_2, ..., \alpha_n\}$, with $\alpha_i \in R$ and let the weighted elementary symmetric polynomials, q_k's, be defined by*

$$\binom{n}{k}q_k = s_k. \qquad (6)$$

Then

$$q_1 \geq q_2^{1/2} \geq q_3^{1/3} \geq \cdots \geq q_n^{1/n}, \qquad (7)$$

$$q_{r-1}q_{r+1} \le q_r^2, \qquad 1 \le r < n. \tag{8}$$

If all α_i's are equal, then the equalities hold and if any single equality holds, then all α_i's are equal.

The norm ω^k is related to q_k as

$$(\omega^k, \omega^k) = \frac{n!k!}{(n-k)!}q_k.$$

From inequalities (7), we obtain,

$$(\omega^k, \omega^k) \le \frac{n!k!}{n^k(n-k)!}(\omega, \omega)^k \tag{9}$$

and equalities hold if and only if ω is strong self-dual.

In $(2n+1)$ dimensions, a strong-self-dual 2-form has a 1-dimensional kernel and it its restriction to the complement of its kernel is a strong self-dual 2-form in $2n$ dimensions. Therefore the inequalities above hold for appropriate values of n and equalities hold again if and only if ω is strong self-dual.

3.2. Equivalence of strong self-duality with previous definitions of self-duality
Using the eigenvalue inequalities, we have shown that (i) a 2-form ω in $2n$ dimensions is strong self-dual if and only if ω^{n-1} is proportional to the Hodge dual of ω, and (ii) a 2-form ω in $4n$ dimensions is strong self-dual if ω^n is self-dual in the Hodge sense . The first condition has been proposed as a definition of self-duality by Trautman [11] while the second one appears in the work of Grossman et al. [8].

These properties have no direct analogues in odd dimensions. The best we can say is the following. Given the 2-form ω in $(2n+1)$-dimensions, the 1-dimensional kernel of the corresponding skew-symmetric matrix determines a 1-form α_ω. Then, if ω is strong self-dual, then

$$\omega^{n-1} \wedge \alpha = k * \omega, \tag{10}$$

where k is a constant.

3.3. Maximal linear subspaces of strong self-dual 2-forms in eight dimensions
Maximal linear spaces of strong self-dual 2-forms on 8 manifolds 7-dimensional. We work with the subspace with basis elements given below.

$$
\begin{aligned}
\omega_1 &= -e^{12} + e^{34} + e^{56} - e^{78}, \\
\omega_2 &= -e^{13} - e^{24} + e^{57} + e^{68}, \\
\omega_3 &= -e^{14} + e^{23} + e^{58} - e^{67}, \\
\omega_4 &= -e^{15} - e^{26} - e^{37} - e^{48}, \\
\omega_5 &= -e^{16} + e^{25} - e^{38} + e^{47}, \\
\omega_6 &= -e^{17} + e^{28} + e^{35} - e^{46}, \\
\omega_7 &= -e^{18} - e^{27} + e^{36} + e^{45}.
\end{aligned}
\tag{11}
$$

3.4. Maximal linear subspaces of strong self-dual 2-forms in 7 dimensions

We recall that maximal linear subspaces of strong self-dual 2-forms in $(2n+1)$ dimensions are inherited from $2n$ or $2n+2$ dimensions. For $2n+1=7$, maximal linear subspaces in $2n=6$ dimensions are 1 dimensional, while the maximal linear subspaces in $2n+2=8$ dimensions are 7-dimensional, as described above. The matrices corresponding these 2-forms can be computed by solving the equations (5) and they are given in [10].

$$\omega_1 = -e^{12} + e^{34} + e^{56},$$

$$\omega_2 = -e^{13} - e^{24} + e^{57},$$

$$\omega_3 = -e^{14} + e^{23} - e^{67},$$

$$\omega_4 = -e^{15} - e^{26} - e^{37},$$

$$\omega_5 = -e^{16} + e^{25} + e^{47},$$

$$\omega_6 = -e^{17} + e^{35} - e^{46},$$

$$\omega_7 = -e^{27} + e^{36} + e^{45}. \tag{12}$$

4. Linear subspaces defined via $n-4$ forms

The self-duality equations given in [7] can be described as eigenspaces of the (self-adjoint) map defined via a 4-form ϕ as follows.

$$T_\Phi(\omega) = *(\Phi \wedge \omega). \tag{13}$$

This definition makes sense in any dimension, provided that such a (globally defined) 4-form exists. In 8-dimensions, the strong self-dual forms given in the previous section are eigenvectors of the map defined via the Bonan form. The restriction of the Bonan form to 7-dimensions gives the dual of the "fundamental 3-form" ϕ. In 7-dimensions ϕ defines a linear map on 2-forms.

4.1. Bonan-form in 8-dimensions

The Bonan-form Φ, which is a $Spin(7)$-invariant 4-form in 8-dimensions [6], can be constructed in terms of strong self-dual 2-forms. We consider a 7-plane $\mathcal{L}$ with an orthonormal basis $\{\eta_1, \eta_2, \ldots, \eta_7\}$ and let

$$\Phi = \eta_1^2 + \eta_2^2 + \cdots + \eta_7^2 \tag{14}$$

This expression is independent of the basis of the 7-plane. To give an explicit expression we can use the orthonormal basis $\eta_i = \frac{1}{2}\omega_i$, with ω_i given by (11) and we obtain

$$\Phi = -\frac{3}{2}(e^{1234} + e^{1256} - e^{1278} + e^{1357} + e^{1368} + e^{1458} - e^{1467}$$

$$-e^{2358} + e^{2367} + e^{2457} + e^{2468} - e^{3456} + e^{3478} + e^{5678}). \tag{15}$$

Furthermore, the ω_i's are eigenvalues of the map (13) defined via the Bonan form, with eigenvalue $9/2$, i.e,.

$$*(\Phi \wedge \omega_i) = \frac{9}{2}\omega_i. \tag{16}$$

4.2. The fundamental 3-*form in* 7-*dimensions*

The properties described above for the Bonan form generalize to 7-dimensions. The Fundamental 3-form ψ is G_2 invariant. We consider again a 7-plane $\mathcal{L}$ with an orthonormal basis $\{\eta_1, \eta_2, \ldots, \eta_7\}$ and let

$$*\psi = \eta_1^2 + \eta_2^2 + \cdots + \eta_7^2 \tag{17}$$

As above, this definition is independent of the basis of the 7-plane. We the orthonormal basis $\eta_i = \frac{1}{\sqrt{3}}\omega_i$, with ω_i given by (12) and we obtain

$$*\psi = -2(e^{1234} + e^{1256} + e^{1357} - e^{1467} + e^{2367} + e^{2457} - e^{3456}). \tag{18}$$

The ω_i's are eigenvalues of the map (13) defined via the the Fundamental 3-form, with eigenvalue 4, i.e,.

$$*(\psi \wedge \omega_i) = 4\omega_i. \tag{19}$$

We finally note that explicit solutions related to these structure were given for $Spin_7$ and G_2 manifolds in[12] and [13] respectively.

References

[1] Bilge A.H., Dereli T. and Koçak Ş., Self-dual Yang-Mills fields in eight dimensions, Lett. Math. Phys., 1996 , V. 36, 301.
[2] Bilge A.H., Dereli T. and Koçak Ş., The geometry of self-dual two-forms, J. Math. Phys., 1997, V. 38, N. 8, 4804.
[3] A.H.Bilge, T.Dereli,Ş.Koçak, The geometry of self-dual gauge fields, *New Symmetries in the Fundamental Interactions Theory*, edited by J. Lukierski, A.Jadczyk (PWN Polish Scientific Publishers,Warszawa,1997) p.45
[4] Bilge A.H., Dereli T. and Koçak Ş., Seiberg-Witten equations on R^8, Proceedings of 5th Gökova Geometry-Topology Conference, Edited by S.Akbulut, T.Önder, R, Stern (TUBITAK, Ankara, 1997), p.87.
[5] Bilge A.H., Dereli T. and Koçak Ş., Maximal linear subspaces of strong self-dual 2-forms and Bona 4-form, Linear Algebra and its Applications 434 (2011) 12001214
[6] Bonan E., Sur les varieties riemanniennes a groupe d'holonomie G_2 ou $Spin(7)$, C.R. Acad. Sci. Paris, 1966, V.262, 127-129.
[7] Corrigan E., Devchand C., Fairlie D. and Nuyts J., First order equations for gauge fields in spaces of dimensions greater than four, Nucl. Phys. B., 1983, V. 214, 452-464.
[8] Grossman B., Kephart T.W. and Statsheff J.D., Solutions to Yang-Mills field equations in eight dimensions and the last Hopf map, Comm. Math. Phys., 1984, V. 96, 431-437.
[9] M. Marcus and H.Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon Inc., Boston, (1964).
[10] Y. Ozdemir, "Spaces of Skew-Symmetric Matrices Satisfying $A^3 = \lambda A$", Linear Algebra and Its Applications, Volume 438, Page: 1365-1371, (2013).
[11] Trautman A., Solutions of the Maxwell and Yang-Mills equations associated with Hopf fiberings, Int. J. Theor. Phys., 1977, V.16, 561-565.
[12] S.Uguz and A.H. Bilge, $(3+3+2)$-warped-like manifolds with $Spin_7$ holonomy, Journal of Geometry and Physics, Volume 61, Issue 6, p. 1093-1103.
[13] S.Uguz, "Special warped-like product manifolds with (week) G_2 holonomy", Ukranian Mathematical Journal, Volume 65, No.8, pp.1126-1140, (2013).