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Integrable G_2 Structures on 7-dimensional 3-Sasakian Manifolds

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Abstract: It is known that there exist canonical and nearly parallel G_2 structures on 7-dimensional 3-Sasakian manifolds. In this paper, we investigate the existence of G_2 structures which are neither canonical nor nearly parallel. We obtain eight new G_2 structures on 7-dimensional 3-Sasakian manifolds which are of general type according to the classification of G_2 structures by Fernandez and Gray. Then by deforming the metric determined by the G_2 structure, we give integrable G_2 structures. On a manifold with integrable G_2 structure, there exists a uniquely determined metric covariant derivative with anti-symetric torsion. We write torsion tensors corresponding to metric covariant derivatives with skew-symmetric torsion. In addition, we investigate some properties of torsion tensors.

7 Boyutlu 3-Sasaki Manifoldları Üzerinde İntegrallenebilir G_2 Yapılar

Anahtar Kelimeler
G_2 grubu,
G_2 yapı,
3-Sasaki yapı

Özet: 7-boyutlu 3-Sasaki manifoldlar üzerinde kanonik ve hemen-hemen paralel G_2 yapıların varlığı bilinmektedir. Bu çalışmada kanonik veya hemen-hemen paralel olmayan G_2 yapıların varlığı incelenmiştir. 7-boyutlu 3- Sasaki manifoldları üzerinde, Fernandez ve Gray'in G_2 yapı sınıflandırmasına göre en geniş sınıfta yer alan sekiz tane yeni G_2 yapı elde edilmiştir. Daha sonra ise, elde edilen G_2 yapıların ürettikleri metrikler deforme edilerek, integrallenebilir G_2 yapılar bulunmuştur. İntegrallenebilir G_2 yapısına sahip bir manifold üzerinde torsiyonu anti-simetrik olan tek türlü belirli bir metrik kovaryant türev vardır. Her bir integrallenebilir G_2 yapı için, bu kovaryant türevin torsiyonu yazılmış ve buna ek olarak, torsiyonun bazı özellikleri incelenmiştir.

1. Introduction

Let $\{e_1,\dots,e_7\}$ be the standard basis of the real vector space $\mathbb{R}^7$ with dual basis $\{e^1,\dots,e^7\}$. Consider the 3-form

$$\varphi_0 = e^{123}+e^{145}+e^{167}+e^{246}-e^{257}-e^{347}-e^{356}$$

where $e^{ijk}=e^i\wedge e^j\wedge e^k$. φ_0 is called the fundamental 3-form on $\mathbb{R}^7$. The group G_2 is defined as

$$G_2 := \{f\in GL(7,\mathbb{R}) \mid f^*\varphi_0=\varphi_0\}$$

and is a compact, simple and simply connected 14-dimensional Lie subgroup of $GL(7,\mathbb{R})$ [1].
A 7-dimensional smooth manifold M is called a manifold with G_2 structure if there exists a 3-form φ on M which can be locally written as φ_0. The 3-form φ determines a Riemannian metric g, a volume form d_{vol} and a 2-fold vector cross product P on M by

$$(x\lrcorner\,\varphi)\wedge(y\lrcorner\,\varphi)\wedge\varphi = 6g(x,y)d_{vol} \tag{1}$$

$$\varphi(x,y,z)=g(P(x,y),z) \tag{2}$$

for any $x,\ y,\ z\in\Gamma(TM)$ [1].

Let (M,g) be a manifold with G_2 structure φ. Fernández and Gray showed in [2] that for all $m\in M$, the Levi-Civita covariant derivative $\nabla\varphi$ is in the space

$$W=\{\alpha\in T_mM^*\otimes\Lambda^3(T_mM)^* \quad | \quad \alpha(u,x,y,P(x,y))=0,\ \forall\, u,x,y,z\in T_mM\}.$$

This space is written as a direct sum of four G_2-irreducible subspaces W_1, W_2, W_3 and W_4 with corresponding dimensions 1, 14, 27 and 7 as

$$W=W_1\oplus W_2\oplus W_3\oplus W_4.$$

A manifold with G_2 structure is said to be of type $\mathscr{P}$, $\mathscr{W}_i$, $\mathscr{W}_i\oplus\mathscr{W}_j$, $\mathscr{W}_i\oplus\mathscr{W}_j\oplus\mathscr{W}_k$ or $\mathscr{W}$, if $\nabla\varphi$ is in $\{0\}$, W_i, $W_i\oplus W_j$, $W_i\oplus W_j\oplus W_k$ or W, respectively, for $i,\ j,\ k=1,2,3,4$. The defining relations of all 16 classes are given by Fernández and Gray [2] and an equivalent characterization is done by Cabrera using $d\varphi$ and $d*\varphi$ in [3]. This characterization is illustrated in the following table.
Let (M,g) be a Riemannian manifold with Levi-Civita covariant derivative ∇ of g. (M,g) is called Sasakian if there exists a Killing vector field ξ of unit length on M so

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$\mathscr{P}$	$d\varphi = 0$ and $d*\varphi = 0$
$\mathscr{W}_1$	$d\varphi = k*\varphi$ and $d*\varphi = 0$
$\mathscr{W}_2$	$d\varphi = 0$
$\mathscr{W}_3$	$d*\varphi = 0$ and $d\varphi\wedge\varphi = 0$
$\mathscr{W}_4$	$d\varphi = \alpha\wedge\varphi$ and $d*\varphi = \beta\wedge*\varphi$
$\mathscr{W}_1\oplus\mathscr{W}_2$	$d\varphi = k*\varphi$ and $*d*\varphi\wedge*\varphi = 0$
$\mathscr{W}_1\oplus\mathscr{W}_3$	$d*\varphi = 0$
$\mathscr{W}_2\oplus\mathscr{W}_3$	$d\varphi\wedge\varphi = 0$ and $*d\varphi\wedge\varphi = 0$
$\mathscr{W}_1\oplus\mathscr{W}_4$	$d\varphi = \alpha\wedge\varphi + f*\varphi$ and $d*\varphi = \beta\wedge*\varphi$
$\mathscr{W}_2\oplus\mathscr{W}_4$	$d\varphi = \alpha\wedge\varphi$
$\mathscr{W}_3\oplus\mathscr{W}_4$	$d\varphi\wedge\varphi = 0$ and $d*\varphi = \beta\wedge*\varphi$
$\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_3$	$*d\varphi\wedge\varphi = 0$ or $*d*\varphi\wedge*\varphi = 0$
$\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_4$	$d\varphi = \alpha\wedge\varphi + f*\varphi$
$\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$	$d*\varphi = \beta\wedge*\varphi$
$\mathscr{W}_2\oplus\mathscr{W}_3\oplus\mathscr{W}_4$	$d\varphi\wedge\varphi = 0$
$\mathscr{W}$	no relation

Table 1. Defining relations for classes of G_2 structures

that the tensor field Φ of type (1,1), defined by $\Phi(x) = \nabla_x\xi$, satisfies the condition

$$(\nabla_x\Phi)(y) = g(\xi,y)x - g(x,y)\xi$$

for all x, $y\in\Gamma(TM)$. The triple (ξ,η,Φ), where η is the metric dual of ξ, is said to be a Sasakian structure on (M,g) [4].
(M,g) is called 3-Sasakian if there exist three Sasakian structures $(\xi_i,\eta_i,\Phi_i)_{i=1,2,3}$ on (M,g) with properties

$$g(\xi_i,\xi_j) = \delta_{ij}$$

for $i,j = 1,2,3$ and

$$[\xi_1,\xi_2] = 2\xi_3,\ [\xi_2,\xi_3] = 2\xi_1,\ [\xi_3,\xi_1] = 2\xi_2.$$

$(\xi_i,\eta_i,\Phi_i)_{i=1,2,3}$ is called a 3-Sasakian structure on (M,g) [4].
It is known that the vertical subbundle $T^v\subset TM$ is spanned by $\{\xi_1,\xi_2,\xi_3\}$. The subbundle T^v and its complement, the horizontal subbundle T^h, are both invariant under Φ_1, Φ_2, Φ_3 [5].
Let (M,g) be a 7-dimensional, compact, simply-connected 3-Sasakian manifold with Sasakian structure (ξ_i,η_i,Φ_i) for $i\in\{1,2,3\}$. Then there exists a local orthonormal frame $\{e_1,\cdots,e_7\}$ such that $e_1 = \xi_1$, $e_2 = \xi_2$, $e_3 = \xi_3$ and Φ_i act on $T^h := span\{e_4,\cdots,e_7\}$ by matrices given in [5]. Denote the corresponding co-frame by $\{\eta_1,\cdots,\eta_7\}$. The differentials of 1-forms η_1, η_2 and η_3 are computed in [5] according to this frame.
A 7-dimensional 3-Sasakian manifold admits three nearly parallel (of type $\mathscr{W}_1$) G_2 structures φ_i, $i = 1,2,3$, given in [5, 6].
Consider also the 3-form φ defined globally on M by

$$\varphi := F_1 + F_2,$$

where

$$F_1 := \eta_1\wedge\eta_2\wedge\eta_3,$$

$$F_2 := \frac{1}{2}(\eta_1\wedge d\eta_1 + \eta_2\wedge d\eta_2 + \eta_3\wedge d\eta_3) + 3\eta_1\wedge\eta_2\wedge\eta_3.$$

It is proved in [5] that φ gives a G_2 structure on M of type $\mathscr{W}_1\oplus\mathscr{W}_3$ such that $d*\varphi = 0$ and $d\varphi = 12*F_1 + 4*F_2$. This is called the canonical G_2 structure of the 7-dimensional 3-Sasakian manifold.
The canonical G_2 structure φ is deformed in [5, 6] to get a nearly parallel G_2 structure in the following way:
Let $s>0$ and consider the Riemannian metric g^s defined by

$$g^s(x,y) := \begin{cases} g(x,y) & \text{if } x \text{ (or } y) \in T^h \\ s^2 g(x,y) & \text{if } x \text{ and } y\in T^v \end{cases}$$

The set $\{\xi_1/s,\xi_2/s,\xi_3/s,e_4,e_5,e_6,e_7\}$ is a g^s-orthonormal frame and the corresponding co-frame is $\{s\eta_1,s\eta_2,s\eta_3,\eta_4,\eta_5,\eta_6,\eta_7\}$.
Define the 3-form

$$\varphi^s := F_1^s + F_2^s,$$

where

$$F_1^s := s^3F_1,\ \ F_2^s := sF_2.$$

It is shown in [5, 6] that the 3-form φ^s gives a G_2 structure on (M,g^s) and this structure is nearly parallel iff $s = 1/\sqrt{5}$.

2. Canonical and Nearly Parallel G_2 Structures

Let (M,g) be a 7-dimensional 3-Sasakian manifold with the 3-Sasakian structure $(\xi_i,\eta_i,\Phi_i)_{i=1,2,3}$. Consider the 3-form

$$\begin{aligned}\omega &= a\eta_1\wedge d\eta_1 + b\eta_2\wedge d\eta_2 + c\eta_3\wedge d\eta_3 \\ &\quad + f\eta_1\wedge\eta_2\wedge\eta_3,\end{aligned} \tag{3}$$

where a, b, c, f are arbitrary constants. This 3-form gives a G_2 structure on M iff the equation (1) holds. Choose the orthonormal frame given in [5]. Then following relations are obtained among coefficients of the 3-form (3):

$$\begin{aligned} a^2(-2(a+b+c)+f) &= \frac{1}{4}, \\ b^2(-2(a+b+c)+f) &= \frac{1}{4}, \\ c^2(-2(a+b+c)+f) &= \frac{1}{4}, \\ abc &= \frac{1}{8}. \end{aligned}$$

Solutions of the system above are

1. $a = 1/2, b = 1/2, c = 1/2, f = 4,$
2. $a = 1/2, b = -1/2, c = -1/2, f = 0,$
3. $a = -1/2, b = 1/2, c = -1/2, f = 0,$
4. $a = -1/2, b = -1/2, c = 1/2, f = 0.$

Hence we express the fundamental 3-forms on M corresponding to the same metric g:

$$\omega_1 = \frac{1}{2}\eta_1\wedge d\eta_1 + \frac{1}{2}\eta_2\wedge d\eta_2 + \frac{1}{2}\eta_3\wedge d\eta_3 + 4\eta_1\wedge\eta_2\wedge\eta_3,$$

$$\omega_2 = \frac{1}{2}\eta_1\wedge d\eta_1 - \frac{1}{2}\eta_2\wedge d\eta_2 - \frac{1}{2}\eta_3\wedge d\eta_3,$$

$$\omega_3 = -\frac{1}{2}\eta_1\wedge d\eta_1 + \frac{1}{2}\eta_2\wedge d\eta_2 - \frac{1}{2}\eta_3\wedge d\eta_3,$$

$$\omega_4 = -\frac{1}{2}\eta_1 \wedge d\eta_1 - \frac{1}{2}\eta_2 \wedge d\eta_2 + \frac{1}{2}\eta_3 \wedge d\eta_3.$$

The 3-form ω_1 is called the canonical G_2 structure on M and is cocalibrated [5] (i.e. of type $\mathscr{W}_1 \oplus \mathscr{W}_3$, [2]). Other 3-forms are nearly parallel (i.e. of type $\mathscr{W}_1$) and are constructed in [5]. To sum up, there is no G_2 structure on M of the form (3) except the ones constructed in [5].

3. New G_2 Structures

Consider a 7-dimensional 3-Sasakian manifold (M,g) whose 3-Sasakian structure is $(\xi_i, \eta_i, \Phi_i)_{i=1,2,3}$. Take the linear combination

$$\begin{aligned} \varphi \;=\; & a\, d\eta_1 \wedge \eta_2 + b\, \eta_1 \wedge d\eta_2 + c\, d\eta_1 \wedge \eta_3 \\ & + d\, \eta_1 \wedge d\eta_3 + e\, d\eta_2 \wedge \eta_3 + f\, \eta_2 \wedge d\eta_3 + h\, \eta_{123} \end{aligned} \tag{4}$$

for $a,\ b,\ c,\ d,\ e,\ f,\ h \in \mathbb{R}$. To get a G_2 structure on M, equations below should be satisfied:

$$\begin{aligned} h(b^2+d^2) &= \frac{1}{4}, \\ h(a^2+f^2) &= \frac{1}{4}, \\ h(c^2+e^2) &= \frac{1}{4}, \\ dfh &= 0, \\ beh &= 0, \\ ach &= 0, \\ ade+bcf &= \frac{1}{8}. \end{aligned}$$

Following G_2 structures are obtained:

1. $a = d = e = 0,\ h = 1$
   (a) $b = 1/2,\ c = 1/2,\ f = 1/2$,
   $\varphi_1 = \frac{1}{2}\ \eta_1 \wedge d\eta_2 + \frac{1}{2}\ d\eta_1 \wedge \eta_3 + \frac{1}{2}\ \eta_2 \wedge d\eta_3 + \eta_{123}$
   (b) $b = -1/2,\ c = -1/2,\ f = 1/2$,
   $\varphi_2 = -\frac{1}{2}\ \eta_1 \wedge d\eta_2 - \frac{1}{2}\ d\eta_1 \wedge \eta_3 + \frac{1}{2}\ \eta_2 \wedge d\eta_3 + \eta_{123}$
   (c) $b = -1/2,\ c = 1/2,\ f = -1/2$,
   $\varphi_3 = -\frac{1}{2}\ \eta_1 \wedge d\eta_2 + \frac{1}{2}\ d\eta_1 \wedge \eta_3 - \frac{1}{2}\ \eta_2 \wedge d\eta_3 + \eta_{123}$
   (d) $b = 1/2,\ c = -1/2,\ f = -1/2$,
   $\varphi_4 = \frac{1}{2}\ \eta_1 \wedge d\eta_2 - \frac{1}{2}\ d\eta_1 \wedge \eta_3 - \frac{1}{2}\ \eta_2 \wedge d\eta_3 + \eta_{123}$
2. $b = c = f = 0,\ h = 1$
   (a) $a = 1/2,\ d = 1/2,\ e = 1/2$,
   $\varphi_5 = \frac{1}{2}\ d\eta_1 \wedge \eta_2 + \frac{1}{2}\ \eta_1 \wedge d\eta_3 + \frac{1}{2}\ d\eta_2 \wedge \eta_3 + \eta_{123}$
   (b) $a = -1/2,\ d = -1/2,\ e = 1/2$,
   $\varphi_6 = -\frac{1}{2}\ d\eta_1 \wedge \eta_2 - \frac{1}{2}\ \eta_1 \wedge d\eta_3 + \frac{1}{2}\ d\eta_2 \wedge \eta_3 + \eta_{123}$
   (c) $a = -1/2,\ d = 1/2,\ e = -1/2$,
   $\varphi_7 = -\frac{1}{2}\ d\eta_1 \wedge \eta_2 + \frac{1}{2}\ \eta_1 \wedge d\eta_3 - \frac{1}{2}\ d\eta_2 \wedge \eta_3 + \eta_{123}$
   (d) $a = 1/2,\ d = -1/2,\ e = -1/2$,
   $\varphi_8 = \frac{1}{2}\ d\eta_1 \wedge \eta_2 - \frac{1}{2}\ \eta_1 \wedge d\eta_3 - \frac{1}{2}\ d\eta_2 \wedge \eta_3 + \eta_{123}$.

Now we determine the class of G_2 structures φ_i, $i = 1, \ldots, 8$.

Theorem 3.1. *The eight G_2 structures φ_i, obtained above, are of type $\mathscr{W}$.*

Proof. We write the proof for φ_1 in details and computations for other G_2 structures are similar.
The exterior derivative $d\varphi_1$ of the 3-form

$$\varphi_1 = \frac{1}{2}\ \eta_1 \wedge d\eta_2 + \frac{1}{2}\ d\eta_1 \wedge \eta_3 + \frac{1}{2}\ \eta_2 \wedge d\eta_3 +\ \eta_{123}$$

is

$$\begin{aligned} d\varphi_1 \;=\; & \frac{1}{2} d\eta_1 \wedge d\eta_2 + \frac{1}{2} d\eta_1 \wedge d\eta_3 + \frac{1}{2} d\eta_2 \wedge d\eta_3 \\ & + d\eta_1 \wedge \eta_{23} - \eta_{13} \wedge d\eta_2 + \eta_{12} \wedge d\eta_3. \end{aligned}$$

In local coordinates,

$$\begin{aligned} d\varphi_1 \;=\; & 2\{\eta_{1245} + \eta_{1246} - \eta_{1247} - \eta_{1256} - \eta_{1257} + \eta_{1267} \\ & - \eta_{1345} + \eta_{1346} - \eta_{1347} - \eta_{1356} - \eta_{1357} - \eta_{1367} \\ & - \eta_{2345} + \eta_{2346} + \eta_{2347} + \eta_{2356} - \eta_{2357} - \eta_{2367}\}. \end{aligned}$$

Since $d\varphi_1 \neq 0$ locally, we have $d\varphi_1 \neq 0$ on M. Thus $\varphi_1 \notin \mathscr{W}_2$ and $\varphi_1 \notin \mathscr{P}$.
φ_1 is locally written as

$$\varphi_1 = \eta_{123} - \eta_{146} + \eta_{157} - \eta_{247} - \eta_{256} - \eta_{345} - \eta_{367}.$$

The Hodge-star of φ_1 is

$$*\varphi_1 = \frac{1}{2}\eta_{12} \wedge d\eta_1 - \frac{1}{2}\eta_{13} \wedge d\eta_3 + \frac{1}{2}\eta_{23} \wedge d\eta_2 + *\eta_{123}.$$

Comparing $d\varphi_1$ and $*\varphi_1$, we see that there is no constant k with the property that $d\varphi_1 = k * \varphi_1$. Then φ_1 can not be an element of $\mathscr{W}_1$, $\mathscr{W}_1 \oplus \mathscr{W}_2$.
Let α be a 1-form on M such that $d\varphi_1 = \alpha \wedge \varphi_1$. Then α can locally be written as $\alpha = \sum \alpha_i \eta_i$, where α_i are smooth functions on M. The coefficient of η_{1245} in $d\varphi_1$ is 2, while that of $\alpha \wedge \varphi_1$ is 0. Thus there does not exist such a 1-form α on M. This implies $\varphi_1 \notin \mathscr{W}_4$ and $\varphi_1 \notin \mathscr{W}_2 \oplus \mathscr{W}_4$.
In addition, $d\varphi_1 \wedge \varphi_1 = -12\eta_{1234567} = -12 d_{vol} \neq 0$. As a result φ_1 is not in $\mathscr{W}_3$, $\mathscr{W}_2 \oplus \mathscr{W}_3$, $\mathscr{W}_3 \oplus \mathscr{W}_4$ or $\mathscr{W}_2 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4$.

$$\begin{aligned} 2d * \varphi_1 \;=\; & \eta_2 \wedge d\eta_1 \wedge d\eta_1 - \eta_1 \wedge d\eta_1 \wedge d\eta_2 \\ & - \eta_3 \wedge d\eta_1 \wedge d\eta_3 + \eta_1 \wedge d\eta_3 \wedge d\eta_3 \\ & + \eta_3 \wedge d\eta_2 \wedge d\eta_2 - \eta_2 \wedge d\eta_2 \wedge d\eta_3. \end{aligned}$$

or in local coordinates,

$$\begin{aligned} d * \varphi_1 \;=\; & 2\{-\eta_{12345} - \eta_{12346} - \eta_{12347} \\ & - \eta_{12356} + \eta_{12357} - \eta_{12367}\} \\ & + 4\{\eta_{14567} + \eta_{24567} + \eta_{34567}\}. \end{aligned}$$

Thus $d * \varphi_1 \neq 0$ and $\varphi_1 \notin \mathscr{W}_1 \oplus \mathscr{W}_3$.
Assume that there exists a 1-form β on M satisfying $d * \varphi_1 = \beta \wedge *\varphi_1$. Writing β locally as $\beta = \sum \beta_i \eta_i$ and

comparing coefficients of η_{12357} and η_{14567} in $d*\varphi_1$ and $\beta\wedge*\varphi_1$ yield $\beta_1=2$ and $\beta_1=4$. Therefore such β does not exist. This yields that φ_1 is not in $\mathscr{W}_1\oplus\mathscr{W}_4$ or $\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$.
Let α be a 1-form and f be a smooth function on M such that $d\varphi_1=\alpha\wedge\varphi_1+f*\varphi_1$. The coefficients of η_{1245} and η_{4567} in local expressions of $d\varphi_1$ and $\alpha\wedge\varphi_1+f*\varphi_1$ imply that

$$f=-2 \text{ and } f=0,$$

a contradiction. Hence $\varphi_1\notin\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_4$.
We also have

$$*d\varphi_1\wedge\varphi_1=8\{-\eta_{124567}+\eta_{134567}-\eta_{234567}\}\neq 0.$$

This gives that φ_1 can not be an element of $\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_3$. Since the 3-form φ_1 does not satisfy any of defining relations of G_2 structures, it is in the widest class $\mathscr{W}$. □

4. Deformations of New G_2 Structures

In this section, we deform G_2 structures obtained in Section 3 and get new G_2 structures of type $\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$. Manifolds which are elements of the class $\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$ are called integrable G_2 manifolds or G_2 manifolds with anti-symmetric torsion [7, 8]. A manifold with G_2 structure has a uniquely determined metric covariant derivative preserving the G_2 structure and having anti-symmetric torsion if and only if it is an integrable manifold [7].
Let (M,g) be a 7-dimensional 3-Sasakian manifold with the 3-Sasakian structure $(\xi_i,\eta_i,\Phi_i)_{i=1,2,3}$. Consider the Riemannian metric g^s on M given by

$$g^s(x,y):=\begin{cases} g(x,y) & x \text{ (or } y)\in T^h \\ s^2g(x,y) & x \text{ and } y\in T^v,\end{cases}$$

where $s>0$ and the G_2 structures

$$\varphi_i=a\,\eta_1\wedge d\eta_2+b\,d\eta_1\wedge\eta_3+c\,\eta_2\wedge d\eta_3+\,\eta_{123}$$

obtained in Section 3, where a, b, $c=\pm\frac{1}{2}$ and $i=1,\dots,8$.
Let

$$F_1=\eta_{123} \text{ and } F_2=a\,\eta_1\wedge d\eta_2+b\,d\eta_1\wedge\eta_3+c\,\eta_2\wedge d\eta_3$$

and define the 3-forms φ_i^s by

$$\varphi_i^s:=F_1^s+F_2^s,$$

where

$$F_1^s:=s^3F_1,\ \ F_2^s:=sF_2.$$

Theorem 4.1. *The 3-forms φ_i^s are G_2 structures on $M^s:=(M,g^s)$. These 3-forms are of type $\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$ iff $s=\frac{1}{\sqrt{2}}$ and are in the widest class $\mathscr{W}$ iff $s\neq\frac{1}{\sqrt{2}}$.*

Proof. Take the g-orthonormal frame $\{e_1,\cdots,e_7\}$ given in [5] such that $e_1=\xi_1$, $e_2=\xi_2$ and $e_3=\xi_3$. Then

$$\{\xi_1/s,\xi_2/s,\xi_3/s,e_4,e_5,e_6,e_7\}$$

is a g^s-orthonormal frame with the corresponding coframe

$$\{s\eta_1,s\eta_2,s\eta_3,\eta_4,\eta_5,\eta_6,\eta_7\}.$$

Denote $\{\xi_1/s,\xi_2/s,\xi_3/s,e_4,e_5,e_6,e_7\}$ and $\{s\eta_1,s\eta_2,s\eta_3,\eta_4,\eta_5,\eta_6,\eta_7\}$ both by $\{Z_1,\cdots,Z_7\}$ with the notation in [6].
We give the proof for φ_1^s, other proofs are similar. To give a G_2 structure on (M,g^s), the 3-form φ_1^s should satisfy

$$(x\lrcorner\varphi_1^s)\wedge(y\lrcorner\varphi_1^s)\wedge\varphi_1^s=6g^s(x,y)d^s_{vol} \tag{5}$$

for $x,y\in\Gamma(TM)$. Choose the g^s-orthonormal frame $\{Z_1,\cdots,Z_7\}$. Then since

$$\begin{aligned}
Z_1\lrcorner\varphi_1^s &= s^2\eta_{23}-\eta_{46}+\eta_{57},\\
Z_2\lrcorner\varphi_1^s &= -s^2\eta_{13}-\eta_{47}-\eta_{56},\\
Z_3\lrcorner\varphi_1^s &= s^2\eta_{12}-\eta_{45}-\eta_{67},\\
Z_4\lrcorner\varphi_1^s &= s\eta_{16}+s\eta_{27}+s\eta_{35},\\
Z_5\lrcorner\varphi_1^s &= -s\eta_{17}+s\eta_{26}-s\eta_{34},\\
Z_6\lrcorner\varphi_1^s &= -s\eta_{14}-s\eta_{25}+s\eta_{37},\\
Z_7\lrcorner\varphi_1^s &= s\eta_{15}-s\eta_{24}-s\eta_{36},
\end{aligned}$$

we have

$$(Z_i\lrcorner\varphi_1^s)\wedge(Z_i\lrcorner\varphi_1^s)\wedge\varphi_1^s=6g^s(Z_i,Z_i)d^s_{vol}=6s^3g(e_i,e_i)d_{vol}$$

and

$$(Z_i\lrcorner\varphi_1^s)\wedge(Z_j\lrcorner\varphi_1^s)\wedge\varphi_1^s=0$$

for $i\neq j$ and thus φ_1^s gives a G_2 structure on M^s.
The exterior derivative of the 3-form

$$\varphi_1^s=+s^3\eta_{123}+\frac{s}{2}\,\eta_1\wedge d\eta_2+\frac{s}{2}\,d\eta_1\wedge\eta_3+\frac{s}{2}\,\eta_2\wedge d\eta_3$$

is

$$\begin{aligned}
d\varphi_1^s &= s^3d\eta_1\wedge\eta_{23}-s^3\eta_{13}\wedge d\eta_2+s^3\eta_{12}\wedge d\eta_3\\
&\quad+\frac{s}{2}d\eta_1\wedge d\eta_2+\frac{s}{2}d\eta_1\wedge d\eta_3+\frac{s}{2}d\eta_2\wedge d\eta_3.
\end{aligned}$$

Since

$$\begin{aligned}
d\varphi_1^s &= 2s\{\eta_{1245}+\eta_{1246}-s^2\eta_{1247}-s^2\eta_{1256}-\eta_{1257}\\
&\quad+\eta_{1267}-\eta_{1345}+s^2\eta_{1346}-\eta_{1347}-\eta_{1356}\\
&\quad-s^2\eta_{1357}-\eta_{1367}-s^2\eta_{2345}+\eta_{2346}+\eta_{2347}\\
&\quad+\eta_{2356}-\eta_{2357}-s^2\eta_{2367}\}
\end{aligned}$$

locally, $d\varphi_1^s\neq 0$ for all $s>0$. Thus φ_1^s is never of type $\mathscr{P}$ or $\mathscr{W}_2$.
Assume that α is a 1-form on M such that $d\varphi_1^s=\alpha\wedge\varphi_1^s$. This 1-form may be written as

$$\begin{aligned}
\alpha=\sum\alpha_iZ_i &= s\alpha_1\eta_1+s\alpha_2\eta_2+s\alpha_3\eta_3+\alpha_4\eta_4\\
&\quad+\alpha_5\eta_5+\alpha_6\eta_6+\alpha_7\eta_7.
\end{aligned}$$

In addition, since

$$\varphi_1^s=s^3\eta_{123}-s\eta_{146}+s\eta_{157}-s\eta_{247}-s\eta_{256}-s\eta_{345}-s\eta_{367},$$

we have

$$\begin{aligned}\alpha\wedge\varphi_1^s &= -s^3\alpha_4\eta_{1234}-s^3\alpha_5\eta_{1235}-s^3\alpha_6\eta_{1236}\\ &\quad -s^3\alpha_7\eta_{1237}+s^2\alpha_2\eta_{1246}-s^2\alpha_1\eta_{1247}\\ &\quad -s^2\alpha_1\eta_{1256}-s^2\alpha_2\eta_{1257}-s^2\alpha_1\eta_{1345}\\ &\quad +s^2\alpha_3\eta_{1346}-s^2\alpha_3\eta_{1357}-s^2\alpha_1\eta_{1367}\\ &\quad -s\alpha_5\eta_{1456}-s\alpha_4\eta_{1457}+s\alpha_7\eta_{1467}\\ &\quad +s\alpha_6\eta_{1567}-s^2\alpha_2\eta_{2345}+s^2\alpha_3\eta_{2347}\\ &\quad +s^2\alpha_3\eta_{2356}-s^2\alpha_2\eta_{2367}+s\alpha_4\eta_{2456}\\ &\quad -s\alpha_5\eta_{2457}-s\alpha_6\eta_{2467}+s\alpha_7\eta_{2567}\\ &\quad +s\alpha_6\eta_{3456}+s\alpha_7\eta_{3457}+s\alpha_4\eta_{3467}\\ &\quad +s\alpha_5\eta_{3567}.\end{aligned}$$

Comparing the coefficients of η_{1245} in $d\varphi_1^s$ and $\alpha\wedge\varphi_1^s$ implies $2s=0$. That is, there is not such 1-form and thus $M^s\notin\mathscr{W}_4$ and $M^s\notin\mathscr{W}_2\oplus\mathscr{W}_4$.

Since $d\varphi_1^s\wedge\varphi_1^s=-12s^2\eta_{1234567}$, we get $d\varphi_1^s\wedge\varphi_1^s\neq 0$ for all positive s and classes $\mathscr{W}_3$, $\mathscr{W}_2\oplus\mathscr{W}_3$, $\mathscr{W}_3\oplus\mathscr{W}_4$ and $\mathscr{W}_2\oplus\mathscr{W}_3\oplus\mathscr{W}_4$ are eliminated.

The Hodge-star $*_s\varphi_1^s$ of the 3-form φ_1^s is

$$\begin{aligned}*_s\varphi_1^s &= -Z_{1245}-Z_{1267}+Z_{1347}+Z_{1356}\\ &\quad -Z_{2346}+Z_{2357}+Z_{4567}\\ &= *\eta_{123}-s^2\eta_{1245}-s^2\eta_{1267}+s^2\eta_{1347}\\ &\quad +s^2\eta_{1356}-s^2\eta_{2346}+s^2\eta_{2357},\end{aligned}$$

which is globally

$$*_s\varphi_1^s=*F_1+\frac{s^2}{2}\eta_{23}\wedge d\eta_2-\frac{s^2}{2}\eta_{13}\wedge d\eta_3+\frac{s^2}{2}\eta_{12}\wedge d\eta_1.$$

The coefficient of η_{4567} in $d\varphi_1^s$ is 0, while in $*_s\varphi_1^s$ it is 1, so there is no constant k with the property that $d\varphi_1^s=k*_s\varphi_1^s$. Therefore the G_2 structure can not belong to $\mathscr{W}_1$ and $\mathscr{W}_1\oplus\mathscr{W}_2$.

$$\begin{aligned}d*_s\varphi_1^s &= \frac{s^2}{2}\{\eta_2\wedge d\eta_1\wedge d\eta_1-\eta_1\wedge d\eta_1\wedge d\eta_2\\ &\quad -\eta_3\wedge d\eta_1\wedge d\eta_3+\eta_1\wedge d\eta_3\wedge d\eta_3\\ &\quad +\eta_3\wedge d\eta_2\wedge d\eta_2-\eta_2\wedge d\eta_2\wedge d\eta_3\}\end{aligned}$$

and

$$\begin{aligned}d*_s\varphi_1^s &= 2s^2\{-\eta_{12345}-\eta_{12346}-\eta_{12347}\\ &\quad -\eta_{12356}+\eta_{12357}-\eta_{12367}\}\\ &\quad +4s^2\{\eta_{14567}+\eta_{24567}+\eta_{34567}\}\end{aligned}$$

give $d*_s\varphi_1^s\neq 0$, and $M^s\notin\mathscr{W}_1\oplus\mathscr{W}_3$.

Let α be a 1-form and f be a smooth function on M satisfying $d\varphi_1^s=\alpha\wedge\varphi_1^s+f*_s\varphi_1^s$. Comparing coefficients of η_{1245} and η_{4567} in $d\varphi_1^s$ and $\alpha\wedge\varphi_1^s+f*_s\varphi_1^s$ respectively, we conclude that

$$f=-2/s \text{ and } f=0.$$

Thus $\mathscr{W}_1\oplus\mathscr{W}_4$ and $\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_4$ are also eliminated.

In addition, in local coordinates,

$$\begin{aligned}*_s d\varphi_1^s &= -2sZ_{145}+\frac{2}{s}Z_{146}+\frac{2}{s}Z_{147}+\frac{2}{s}Z_{156}\\ &\quad -\frac{2}{s}Z_{157}-2sZ_{167}+\frac{2}{s}Z_{245}-2sZ_{246}\\ &\quad +\frac{2}{s}Z_{247}+\frac{2}{s}Z_{256}+2sZ_{257}+\frac{2}{s}Z_{267}\\ &\quad +\frac{2}{s}Z_{345}+\frac{2}{s}Z_{346}-2sZ_{347}-2sZ_{356}\\ &\quad -\frac{2}{s}Z_{357}+\frac{2}{s}Z_{367}\\ &= -2s^2\eta_{145}+2\eta_{146}+2\eta_{147}+2\eta_{156}-2\eta_{157}\\ &\quad -2s^2\eta_{167}+2\eta_{245}-2s^2\eta_{246}+2\eta_{247}+2\eta_{256}\\ &\quad +2s^2\eta_{257}+2\eta_{267}+2\eta_{345}+2\eta_{346}-2s^2\eta_{347}\\ &\quad -2s^2\eta_{356}-2\eta_{357}+2\eta_{367}\end{aligned}$$

and

$$(*_s d\varphi_1^s)\wedge\varphi^s = 4s(1+s^2)\{\eta_{134567}-\eta_{124567}-\eta_{234567}\}.$$

Thus $(*_s d\varphi_1^s)\wedge\varphi_1^s$ is non-zero. The 3-form φ_1^s is not in $\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_3$.

Let β be a 1-form on M such that $d*_s\varphi_1^s=\beta\wedge*_s\varphi_1^s$. Then β can be locally written as

$$\begin{aligned}\beta=\sum\beta_iZ_i &= s\beta_1\eta_1+s\beta_2\eta_2+s\beta_3\eta_3+\beta_4\eta_4\\ &\quad +\beta_5\eta_5+\beta_6\eta_6+\beta_7\eta_7\end{aligned}$$

for $\beta_i\in C^\infty(M)$. Since

$$\begin{aligned}\beta\wedge*_s\varphi_1^s &= -s^3\beta_3\eta_{12345}-s^3\beta_1\eta_{12346}-s^3\beta_2\eta_{12347}\\ &\quad -s^3\beta_2\eta_{12356}+s^3\beta_1\eta_{12357}-s^3\beta_3\eta_{12367}\\ &\quad -s^2\beta_6\eta_{12456}-s^2\beta_7\eta_{12457}-s^2\beta_4\eta_{12467}\\ &\quad -s^2\beta_5\eta_{12567}+s^2\beta_4\eta_{13456}-s^2\beta_5\eta_{13457}\\ &\quad -s^2\beta_6\eta_{13467}+s^2\beta_7\eta_{13567}+s\beta_1\eta_{14567}\\ &\quad +s^2\beta_5\eta_{23456}+s^2\beta_4\eta_{23457}-s^2\beta_7\eta_{23467}\\ &\quad -s^2\beta_6\eta_{23567}+s\beta_2\eta_{24567}+s\beta_3\eta_{34567},\end{aligned}$$

$d*_s\varphi_1^s=\beta\wedge*_s\varphi_1^s$ holds if and only if $\frac{2}{s}=4s$, $\beta_1=\beta_2=\beta_3=4s$ and $\beta_4=\beta_5=\beta_6=\beta_7=0$. The identity $\frac{2}{s}=4s$ gives $s=\frac{1}{\sqrt{2}}$. For $s=\frac{1}{\sqrt{2}}$ and $\beta=2(\eta_1+\eta_2+\eta_3)$, the equation $d*_s\varphi_1^s=\beta\wedge*_s\varphi_1^s$ holds globally. As a result, the G_2 structure φ_1^s is of type $\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$ iff $s=\frac{1}{\sqrt{2}}$. If $s\neq\frac{1}{\sqrt{2}}$, then φ_1^s is in the widest class $\mathscr{W}$.

For G_2 structures φ_i^s, where $i=2,\dots,8$, we only write 1-forms β_i such that $d*_s\varphi_i^s=\beta_i\wedge*_s\varphi_i^s$. These 1-forms are

$$\beta_2=2\eta_1-2\eta_2-2\eta_3,\;\;\beta_3=-2\eta_1+2\eta_2-2\eta_3,$$

$$\beta_4=-2\eta_1-2\eta_2+2\eta_3,\;\;\beta_5=-2\eta_1-2\eta_2-2\eta_3,$$

$$\beta_6=-2\eta_1+2\eta_2+2\eta_3,\;\;\beta_7=2\eta_1-2\eta_2+2\eta_3,$$

$$\beta_8=2\eta_1+2\eta_2-2\eta_3.$$

□

Let $s=\frac{1}{\sqrt{2}}$. Then G_2 structures φ_i^s are in the class $\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$. Thus there exists a uniquely determined metric covariant derivative ∇_i with anti-symmetric torsion tensor

T_i on each (M, g^s, φ_i^s) preserving the G_2 structure and T_i are given in [7] by the formula

$$T_i = \frac{1}{6} g^s(d\varphi_i^s, *_s\varphi_i^s)\varphi_i^s - *_s d\varphi_i^s + *_s(\beta \wedge \varphi_i^s).$$

We directly compute T_i for each i and obtain the same anti-symmetric torsion tensor for each covariant derivative ∇_i:
For each $i \in \{1, \ldots, 8\}$, we have

$$T_i = \frac{1}{2}\eta_1 \wedge d\eta_1 + \frac{1}{2}\eta_2 \wedge d\eta_2 + \frac{1}{2}\eta_3 \wedge d\eta_3 + 2\eta_{123}.$$

Now we introduce some properties of T.

1. The torsion tensor T is not ∇-parallel:

   Choose the g^s-orthonormal frame $\{Z_1, \cdots, Z_7\}$ on an open subset of M. By the Kozsul formula we write the Levi-Civita covariant derivative ∇^{g^s} in local coordinates. Since $\nabla = \nabla^{g^s} + T/2$ and

   $$\begin{aligned} T &= \tfrac{1}{2s^2} Z_1 \wedge dZ_1 + \tfrac{1}{2s^2} Z_2 \wedge dZ_2 \\ &\quad + \tfrac{1}{2s^2} Z_3 \wedge dZ_3 + \tfrac{2}{s^3} Z_{123} \\ &= -\tfrac{1}{s^3} Z_{123} - \tfrac{1}{s} Z_{145} - \tfrac{1}{s} Z_{167} - \tfrac{1}{s} Z_{246} \\ &\quad + \tfrac{1}{s} Z_{257} - \tfrac{1}{s} Z_{347} - \tfrac{1}{s} Z_{356}, \end{aligned}$$

   we compute ∇ locally and we obtain

   $$\nabla_{Z_1} T(Z_2, Z_4, Z_7) = 4 \neq 0.$$

2. The torsion tensor is not closed:

   $$dT = \sum (Z_i \lrcorner T) \wedge (Z_i \lrcorner T) = 12 *_s Z_{123} = 12 * \eta_{123}.$$

Note: Take a 7-dimensional 3-Sasakian manifold (M, g) with the 3-Sasakian structure $(\xi_i, \eta_i, \Phi_i)_{i=1,2,3}$. Consider the 3-form

$$\begin{aligned} \varphi &= a\, \eta_1 \wedge d\eta_1 + b\, \eta_2 \wedge d\eta_2 + c\, \eta_3 \wedge d\eta_3 \\ &\quad + d\, \eta_1 \wedge \eta_2 \wedge \eta_3 + e\, \eta_1 \wedge d\eta_2 + f\, \eta_1 \wedge d\eta_3 \\ &\quad + g\, \eta_2 \wedge d\eta_1 + h\, \eta_2 \wedge d\eta_3 + k\, \eta_3 \wedge d\eta_1 \\ &\quad + l\, \eta_3 \wedge d\eta_2 \end{aligned} \tag{6}$$

for $a, b, c, d, e, f, g, h, k, l \in \mathbb{R}$. To get a G_2 structure on M determined by φ, following equations should be satisfied:

$$(a^2 + e^2 + f^2)(-2(a+b+c)+d) = \frac{1}{4}, \tag{7}$$
$$(c^2 + k^2 + l^2)(-2(a+b+c)+d) = \frac{1}{4}, \tag{8}$$
$$(b^2 + h^2 + g^2)(-2(a+b+c)+d) = \frac{1}{4}, \tag{9}$$
$$(-2(a+b+c)+d)(eb + ag + hf) = 0, \tag{10}$$
$$(-2(a+b+c)+d)(cf + el + ak) = 0, \tag{11}$$
$$(-2(a+b+c)+d)(hc + kg + bl) = 0, \tag{12}$$
$$abc - ceg - bfk - ahl + fgl + hek = \frac{1}{8}. \tag{13}$$

This system has infinitely many solutions. Now we will determine under which conditions G_2 structures obtained are closed or co-closed.

$$\begin{aligned} d\varphi &= a\, d\eta_1 \wedge d\eta_1 + b\, d\eta_2 \wedge d\eta_2 + c\, d\eta_3 \wedge d\eta_3 \\ &\quad + d\,(\, d\eta_1 \wedge \eta_{23} - \, d\eta_2 \wedge \eta_{13} + \, d\eta_3 \wedge \eta_{12}) \\ &\quad + (e+g)\, d\eta_1 \wedge d\eta_2 + (f+k)\, d\eta_1 \wedge d\eta_3 \\ &\quad + (h+l)\, d\eta_2 \wedge d\eta_3. \end{aligned}$$

If φ is closed, then it is closed according to the local frame chosen. Thus

$$\begin{aligned} d\varphi &= (8a - 2d)\{\eta_{2345} + \eta_{2367}\} \\ &\quad + (8b - 2d)\{\eta_{1357} - \eta_{1346}\} \\ &\quad + (8c - 2d)\{\eta_{1247} + \eta_{1256}\} + 8(a+b+c)\eta_{4567} \\ &\quad + 4(e+g)\{\eta_{2346} - \eta_{2357} - \eta_{1345} - \eta_{1367}\} \\ &\quad + 4(f+k)\{\eta_{2347} + \eta_{2356} + \eta_{1245} + \eta_{1267}\} \\ &\quad + 4(h+l)\{-\eta_{1347} - \eta_{1356} + \eta_{1246} - \eta_{1257}\} \\ &= 0. \end{aligned}$$

All coefficients should be zero. First we get

$$d = 4a = 4b = 4c.$$

Since $a = b = c$ and $a + b + c = 0$, this gives

$$a = b = c = d = 0.$$

In addition,

$$f = -k, \; e = -g, \; h = -l$$

imply

$$abc - ceg - bfk - ahl + fgl + hek = 0.$$

This can not hold, since for a G_2 structure we have (13). As a result, there is no closed G_2 structure φ on a 7-dimensional 3-Sasakian manifold of the form (6).
Now we investigate the existence of co-closed G_2 structures. We have

$$\begin{aligned} d*\varphi &= e\, \eta_3 \wedge d\eta_2 \wedge d\eta_2 - e\, \eta_2 \wedge d\eta_2 \wedge d\eta_3 \\ &\quad + f\, \eta_3 \wedge d\eta_2 \wedge d\eta_3 - f\, \eta_2 \wedge d\eta_3 \wedge d\eta_3 - \\ &\quad g\, \eta_3 \wedge d\eta_1 \wedge d\eta_1 + g\, \eta_1 \wedge d\eta_1 \wedge d\eta_3 \\ &\quad - h\, \eta_3 \wedge d\eta_1 \wedge d\eta_3 + h\, \eta_1 \wedge d\eta_3 \wedge d\eta_3, \end{aligned}$$

or, in local coordinates,

$$\begin{aligned} d*\varphi &= -4h\eta_{12345} + 4f\eta_{12346} + 4(g-e)\eta_{12347} \\ &\quad + 4(g-e)\eta_{12356} - 4f\eta_{12357} + 4h\eta_{12367} \\ &\quad + 8h\eta_{14567} - 8f\eta_{24567} + 8(e-g)\eta_{34567}. \end{aligned}$$

To get a co-closed G_2 structure, equations (7)-(13) together with

$$e = g, \; f = h = 0$$

should be satisfied. Only G_2 structures with these properties are the cocalibrated G_2 structure and three nearly parallel G_2 structures all given in [5].

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